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A Lattice-Theoretical Framework for Annular Filters in Morphological Image Processing

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Abstract. We study the idempotence of operators of the form $\varepsilon \vee \mathbf{id} \wedge \delta$ (where $\varepsilon \leq \delta$ and both ε and δ are increasing) on a modular lattice $\mathscr{L}$, in relation to the idempotence of the operators $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$. We consider also the conditions under which $\varepsilon \vee \mathbf{id} \wedge \delta$ is the composition of $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$. The case where δ is a dilation and ε an erosion is of special interest. When $\mathscr{L}$ is a complete lattice on which Minkowski operations can be defined, we obtain very precise conditions for the idempotence of these operators. Here $\mathbf{id} \wedge \delta$ is called an *annular opening*, $\varepsilon \vee \mathbf{id}$ is called an *annular closing*, and $\varepsilon \vee \mathbf{id} \wedge \delta$ is called an *annular filter*. Our theory can be applied to the design of idempotent morphological filters removing isolated spots in digital pictures.

Keywords: Modular lattice, Idempotent operators, Image processing, Mathematical morphology, Erosion, Dilation, Annular filters

1 Introduction

Mathematical morphology is a branch of image processing and analysis which originates from a set-theoretical approach where a figure is an element of $\mathscr{P}(E)$, the set of subsets of a space E (which can be the Euclidean space $\mathbb{R}^d$ or the digital space $\mathbb{Z}^d$), and the shape of that figure is studied through its interactions (unions and intersections) with the translates of a probe called *structuring element*; the latter is generally a compact set [9,14]. In order to apply this approach to $\mathrm{Fun}(E, \overline{\mathbb{R}})$, the family of numerical functions $F : E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

(modeling grey-level images), structuring elements become *structuring functions*, which are numerical functions whose support (defined here as the set of points where the function's value is $> -\infty$) is generally compact. The structuring function acts as a probe by being translated both in the space of points and in the set of numerical values representing grey-levels of the image points; then the interactions between the numerical function representing the image and the translates of the structuring function are realized through lattice-theoretical join and meet operations (which generalize unions and intersections). This has led to an algebraic theory of morphological operations, based on lattice theory [15, 7, 4]: in such an approach, images are modeled as elements of a complete lattice, and morphological image operations are transformations on that complete lattice, which satisfy some specified algebraic properties pertaining to order and composition.

Two basic complete lattices are $\mathscr{P}(E)$ (see above) and $\mathrm{Fun}(E, \mathscr{T})$, the grey-level images defined on E with grey-values in some other complete lattice $\mathscr{T}$. If $\mathscr{T} = \overline{\mathbb{R}}$, we shall write $\mathrm{Fun}(E)$ rather than $\mathrm{Fun}(E, \overline{\mathbb{R}})$.

A classical morphological operator is the *opening* by a structuring element B. In the set-theoretical setting, it associates to every set $X \subseteq E$ the union of all translates of B included in X. A similar definition holds in the case of a complete lattice with a group of automorphisms in place of the translations [13]. The behaviour of this opening is to remove from a set X all portions which are too narrow to contain a translate of B; for grey-level images (numerical functions), it darkens light portions which are too narrow to contain a translate of B. The dual operator is the *closing* by B; it removes narrow holes from a set, and in a grey-level image it lightens dark narrow image portions.

The opening by a structuring element is an *algebraic opening* [15], in other words it is idempotent (equal to its auto-composition), increasing (isotone), and anti-extensive (it decreases every object). It is however not the only type of algebraic opening. Another type of opening has been considered, which removes points from a set on the basis of their isolation. It was introduced by Serra in [15, pp. 107,108]. Let E be a Euclidean or digital space, and take a symmetric structuring element B which does not contain the origin; then the set operator on $\mathscr{P}(E)$ given by

$$X \mapsto X \cap (X \oplus B), \tag{1.1}$$

where $\oplus$ is the Minkowski addition, replaces a set X by the union of all pairs $\{p, q\}$ inside X such that p and q are *adjacent* in the sense that $p - q \in B$; as B is symmetric, this adjacency relation is symmetric. This operation is an algebraic opening, and it removes from a set X all *isolated* points, where a point $p \in X$ is called isolated if there is no point $q \in X$ such that p is adjacent to q. In [15], the effect of this operation on a natural image was illustrated in the case where the structuring element B was a ring, and this led to it being called the *annular opening*. One can consider the dual operation

$$X \mapsto X \cup (X \ominus B), \tag{1.2}$$

where $\ominus$ is the Minkowski subtraction; it is an algebraic closing, and so it is called the *annular closing*. Its effect is to add to a set X all isolated points from the background X^c, in other words to remove isolated hole points.

The above-defined annular opening and closing are translation-invariant; in fact the adjacency relation is invariant under translations. As explained in the Introduction of [8], it is easy to generalize them by taking an arbitrary symmetric relation $\sim$ on the space E, which is not necessarily translation-invariant; then the annular closing removes from a set X all points $p \in X$ such that there is no point $q \in X$ with $p \sim q$ (i.e., p is adjacent to q); dually the annular closing adds to a set X all points $p \notin X$ such that there is not point $q \notin X$ with $p \sim q$.

It is known that any increasing operator for binary images extends to a "flat" operator for grey-level images [3,14]; the set structuring element involved in such an operator is then considered as a "flat" structuring function. When annular openings are applied to grey-level images, isolated light spots are removed. Grey-level annular openings with non-flat structuring elements were introduced by the authors in [13]. Given a grey-level structuring function A whose support is symmetric (in other words, for every point x we have $A(x) > -\infty \Leftrightarrow A(-x) > -\infty$), and such that every point x in that support (i.e., with $A(x) > -\infty$) satisfies

$$A(x) + A(-x) \geq 0, \tag{1.3}$$

we consider the operator on grey-level functions

$$I \mapsto I \wedge (I \oplus A), \tag{1.4}$$

where $\wedge$ is the meet operation and $\oplus$ is the generalization of the Minkowski addition to numerical functions [7] (in other words, a sup-convolution). This operator is an algebraic opening, and it can also be called an annular opening. We have dually the annular closing

$$I \mapsto I \vee (I \ominus A), \tag{1.5}$$

where $\vee$ is the join operation and $\ominus$ is the generalization of the Minkowski subtraction to numerical functions [7] (in fact, a form of inf-correlation of A and I). In the case where A is constantly zero on its support B, we get the "flat" operators associated to the set-theoretical ones defined in (1.1) and (1.2).

It is more difficult here (for grey-level images) to interpret the behaviour of annular openings and closings in terms of an adjacency relation. Here we have numerical functions in place of sets, and so the basic constituent of an image is not a point, but the pair (p, v) associating a *finite* grey-level value $v \in \mathbb{R}$ to a point $p \in E$. If we write $I(p)$ for the grey-level associated by the image I to the point p, then the notion that the pair (p, v) is "in I" must be interpreted as $v \leq I(p)$; technically speaking, this means that (p, v) belongs to the so-called *umbra* of I [11]. Now two pairs (p, v) and (q, w) (where $p, q \in E$ and $v, w \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$) can be considered as "adjacent" if $p - q$ is in the support of A (that is, $A(p - q) > -\infty$), and we have both inequalities:

$$v - w \leq A(p - q) \qquad \text{and} \qquad w - v \leq A(q - p).$$

It can be shown [13] that after the application of the annular opening (1.4) to I, the grey-level associated to p will be the supremum of all finite values v_i such that (p, v_i) is "in I" and there is a pair (q_i, w_i) which is "in I" and "adjacent" to (p, v_i). We illustrate this in Fig. 1 with $E = \mathbb{Z}$ and taking for A the function with support $\{-1, +1\}$, having value $+1$ on it; here the grey-level v associated to pixel p is "isolated" if both neighbours of p have grey-level $< v - 1$, and in this case the annular opening reduces the grey-level of p to one plus the maximum grey-level of its two neighbours.

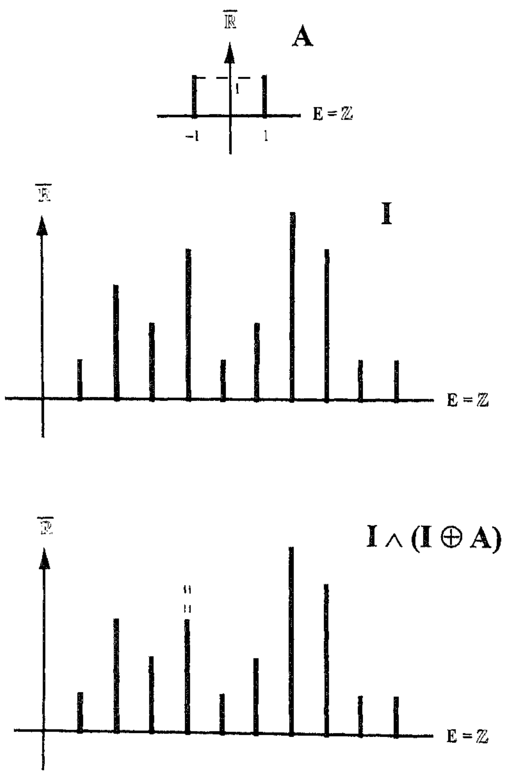

Fig. 1. We assume a discrete space $E = \mathbb{Z}$. Top: the structuring element A, with support $\{-1, +1\}$, and having constant grey-level $+1$ on it. *Middle*: the original function I, representing a one- dimensional signal. *Bottom*: applying the annular opening to I yields $I \wedge (I \oplus A)$, where isolated peaks are reduced to one unit above their surrounding; we show in *dashed lines* the portions where the grey-level in I is larger than in $I \wedge (I \oplus A)$

Note that when we take continuously varying grey-levels in $\overline{\mathbb{R}}$ (and not discrete grey-levels, e.g. in $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$), and a structuring function with infinite support, it may happen that such a supremum v is reached without the existence of a pair (q, w) "in I" and "adjacent" to (p, v). This illustrates one of the main difficulties encountered when extending set transformations into transformations of numerical functions. Further examples of this fact have been given in [11], where we showed that a mathematically consistent treatment of morphological operations on numerical functions $E \to \overline{\mathbb{R}}$ requires the consideration of the complete lattice structure of the space of such functions. Thus all our results concerning morphological operations on grey-level images, including the ones we gave on annular openings [13], are expressed in a wider framework of complete lattices having suitable properties.

The starting point of a recent paper by the authors [8] is the following question: *Is it possible to devise an increasing idempotent operator which would behave as both an annular opening and an annular closing, in other words removing from a set all its isolated points, and at the same time adding to it all isolated points of its complement?* In [8] we generalized annular openings and closings for sets into a more general *annular operator* which removes isolated points both in the foreground X and in the background X^c. Here an isolated point is defined in terms of an underlying adjacency relation between pixels, which may be different for foreground and background pixels. More precisely, let $\overset{0}{\sim}$ and $\overset{1}{\sim}$ be two symmetric relations on E (distinct or not), which stand for background and foreground adjacencies; we derive from them the operators δ_s and ε_s $(s = 0, 1)$ defined by

$$\delta_s(X) = \{y \in E \mid \exists x \in X,\ x \overset{s}{\sim} y\}$$

$$\varepsilon_s(X) = \{y \in E \mid \nexists x \in X^c,\ x \overset{s}{\sim} y\}.$$

Technically speaking, δ_s is a dilation while ε_s is an erosion [7], and the dual of δ_s w.r.t. complementation. Write **id** for the *identity* operator $X \mapsto X$. Assuming that for every $x \in E$ there is some $y \in E$ with $x \overset{0}{\sim} y \overset{1}{\sim} x$, we obtain the *annular operator*

$$\varepsilon_0 \vee (\mathbf{id} \wedge \delta_1) = (\varepsilon_0 \vee \mathbf{id}) \wedge \delta_1, \tag{1.6}$$

which removes from X isolated points (w.r.t. foreground adjacency $\overset{1}{\sim}$) and at the same time adds to X isolated points of X^c (w.r.t. background adjacency $\overset{0}{\sim}$). We showed that under a specific condition in terms of both adjacencies, this operator is idempotent; we call it then an *annular filter*.

The goal of this paper is to extend annular filters to other types of pictorial objects than sets, in particular to numerical functions. In the most general sense the object space consisting of all images is a lattice $\mathscr{L}$. The annular opening and closing become then operators $\mathscr{L} \to \mathscr{L}$ of the form $\mathbf{id} \wedge \delta$ and $\mathbf{id} \vee \varepsilon$, where $\mathbf{id}$ is the identity while δ and ε are a dilation and an erosion [7] respectively (in particular, they are increasing). If $\varepsilon \not\leq \delta$, then the annular opening and closing are incompatible and cannot be combined. This is easily explained in the case where $\mathscr{L} = \mathrm{Fun}(E)$: if $\varepsilon \not\leq \delta$, this means that there is some image $I \in \mathscr{L}$ such that $\varepsilon(I) \not\leq \delta(I)$, so that for some point $p \in E$ we have $\varepsilon(I)(p) > \delta(I)(p)$; now at point p, the behaviour of $\mathbf{id} \wedge \delta$ is to decrease $I(p)$ to $\delta(I)(p)$ whenever $I(p) > \delta(I)(p)$, while the behaviour of $\mathbf{id} \vee \varepsilon$ is to increase $I(p)$ to $\varepsilon(I)(p)$ whenever $\varepsilon(I)(p) > I(p)$; thus for $\varepsilon(I)(p) > I(p) > \delta(I)(p)$, the first operator requires decreasing $I(p)$ to $\delta(I)(p)$, while the second one requires increasing $I(p)$ to $\varepsilon(I)(p)$, and so they are contradictory.

We suppose thus that $\varepsilon \leq \delta$. It is also necessary to assume that the lattice $\mathscr{L}$ is *modular*, so that we have the equality

$$\varepsilon \vee (\mathbf{id} \wedge \delta) = (\varepsilon \vee \mathbf{id}) \wedge \delta,$$

and we can simply write

$$\varepsilon \vee \mathbf{id} \wedge \delta. \tag{1.7}$$

Note that both the lattices $\mathscr{P}(E)$ and $\mathrm{Fun}(E)$ are modular. We obtain in this way the required operator which generalizes the annular operator given in (1.6). To see that (1.7) gives indeed a combination of the behaviours of $\mathbf{id} \wedge \delta$ and $\mathbf{id} \vee \varepsilon$, we consider again the case where $\mathscr{L} = \mathrm{Fun}(E)$. Given a point $p \in E$, we write $i = I(p)$, $e = \varepsilon(I)(p)$, and $d = \delta(I)(p)$; since $e \leq d$, it is easily seen that $e \vee i \wedge d$ is the median of the three values i, e, and d; we have then three cases:

(a) $i < e \leq d$: Here $\mathbf{id} \vee \varepsilon$ increases the value at p from i to e, while $\mathbf{id} \wedge \delta$ does not change i; now $\varepsilon \vee \mathbf{id} \wedge \delta$ changes the value at p into the median value $e \vee i \wedge d = e$.
(b) $e \leq i \leq d$. Here both $\mathbf{id} \wedge \delta$ and $\mathbf{id} \vee \varepsilon$ do not modify the value i at p, and similarly $\varepsilon \vee \mathbf{id} \wedge \delta$ does not modify it, since $e \vee i \wedge d = i$.
(c) $e \leq d < i$. Here $\mathbf{id} \wedge \delta$ decreases the value at p from i to d, while $\mathbf{id} \vee \varepsilon$ does not change i; now $\varepsilon \vee \mathbf{id} \wedge \delta$ changes the value at p into the median value $e \vee i \wedge d = d$.

Hence in the case of numerical functions, the operator $\varepsilon \vee \mathbf{id} \wedge \delta$ combines the behaviours of the annular opening $\mathbf{id} \wedge \delta$ and the annular closing $\mathbf{id} \vee \varepsilon$, and we call it an *annular operator.* As in [8] in the case of set operators arising from adjacency relations, we will aim to find conditions for the idempotence of $\varepsilon \vee \mathbf{id} \wedge \delta$, but this time in the most general setting where the space of pictorial objects is a modular lattice on which we make as few assumptions as possible.

In Section 2 we consider the case where $\mathscr{L}$ is an arbitrary modular lattice (not necessarily complete), and give then conditions for the idempotence of $\varepsilon \vee \mathbf{id} \wedge \delta$; some of these conditions are necessary and sufficient, other ones are only sufficient. We illustrate these results with sets, and show that this gives some theorems obtained in [8].

In Section 3 we suppose that $\mathscr{L}$ is a complete lattice in which the so-called "Basic Assumption" introduced in [7] is satisfied; this assumption is the one which allows us to define on $\mathscr{L}$ the Minkowski addition $\oplus$ and subtraction $\ominus$. We recall and generalize the results obtained in [13] for annular openings in this framework, and obtain a sufficient condition for the idempotence of the annular operator of the form

$$\varepsilon_B \vee \mathbf{id} \wedge \delta_A : X \mapsto (X \ominus B) \vee X \wedge (X \oplus A), \tag{1.8}$$

which is a specialization of (1.7). We explicit this theory with the particular case of $\mathrm{Fun}(E)$.

In Section 4 we examine conditions under which the idempotent annular filter $\varepsilon \vee \mathbf{id} \wedge \delta$ can be obtained as the composition of the annular opening $\mathbf{id} \wedge \delta$ and the annular closing $\mathbf{id} \vee \varepsilon$. Here we assume again that $\mathscr{L}$ is an arbitrary modular lattice. We illustrate our results in the particular case where $\mathscr{L} = \mathscr{P}(E)$.

2 Annular Operators on Modular Lattices

In this section we will give general results concerning the properties of an operator of the form $\varepsilon \vee \mathbf{id} \wedge \delta$ on an arbitrary modular lattice, in particular conditions yielding idempotence. We assume that the reader is acquainted with the basic elements of lattice theory [1]. Refer to [7] for a short reminder.

Let $(\mathscr{L}, \leq)$ be a *lattice*, where $\leq$ is a partial ordering relation on $\mathscr{L}$, and the *join* and *meet* of two elements X, Y of $\mathscr{L}$ are written $X \vee Y$ and $X \wedge Y$ respectively. It is easily checked that for every $L, M, H \in \mathscr{L}$,

$$L \leq H \quad \Longrightarrow \quad L \vee (M \wedge H) \leq (L \vee M) \wedge H.$$

Now $\mathscr{L}$ is said to be *modular* if this inequality becomes an equality:

$$L \leq H \quad \Longrightarrow \quad L \vee (M \wedge H) = (L \vee M) \wedge H. \tag{2.1}$$

In such a case one simply writes $L \vee M \wedge H$. Note that since in every lattice $\mathscr{L}$ the inequalities $L \leq L \vee (M \wedge H)$ and $(L \vee M) \wedge H \leq H$ hold for any $L, M, H \in \mathscr{L}$, it is clear that the equality $L \vee (M \wedge H) = (L \vee M) \wedge H$ can be true only if $L \leq H$. Next, we say that $\mathscr{L}$ is *distributive* if

$$\forall X, Y, Z \in \mathscr{L}, \qquad X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z),$$

or equivalently [1],

$$\forall X, Y, Z \in \mathscr{L}, \qquad X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z).$$

Every distributive lattice is modular. Examples of distributive lattices include $\mathscr{P}(E)$, the family of parts of a set E, ordered by inclusion, or the set $\mathrm{Fun}(E)$ of functions $E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, ordered by setting $F \leq G$ if $F(x) \leq G(x)$ for all $x \in E$. An example of a modular lattice which is not distributive is given by that of vector subspaces of a vector space. Finally, two typical examples of non-modular lattices are the one of convex subsets of a Euclidean space, and the one of partitions of a set, where the ordering relation $P \leq P'$ denotes that partition P is finer than partition P'.

A map $\mathscr{L} \to \mathscr{L}$ is called an *operator*. The set $\mathscr{L}^{\mathscr{L}}$ of operators inherits in a natural way the partial order $\leq$ and the lattice structure of $\mathscr{L}$: $\psi \leq \xi$ means that $\psi(X) \leq \xi(X)$ for every $X \in \mathscr{L}$, while $\psi \vee \xi$ and $\psi \wedge \xi$ are given by $(\psi \vee \xi)(X) = \psi(X) \vee \xi(X)$ and $(\psi \wedge \xi)(X) = \psi(X) \wedge \xi(X)$ for every $X \in \mathscr{L}$. Moreover, whenever $\mathscr{L}$ is modular or distributive, so is $\mathscr{L}^{\mathscr{L}}$. The composition $\psi\xi$ of two operators ψ and ξ is defined by $\psi\xi(X) = \psi(\xi(X))$; furthermore ψ^2 denotes $\psi\psi$. Note that for $\psi, \xi, \eta \in \mathscr{L}^{\mathscr{L}}$ we have $(\psi \vee \xi)\eta = \psi\eta \vee \xi\eta$ and $(\psi \wedge \xi)\eta = \psi\eta \wedge \xi\eta$. The operator ψ is said to be *idempotent* if $\psi^2 = \psi$. Write **id** for the identity operator.

From now on, we assume that the lattice $\mathscr{L}$ is modular. An operator ψ is said to be *increasing* if for every $X, Y \in \mathscr{L}$ satisfying $X \leq Y$, we have $\psi(X) \leq \psi(Y)$. We will study operators of the form

$$(\varepsilon \vee \mathbf{id}) \wedge \delta = \varepsilon \vee (\mathbf{id} \wedge \delta) \qquad \text{where} \quad \varepsilon \leq \delta \tag{2.2}$$

and ε, δ are increasing. The assumption $\varepsilon \leq \delta$ that we use throughout our theorems stands mainly to allow us to define the operator $\varepsilon \vee \mathbf{id} \wedge \delta$ without ambiguity; however, we saw also in the Introduction that when $\varepsilon \not\leq \delta$, the annular opening $\mathbf{id} \wedge \delta$ and the annular closing $\mathbf{id} \vee \varepsilon$ have in practice contradictory behaviours, so that one cannot envisage designing an operator combining their effects. Our main goal is to find conditions which guarantee that such an operator $\varepsilon \vee \mathbf{id} \wedge \delta$ is idempotent. Since $\varepsilon, \delta, \mathbf{id}$ are increasing, we will obtain in such a way an increasing idempotent operator, in other words a *morphological filter* [15]. Moreover, we will generally consider the case where the operators $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$ are idempotent (representing the annular closing and annular opening, respectively). Note that in (2.2) the operators δ and ε play dual roles: if we invert the ordering $\leq$ (interchanging $\vee$ and $\wedge$) and interchange δ and ε, then we will interchange $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$, while $\varepsilon \vee \mathbf{id} \wedge \delta$ will remain the same; thus many of our results will consist of two parts, the second one being the dual of the first. In fact, this is just a manifestation of the *duality principle* in the theory of partially ordered sets.

In the sequel, we will generally use the word *filter* in the sense of a *morphological filter* [15], that is an idempotent increasing operator. This terminology is a variant of the customary one in signal processing, where a filter denotes a linear translation-invariant operator for signals, but has nothing to do with the use of this word "filter" in topology and in lattice theory, as in for example in [1].

There is no precise formal definition of an *annular opening*; in the most general sense, it is an idempotent operator of the form $\mathbf{id} \wedge \delta$, where δ is increasing, but more specifically, one generally assumes here that δ is a *dilation*, that is an operator which distributes the supremum [7]. Similarly, an *annular closing* is in the most general sense an idempotent operator of the form $\mathbf{id} \vee \varepsilon$, where ε is increasing, but more specifically, ε can be assumed to be an *erosion*, that is an operator which distributes the infimum [7]. Finally, we call an *annular filter* an idempotent operator of the form $\varepsilon \vee \mathbf{id} \wedge \delta$, where $\mathbf{id} \wedge \delta$ is an annular opening and $\mathbf{id} \vee \varepsilon$ is an annular closing.

The following result characterizes the idempotence of these two operators $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$:

Lemma 2.1 *Let ε, δ be two operators on the lattice $\mathscr{L}$.*

(*i*) $\varepsilon \vee \mathbf{id}$ *is idempotent if and only if* $\varepsilon(\varepsilon \vee \mathbf{id}) \leq \varepsilon \vee \mathbf{id}$.
(*ii*) $\mathbf{id} \wedge \delta$ *is idempotent if and only if* $\delta(\mathbf{id} \wedge \delta) \geq \mathbf{id} \wedge \delta$.

Proof. (*i*) We have

$$(\varepsilon \vee \mathbf{id})^2 = \left[\varepsilon(\varepsilon \vee \mathbf{id})\right] \vee \left[\mathbf{id}(\varepsilon \vee \mathbf{id})\right] = \left[\varepsilon(\varepsilon \vee \mathbf{id})\right] \vee [\varepsilon \vee \mathbf{id}].$$

Thus the idempotence of $\varepsilon \vee \mathbf{id}$ means that

$$\left[\varepsilon(\varepsilon \vee \mathbf{id})\right] \vee [\varepsilon \vee \mathbf{id}] = \varepsilon \vee \mathbf{id},$$

which is equivalent to $\varepsilon(\varepsilon \vee \mathbf{id}) \leq \varepsilon \vee \mathbf{id}$ (thanks to the equivalence $A \vee B = B \Leftrightarrow A \leq B$). Now (*ii*) is proved in the same way (or follows by duality). ∎

Our next result gives some basic properties of $\varepsilon \vee \mathbf{id} \wedge \delta$ for the case $\varepsilon \leq \delta$.

Lemma 2.2 *Let ε, δ be two operators on the modular lattice $\mathscr{L}$, such that $\varepsilon \leq \delta$, and let $\psi = \varepsilon \vee \mathbf{id} \wedge \delta$. Then:*

(*i*) $\varepsilon \leq \psi$, $\mathbf{id} \wedge \delta \leq \psi$, $\psi \leq \delta$, *and* $\psi \leq \varepsilon \vee \mathbf{id}$.
(*ii*) $\psi \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$ *and* $\mathbf{id} \wedge \psi = \mathbf{id} \wedge \delta$.
(*iii*) *If $\mathscr{L}$ is distributive and an operator ξ satisfies* $\xi \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$ *and* $\mathbf{id} \wedge \xi = \mathbf{id} \wedge \delta$, *then* $\xi = \psi$.
(*iv*) *If* $\varepsilon \vee \mathbf{id}$ *is idempotent, then* $\psi(\varepsilon \vee \mathbf{id}) = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id})$.
(*v*) *If* $\mathbf{id} \wedge \delta$ *is idempotent, then* $\psi(\mathbf{id} \wedge \delta) = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$.

Proof. The equality $\psi = \varepsilon \vee (\mathbf{id} \wedge \delta)$ gives the two inequalities $\varepsilon \leq \psi$ and $\mathbf{id} \wedge \delta \leq \psi$ in (*i*), as well as the equality

$$\psi \vee \mathbf{id} = \varepsilon \vee (\mathbf{id} \wedge \delta) \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$$

in (ii). On the other hand the equality $\psi = (\varepsilon \vee \mathbf{id}) \wedge \delta$ gives the two inequalities $\psi \leq \delta$ and $\psi \leq \varepsilon \vee \mathbf{id}$ in (i), as well as the equality

$$\mathbf{id} \wedge \psi = \mathbf{id} \wedge (\varepsilon \vee \mathbf{id}) \wedge \delta = \mathbf{id} \wedge \delta$$

in (ii). Thus (i) and (ii) hold.

Let the operator ξ satisfy $\xi \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \xi = \mathbf{id} \wedge \delta$; then by (ii) we have $\xi \vee \mathbf{id} = \psi \vee \mathbf{id}$ and $\mathbf{id} \wedge \xi = \mathbf{id} \wedge \psi$. If $\mathscr{L}$ is distributive, it is well-known [1] that this implies that $\psi = \xi$; we repeat here the proof of it:

$$\xi = \xi \wedge (\mathbf{id} \vee \xi) = \xi \wedge (\mathbf{id} \vee \psi) = (\xi \wedge \mathbf{id}) \vee (\xi \wedge \psi)$$
$$= (\psi \wedge \xi) \vee (\psi \wedge \mathbf{id}) = \psi \wedge (\xi \vee \mathbf{id}) = \psi \wedge (\psi \vee \mathbf{id}) = \psi.$$

Thus (iii) holds.

If $\varepsilon \vee \mathbf{id}$ is idempotent, then

$$\psi(\varepsilon \vee \mathbf{id}) = \big[(\varepsilon \vee \mathbf{id}) \wedge \delta\big](\varepsilon \vee \mathbf{id}) = \big[(\varepsilon \vee \mathbf{id})(\varepsilon \vee \mathbf{id})\big] \wedge \big[\delta(\varepsilon \vee \mathbf{id})\big]$$
$$= \big[\mathbf{id}(\varepsilon \vee \mathbf{id})\big] \wedge \big[\delta(\varepsilon \vee \mathbf{id})\big] = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id}),$$

giving (iv). Finally (v) is proved in the same way (or follows by duality). ∎

Note that if $\varepsilon \vee \mathbf{id}$ is idempotent, then (ii) and (iv) combined give

$$\psi(\psi \vee \mathbf{id}) = (\mathbf{id} \wedge \psi)(\psi \vee \mathbf{id}), \tag{2.3}$$

while if $\mathbf{id} \wedge \delta$ is idempotent, then (ii) and (v) combined give

$$\psi(\psi \wedge \mathbf{id}) = (\psi \vee \mathbf{id})(\mathbf{id} \wedge \psi). \tag{2.4}$$

We will now examine conditions for the idempotence of $\varepsilon \vee \mathbf{id} \wedge \delta$ when ε and δ are increasing, and $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$ are themselves idempotent.

Proposition 2.3 *Let ε, δ be two increasing operators on the modular lattice $\mathscr{L}$, such that $\varepsilon \leq \delta$, and let $\psi = \varepsilon \vee \mathbf{id} \wedge \delta$.*

- (i) *If $\psi^2 \geq \psi$, then $\varepsilon \leq \delta\psi$.*
- (ii) *If $\varepsilon \leq \delta\psi$ and $\mathbf{id} \wedge \delta$ is idempotent, then $\psi^2 \geq \psi$.*
- (iii) *If $\psi^2 \leq \psi$, then $\delta \geq \varepsilon\psi$.*
- (iv) *If $\delta \geq \varepsilon\psi$ and $\varepsilon \vee \mathbf{id}$ is idempotent, then $\psi^2 \leq \psi$.*

In particular when both $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$ are idempotent, ψ will be idempotent if and only if we have both $\varepsilon \leq \delta\psi$ and $\delta \geq \varepsilon\psi$.

Proof. (i) By Lemma 2.2 (
hence we get $\varepsilon \leq \psi \leq \psi^2$
(ii) By Lemma 2.2 (i)
that $\varepsilon \leq \psi \wedge \delta\psi = (\mathbf{id} \wedge$
$(\mathbf{id} \wedge \delta)\psi \leq \psi^2$. Combinin
$\mathbf{id} \wedge \delta \leq \psi$, and $\mathbf{id} \wedge \delta$ is inc
$(\mathbf{id} \wedge \delta)\psi \leq \psi^2$; combinin
get $\psi = \varepsilon \vee (\mathbf{id} \wedge \delta) \leq \psi^2$
Now (iii) and (iv) are
(or follow by duality). The
(i, ii, iii, iv). ∎

Corollary 2.4 *Let ε, δ be*
If $\varepsilon \leq \delta(\mathbf{id} \wedge \delta)$ or $\delta \geq \varepsilon($
have:

(i) *If $\varepsilon \leq \delta(\mathbf{id} \wedge \delta)$ and*
(ii) *If $\delta \geq \varepsilon(\varepsilon \vee \mathbf{id})$ and*

Proof. Suppose that $\varepsilon \leq$
$\delta(\mathbf{id} \wedge \delta) \leq \delta$ and so $\varepsilon \leq$
we get $\delta(\mathbf{id} \wedge \delta) \leq \delta\psi$.
idempotent, item (ii) of
The corresponding s
by duality. ∎

Thus by Corollary 2.4,
both $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$
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Before going into
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is called an *inf-overfilt*

Proof. (i) By Lemma 2.2 (i) we have $\varepsilon \leq \psi$, and $\psi \leq \delta$, so that $\psi^2 \leq \delta\psi$; hence we get $\varepsilon \leq \psi \leq \psi^2 \leq \delta\psi$.

(ii) By Lemma 2.2 (i) we have $\varepsilon \leq \psi$, and by hypothesis $\varepsilon \leq \delta\psi$, so that $\varepsilon \leq \psi \wedge \delta\psi = (\mathbf{id} \wedge \delta)\psi$; by Lemma 2.2 (i) again, $\mathbf{id} \wedge \delta \leq \psi$, so that $(\mathbf{id} \wedge \delta)\psi \leq \psi^2$. Combining both inequalities, we get $\varepsilon \leq (\mathbf{id} \wedge \delta)\psi \leq \psi^2$. As $\mathbf{id} \wedge \delta \leq \psi$, and $\mathbf{id} \wedge \delta$ is increasing and idempotent, we get $\mathbf{id} \wedge \delta = (\mathbf{id} \wedge \delta)^2 \leq (\mathbf{id} \wedge \delta)\psi \leq \psi^2$; combining the two inequalities $\mathbf{id} \wedge \delta \leq \psi^2$ and $\varepsilon \leq \psi^2$, we get $\psi = \varepsilon \vee (\mathbf{id} \wedge \delta) \leq \psi^2$.

Now (iii) and (iv) are proved in the same way as (i) and (ii) respectively (or follow by duality). The last sentence of the statement follows by combining (i, ii, iii, iv). ∎

Corollary 2.4 *Let ε, δ be two increasing operators on the modular lattice $\mathscr{L}$. If $\varepsilon \leq \delta(\mathbf{id} \wedge \delta)$ or $\delta \geq \varepsilon(\varepsilon \vee \mathbf{id})$, then $\varepsilon \leq \delta$, and setting $\psi = \varepsilon \vee \mathbf{id} \wedge \delta$, we have:*

(i) *If $\varepsilon \leq \delta(\mathbf{id} \wedge \delta)$ and $\mathbf{id} \wedge \delta$ is idempotent, then $\psi^2 \geq \psi$.*
(ii) *If $\delta \geq \varepsilon(\varepsilon \vee \mathbf{id})$ and $\varepsilon \vee \mathbf{id}$ is idempotent, then $\psi^2 \leq \psi$.*

Proof. Suppose that $\varepsilon \leq \delta(\mathbf{id} \wedge \delta)$. As $\mathbf{id} \wedge \delta \leq \mathbf{id}$ and δ is increasing, we have $\delta(\mathbf{id} \wedge \delta) \leq \delta$ and so $\varepsilon \leq \delta$. As $\mathbf{id} \wedge \delta \leq \varepsilon \vee (\mathbf{id} \wedge \delta) = \psi$, and δ is increasing, we get $\delta(\mathbf{id} \wedge \delta) \leq \delta\psi$. Hence $\varepsilon \leq \delta\psi$. Assuming furthermore that $\mathbf{id} \wedge \delta$ is idempotent, item (ii) of Proposition 2.3 implies that $\psi^2 \geq \psi$, and we get (i).

The corresponding statements for $\delta \geq \varepsilon(\varepsilon \vee \mathbf{id})$, in particular (ii), follow by duality. ∎

Thus by Corollary 2.4, a sufficient condition for the idempotence of ψ is that both $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent, and that the following conditions are satisfied:

$$\varepsilon \leq \delta(\mathbf{id} \wedge \delta) \qquad \text{and} \qquad \delta \geq \varepsilon(\varepsilon \vee \mathbf{id}).$$

As we will see later in this section, this is exactly what happens with the "adjacency triple conditions" given in [8] with $\mathscr{L} = \mathscr{P}(\mathscr{E})$, when δ and ε are the dilation and erosion defined by foreground and background adjacency relations. In the next section, we will consider a particular class of complete lattices where Minkowski operations can be defined, and we will give sufficient conditions in terms of structuring elements for obtaining such conditions as above; this will be illustrated in the case where $\mathscr{L}$ is the lattice of grey-level functions $\mathrm{Fun}(E)$ for $E = \mathbb{R}^d$ or $\mathbb{Z}^d$.

Before going into these particular cases, let us show that these conditions lead also to a new interpretation of the operator $\varepsilon \vee \mathbf{id} \wedge \delta$. We recall two definitions from [13,15]: an increasing operator η satisfying $\eta(\eta \vee \mathbf{id}) = \eta$ is called a *sup-underfilter*, while an increasing operator ζ satisfying $\zeta(\zeta \wedge \mathbf{id}) = \zeta$ is called an *inf-overfilter*. The following result generalizes [13, Proposition 4.2]:

Lemma 2.5 *Let ε, δ be two increasing operators on the lattice $\mathscr{L}$. Let $\eta = \varepsilon(\varepsilon \vee \mathbf{id})$ and $\zeta = \delta(\mathbf{id} \wedge \delta)$.*

(i) *$\varepsilon \vee \mathbf{id}$ is idempotent if and only if $\eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$; then η is a sup-underfilter, that is $\eta(\eta \vee \mathbf{id}) = \eta$.*

(ii) *$\mathbf{id} \wedge \delta$ is idempotent if and only if $\mathbf{id} \wedge \zeta = \mathbf{id} \wedge \delta$; then ζ is an inf-overfilter, that is $\zeta(\mathbf{id} \wedge \zeta) = \zeta$.*

Proof. Note that since ε and δ are increasing, η and ζ will also be increasing.

(i) If $\eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$, then clearly $\varepsilon(\varepsilon \vee \mathbf{id}) = \eta \leq \eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$, and $\varepsilon \vee \mathbf{id}$ is idempotent by Lemma 2.1 (i). Suppose now that $\varepsilon \vee \mathbf{id}$ is idempotent; by Lemma 2.1 (i) we have $\eta = \varepsilon(\varepsilon \vee \mathbf{id}) \leq \varepsilon \vee \mathbf{id}$; as ε is increasing and $\mathbf{id} \leq \varepsilon \vee \mathbf{id}$, we get $\varepsilon = \varepsilon\mathbf{id} \leq \varepsilon(\varepsilon \vee \mathbf{id}) = \eta$; combining the two inequalities gives $\varepsilon \leq \eta \leq \varepsilon \vee \mathbf{id}$, and taking the join of each member with $\mathbf{id}$, we get

$$\varepsilon \vee \mathbf{id} \leq \eta \vee \mathbf{id} \leq \varepsilon \vee \mathbf{id},$$

that is $\eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$. Now assuming $\eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$ and $\varepsilon \vee \mathbf{id}$ being idempotent, we get

$$\eta(\eta \vee \mathbf{id}) = \varepsilon(\varepsilon \vee \mathbf{id})(\varepsilon \vee \mathbf{id}) = \varepsilon(\varepsilon \vee \mathbf{id}) = \eta,$$

that is η is a sup-underfilter.

(ii) is proved in the same way (or follows by duality). ∎

Proposition 2.6 *Let ε, δ be two increasing operators on the modular lattice $\mathscr{L}$, and let $\eta = \varepsilon(\varepsilon \vee \mathbf{id})$ and $\zeta = \delta(\mathbf{id} \wedge \delta)$. Assume that $\varepsilon \leq \zeta$, $\eta \leq \delta$, and both $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent. Then:*

(i) $\varepsilon \leq \delta$ and $\eta \leq \zeta$.

(ii) $\eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \zeta = \mathbf{id} \wedge \delta$.

(iii) $\eta \vee \mathbf{id} \wedge \zeta = \eta \vee \mathbf{id} \wedge \delta = \varepsilon \vee \mathbf{id} \wedge \zeta = \varepsilon \vee \mathbf{id} \wedge \delta$.

(iv) $\varepsilon \vee \mathbf{id} \wedge \delta$ is idempotent.

(v) η is a sup-underfilter, and ζ is an inf-overfilter.

Proof. By Lemma 2.5, (ii) and (v) follow from the idempotence of $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$. Now the conditions of Corollary 2.4 are satisfied, so that $\varepsilon \leq \delta$ and (iv) holds. It remains to be shown that $\eta \leq \zeta$ and that (iii) holds. We have

$$\begin{aligned}
(\eta \vee \mathbf{id}) \wedge \zeta &= (\varepsilon \vee \mathbf{id}) \wedge \zeta && \text{since } \eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id},\\
&= \varepsilon \vee (\mathbf{id} \wedge \zeta) && \text{since } \varepsilon \leq \zeta,\\
&= \varepsilon \vee (\mathbf{id} \wedge \delta) && \text{since } \mathbf{id} \wedge \zeta = \mathbf{id} \wedge \delta,\\
&= (\varepsilon \vee \mathbf{id}) \wedge \delta && \text{since } \varepsilon \leq \delta,\\
&= (\eta \vee \mathbf{id}) \wedge \delta && \text{since } \eta \vee \mathbf{id} = \varepsilon \vee \mathbf{id},\\
&= \eta \vee (\mathbf{id} \wedge \delta) && \text{since } \eta \leq \delta,\\
&= \eta \vee (\mathbf{id} \wedge \zeta) && \text{since } \mathbf{id} \wedge \zeta = \mathbf{id} \wedge \delta,
\end{aligned}$$

from which we derive that

$$\eta \leq \eta \vee (\mathbf{id} \wedge \zeta) = (\eta \vee \mathbf{id}) \wedge \zeta \leq \zeta.$$

Therefore $\eta \leq \zeta$ and (iii) holds. ∎

Let us illustrate the above results in the case of sets. We will get in this way some of the results of [8]. Let E be any set. A map

$$A : E \to \mathcal{P}(E) : x \mapsto A(x), \tag{2.5}$$

will be called a *variable structuring element*. It leads to the two operators $\delta_A, \varepsilon_A : \mathcal{P}(E) \to \mathcal{P}(E)$ given by

$$\delta_A(X) = \cup_{x \in X} A(x) \tag{2.6}$$

and

$$\varepsilon_A(X) = \{z \in E \mid A(z) \subseteq X\}. \tag{2.7}$$

It is easily seen that these two operators satisfy the property

$$\forall X, Y \in \mathcal{P}(E), \qquad \delta_A(X) \subseteq Y \iff \forall x \in X,\ A(x) \subseteq Y \iff X \subseteq \varepsilon_A(Y),$$

which means that $(\varepsilon_A, \delta_A)$ is an *adjunction* [7], and implies that δ_A is a *dilation* (it distributes the union), while ε_A is an *erosion* (it distributes the intersection); we call δ_A and ε_A the dilation and the erosion by A. Note that every adjunction on $\mathcal{P}(E)$ takes this form, so that dilations and erosions on $\mathcal{P}(E)$ are those involving a variable structuring function.

To the variable structuring element A, we associate its *transpose* $\check{A}$, which is the variable structuring element defined by

$$\check{A}(x) = \{y \in E \mid x \in A(y)\}, \tag{2.8}$$

in other words by the equivalence

$$x \in A(y) \iff y \in \check{A}(x). \tag{2.9}$$

This implies in particular that $\check{\check{A}} = A$. Now it is easy to check that $\varepsilon_{\check{A}}$ is the dual by complementation of δ_A, which means that for every $X \in \mathcal{P}(E)$:

$$\left(\delta_A(X^c)\right)^c = \varepsilon_{\check{A}}(X) \qquad \text{and} \qquad \left(\varepsilon_{\check{A}}(X^c)\right)^c = \delta_A(X),$$

where X^c denotes the complement of X in E; similarly ε_A is the dual by complementation of $\delta_{\check{A}}$.

Let us say that A is *symmetric* if $A = \check{A}$. It is easy to show that if A is symmetric, then $\mathbf{id} \wedge \delta_A$ and $\mathbf{id} \vee \varepsilon_A$ are idempotent. This was implicitly shown in the discussion following [13, Proposition 3.1] for the idempotence of $\mathbf{id} \wedge \delta_A$, while the idempotence of $\mathbf{id} \vee \varepsilon_A$ follows by duality. These two operators $\mathbf{id} \wedge \delta_A$ and $\mathbf{id} \vee \varepsilon_A$ are the *annular opening* and *closing* by the variable structuring element A. Note also that if A is not symmetric and every $p \in E$ satisfies $p \notin A(p)$, then $\mathbf{id} \wedge \delta_A$ is not idempotent, for taking $y \in A(x)$ with $x \notin A(y)$, we have $(\mathbf{id} \wedge \delta_A)(\{x, y\}) = \{y\}$ and $(\mathbf{id} \wedge \delta_A)(\{y\}) = \emptyset$.

Now a symmetric variable structuring element A corresponds to a symmetric adjacency relation $\sim$ on E, defined by:

$$x \sim y \iff x \in A(y) \iff y \in A(x). \tag{2.10}$$

Then δ_A and ε_A coincide with the dilation and erosion defined in [5,8] from this adjacency relation.

Suppose next that we have two adjacency relations $\overset{0}{\sim}$ and $\overset{1}{\sim}$ (corresponding by (2.10) to two symmetric variable structuring elements A_0 and A_1), and let $(\varepsilon_i, \delta_i)$ be the adjunction associated to $\overset{i}{\sim}$ for $i = 0, 1$ (in other words, $(\varepsilon_i, \delta_i) = (\varepsilon_{A_i}, \delta_{A_i})$ with δ_{A_i} and ε_{A_i} given as in (2.6) and (2.7) respectively). We define the compound adjacency relation $\overset{0,1}{\sim}$ by $x \overset{0,1}{\sim} y$ if both $x \overset{0}{\sim} y$ and $x \overset{1}{\sim} y$ hold; thus the symmetric adjacency relation $\overset{0,1}{\sim}$ corresponds to the symmetric variable structuring element $x \mapsto A_0(x) \cap A_1(x)$.

In [8, Assumption 5.1] we postulated that for every $x \in E$ there is some $y \in E$ such that $x \overset{0,1}{\sim} y$, in other words that $A_0(x) \cap A_1(x) \neq \emptyset$. It is then easily shown [8, Proposition 5.4] that this implies that $\varepsilon_0 \leq \delta_1$ and $\varepsilon_1 \leq \delta_0$. Hence we define the set operator $\psi = \varepsilon_0 \vee \mathbf{id} \wedge \delta_1$, which is the annular operator removing from a binary image foreground points which are isolated from the point of view of $\overset{1}{\sim}$, as well as background points which are isolated from the point of view of $\overset{0}{\sim}$. We gave in [8] conditions on the adjacency relations $\overset{0}{\sim}$ and $\overset{1}{\sim}$ for the idempotence of ψ; we will see that they are particular cases of the conditions given in Proposition 2.3 and Corollary 2.4. Consider a triple $x, y, z \in E$ such that

$$x \overset{0,1}{\sim} y, \quad x \overset{0}{\sim} z, \quad \text{and } y \overset{1}{\sim} z. \tag{2.11}$$

Then we say [8] that x is a *0-triple point*, y is a *1-triple point*, and z is a *0/1-triple point*; see Fig. 2. More precisely: a point $x \in E$ is called a *0-triple point* if there exist $y, z \in E$ such that (2.11) holds, a point $y \in E$ is called a *1-triple point* if there exist $x, z \in E$ such that (2.11) holds, and a point $z \in E$ is called a *0/1-triple point* if there exist $x, y \in E$ such that (2.11) holds. We showed in [8, Corollary 5.7] that if every point in E is a 0-triple point (resp., a 1-triple point), then $\psi^2 \geq \psi$ (resp., $\psi^2 \leq \psi$). Let us explain how this result is a consequence of Corollary 2.4.

Let x be a 0-triple point, and let y, z be as in (2.11). Given a set X such that $x \in \varepsilon_0(X)$, we have $A_0(x) \subseteq X$, and as $x \overset{0}{\sim} y, z$, we get $y, z \in X$; as $y \overset{1}{\sim} z$, this gives $y \in \delta_1(X)$, so that $y \in X \cap \delta_1(X)$, and as $x \overset{1}{\sim} y$, we deduce then that $x \in \delta_1\big(X \cap \delta_1(X)\big)$. Thus if every point of E is a 0-triple point, then $\varepsilon_0 \leq \delta_1(\mathbf{id} \wedge \delta_1)$, and as $\mathbf{id} \wedge \delta_1$ is idempotent (by the symmetry of $\overset{1}{\sim}$), item (i) of Corollary 2.4 gives $\psi^2 \geq \psi$.

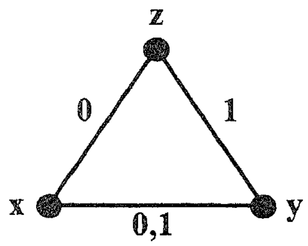

Fig. 2. An adjacency triple configuration; here x is a 0-triple point, y is a 1-triple point, and z is a 0/1-triple point

Let y be a 1-triple point, and let x, z be as in (2.11). Given a set X such that $y \in \varepsilon_0\big(X \cup \varepsilon_0(X)\big)$, we have $A_0(y) \subseteq X \cup \varepsilon_0(X)$, and as $y \overset{0}{\sim} x$, we get $x \in X \cup \varepsilon_0(X)$; now if $x \notin X$, then we have $x \in \varepsilon_0(X)$, in other words $A_0(x) \subseteq X$, and as $x \overset{0}{\sim} z$, this gives thus $z \in X$; hence $x \in X$ or $z \in X$, and as $y \overset{1}{\sim} x, z$, we deduce that $y \in \delta_1(X)$. Thus if every point of E is a 1-triple point, then $\varepsilon_0(\varepsilon_0 \vee \mathbf{id}) \leq \delta_1$, and as $\varepsilon_0 \vee \mathbf{id}$ is idempotent (by the symmetry of $\overset{0}{\sim}$), item (ii) of Corollary 2.4 gives $\psi^2 \leq \psi$.

Therefore the sufficient condition given in [8] for the idempotence of ψ (namely, that every point of E is both a 0-triple point and a 1-triple point), reduces to Corollary 2.4, thanks to the above argument. In the next section, where we will consider complete lattices equipped with Minkowski operations, we will use similar arguments in order to obtain the sufficient conditions given in Corollary 2.4.

We showed in [8, Proposition 5.9] that if $\psi^2 \geq \psi$ (resp., $\psi^2 \leq \psi$), then every point of E is either a 0-triple point or a 0/1-triple point (resp., either a 1-triple point or a 0/1-triple point). Indeed, if $\psi^2 \geq \psi$, then item (i) of Proposition 2.3 gives $\varepsilon_0 \leq \delta_1\psi$, in other words that for every $X \subseteq E$,

$$\varepsilon_0(X) \subseteq \delta_1(\psi(X)) = \delta_1\Big(\varepsilon_0(X) \cup \big[X \cap \delta_1(X)\big]\Big) = \delta_1(\varepsilon_0(X)) \cup \delta_1\big[X \cap \delta_1(X)\big];$$

now every $x \in E$ satisfies $x \in \varepsilon_0(A_0(x))$, so that we have either $x \in \delta_1(\varepsilon_0(A_0(x)))$ or $x \in \delta_1\big[A_0(x) \cap \delta_1(A_0(x))\big]$. The fact that $x \in \delta_1\big[A_0(x) \cap \delta_1(A_0(x))\big]$ means exactly that $x \overset{1}{\sim} y$ for some $y \in A_0(x)$ such that there is $z \in A_0(x)$ with $y \overset{1}{\sim} z$, in other words x is a 0-triple point (see (2.11)). The fact that $x \in \delta_1(\varepsilon_0(A_0(x)))$ means that there is some $y \in E$ such that $A_0(y) \subseteq A_0(x)$ and $y \overset{1}{\sim} x$; since there is some $z \in E$ with $y \overset{0,1}{\sim} z$, we get $x \overset{1}{\sim} y \overset{0,1}{\sim} z \overset{0}{\sim} x$, and so x is a 0/1-triple point.

Thus we have shown by using item (i) of Proposition 2.3 that if $\psi^2 \geq \psi$, then every point of E is either a 0-triple point or a 0/1-triple point. Similarly item (iii) of Proposition 2.3 allows us to show that if $\psi^2 \leq \psi$, then every point of E is either a 1-triple point or a 0/1-triple point.

3 Minkowski lattices and grey-level functions

As explained above, when the lattice of pictorial objects is $\mathscr{P}(E)$, it is possible to construct annular openings and closings, as well as their generalization into annular filters, with the use of foreground and background adjacency relations $\overset{1}{\sim}$ and $\overset{0}{\sim}$ on E. In the case of annular openings and closings, this description is in some way a mathematical characterization, because a dilation δ gives $\mathbf{id} \wedge \delta$ idempotent if and only if it arises from an adjacency relation, except possibly in the case where $\mathbf{id} \wedge \delta$ preserves a singleton, and dually an erosion ε gives $\mathbf{id} \vee \varepsilon$ idempotent if and only if it arises from an adjacency relation, except possibly in the case where $\mathbf{id} \vee \varepsilon$ preserves the complement of a singleton.

The structure of the lattice of grey-level functions $\mathrm{Fun}(E, \overline{\mathbb{R}})$ or $\mathrm{Fun}(E, \overline{\mathbb{Z}})$ (where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ and $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$) is much more complex than that of $\mathscr{P}(E)$ [11]. Therefore it is difficult to give a similar description of annular operators on grey-level functions in terms of adjacency relations (this time between pairs (p, t), where p is a point and t is a grey-level). Moreover, the computer implementation of morphological operations is more complex for grey-level images than for binary ones; therefore such a general approach becomes rather technical in practice. We have thus decided to make some restrictions, namely that our operators are translation-invariant.

Indeed, in [13] we took E having the structure of an abelian group (such as $\mathbb{Z}^d$ or $\mathbb{R}^d$), and then we could characterize translation-invariant annular openings of the form $\mathbf{id} \wedge \delta_A$ for grey-level functions: this operator is idempotent if and only if the support of grey-level structuring function A (that is, the set of points x such that $A(x) > -\infty$) is symmetric, and every point x of that support satisfies $A(x) + A(-x) \geq 0$ (cfr. (1.3)). This condition on the structuring function A is the generalization of the requirement of symmetry for the structuring element involved in Serra's original annular opening for sets. Now this characterization given in [13] of translation-invariant annular openings for grey-level functions was a particular case of a similar characterization given in a wider theoretical framework; more precisely, we considered as object space an arbitrary complete lattice having certain general properties (the so-called "Basic Assumption" in [7]), which allow the definition of the Minkowski operations $\oplus$ and $\ominus$ on that lattice, in such a way that the usual properties of $\oplus$ and $\ominus$ are satisfied. Particular cases where this theory can be applied include $\mathscr{P}(E)$ (where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$), $\mathrm{Fun}(E, \overline{\mathbb{R}})$, and $\mathrm{Fun}(E, \overline{\mathbb{Z}})$, but also more exotic cases, such as the lattice of convex sets, or the one of closed sets, etc.

In this section, we will generalize the result of [13] to annular filters, and study the idempotence of operators of the form $\mathbf{id} \wedge \delta_A$ (annular opening), $\mathbf{id} \vee \varepsilon_B$ (annular closing), and $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$ (annular filter), where δ_A is the dilation $X \mapsto X \oplus A$ and ε_B is the erosion $X \mapsto X \ominus B$; the space of pictorial objects will be assumed to be a complete lattice satisfying certain conditions, in particular the so-called "Basic Assumption" which allow the definition of Minkowski operations $\oplus$ and $\ominus$. The arguments we will use will have a superficial similarity to those given above with adjacency relations and the adjacency triple conditions (cfr. (2.11)).

Throughout this section we assume that $\mathcal{L}$ is a *complete lattice* [1,7], which means that every part $\mathcal{S}$ of $\mathcal{L}$ has a least upper bound $\bigvee \mathcal{S}$, called the *supremum of* $\mathcal{S}$, and a greatest lower bound $\bigwedge \mathcal{S}$, called the *infimum of* $\mathcal{S}$; in particular $\mathcal{L}$ has a *greatest element* I and a *least element* O defined by

$$I = \bigvee \mathcal{L} = \bigwedge \emptyset \qquad \text{and} \qquad O = \bigwedge \mathcal{L} = \bigvee \emptyset.$$

The binary join and meet operations $\vee$, $\wedge$ are particular cases of the supremum and infimum operations $\bigvee$, $\bigwedge$, in the sense that for $A, B \in \mathcal{L}$ we have

$$A \vee B = \bigvee \{A, B\} \qquad \text{and} \qquad A \wedge B = \bigwedge \{A, B\}.$$

We will make several technical assumptions on the complete lattice $\mathcal{L}$, but this requires first recalling a few definitions, mostly from [7,13]:

Definition 3.1 Let $\mathcal{L}$ be a complete lattice.

(i) Given a part $\mathcal{Y}$ of $\mathcal{L}$ and $X \in \mathcal{L}$, we write

$$\mathcal{Y}(X) = \{Y \in \mathcal{Y} \mid Y \leq X\}.$$

(ii) A part $\mathcal{Y}$ of $\mathcal{L}$ is said *sup-generating* if every $X \in \mathcal{L}$ is the supremum of some part of $\mathcal{Y}$, in other words if

$$\forall X \in \mathcal{L}, \qquad X = \bigvee \mathcal{Y}(X).$$

Note that we generally assume that $O \notin Y$, because O would be redundant in that sup-generating family.

(iii) A part $\mathcal{Y}$ of $\mathcal{L}$ is called *lower* if for every $Y \in \mathcal{Y}$ and $Z \in \mathcal{L}$ such that $Z \leq Y$, we have either $Z \in \mathcal{Y}$ or $Z = O$. (Note that this definition here is slightly different from that given in [13, Definition 3.1 (ii)], because there we required that $Z \leq Y \in \mathcal{Y}$ implies $Z \in \mathcal{Y}$; thus we say here that $\mathcal{Y}$ is lower whenever we would have said in [13] that $\mathcal{Y} \vee \{O\}$ is lower.)

(iv) We say that $\mathcal{L}$ is *infinite supremum distributive* (in brief, *ISD*), if the meet operation $\wedge$ distributes the supremum operation $\bigvee$, in other words for every $X \in \mathcal{L}$ and every non-empty family Y_i, $i \in \mathcal{I}$, of elements of $\mathcal{L}$, we have:

$$X \wedge \Big(\bigvee_{i \in \mathcal{I}} Y_i\Big) = \bigvee_{i \in \mathcal{I}} (X \wedge Y_i).$$

(v) We say that $\mathcal{L}$ is *infinite infimum distributive* (in brief, *IID*), if the join operation $\vee$ distributes the infimum operation $\bigwedge$, in other words for every $X \in \mathcal{L}$ and every non-empty family Y_i, $i \in \mathcal{I}$, of elements of $\mathcal{L}$, we have:

$$X \vee \Big(\bigwedge_{i \in \mathcal{I}} Y_i\Big) = \bigwedge_{i \in \mathcal{I}} (X \vee Y_i).$$

(*vi*) An *automorphism* of $\mathscr{L}$ is a permutation τ of $\mathscr{L}$ such that for every $X, Y \in \mathscr{L}, X \leq Y \Leftrightarrow \tau(X) \leq \tau(Y)$.

There is no standard terminology concerning ISD and IID, some mathematicians would exchange the definitions (*iv*) and (*v*) we gave for them. Note that an ISD (or IID) complete lattice is distributive, and we recall that every distributive lattice is modular. In order to study $\varepsilon \vee \mathbf{id} \wedge \delta$ (where $\varepsilon \leq \delta$), we assumed in the previous section that $\mathscr{L}$ is modular; in this section we will rather use the stronger condition of distributivity, and even in some cases ISD or IID. Now we recall the conditions which allow the definition of Minkowski operations on a complete lattice:

Definition 3.2 A *Minkowski lattice* is a triple $(\mathscr{L}, \ell, \mathbf{T})$, where $\mathscr{L}$ is a complete lattice, ℓ is a sup-generating part of $\mathscr{L}$, and $\mathbf{T}$ is a set of automorphisms of $\mathscr{L}$, satisfying the following three conditions:

(*a*) $\mathbf{T}$ is an *abelian group* for the law of composition, that is: $\mathbf{id} \in \mathbf{T}$ and for $\sigma, \tau \in \mathbf{T}, \sigma^{-1} \in \mathbf{T}, \sigma\tau \in \mathbf{T}$, and $\sigma\tau = \tau\sigma$.
(*b*) ℓ is *invariant* under $\mathbf{T}$, that is: for $x \in \ell$ and $\tau \in \mathbf{T}$, $\tau(x) \in \ell$.
(*c*) $\mathbf{T}$ is *transitive* on ℓ, that is: for every $x, y \in \ell$, there is some $\tau \in \mathbf{T}$ such that $\tau(x) = y$.

Furthermore, we say that the Minkowski lattice $(\mathscr{L}, \ell, \mathbf{T})$ is *lower* if the sup-generating set ℓ is lower, in other words for $x \in \ell$ and $y \leq x$, either $y = O$ or $y \in \ell$.

Note that by (*c*) we have always $O \notin \ell$. Also we use lower-case letters to designate elements of ℓ (other elements of $\mathscr{L}$ are designated by upper-case letters). In [7], we showed by a standard group-theoretic argument that in (*c*), the automorphism $\tau \in \mathbf{T}$ such that $\tau(x) = y$ is in fact *unique*. Thus, fixing some "origin" $o \in \ell$, we obtain a bijection $\mathbf{T} \to \ell : \tau \mapsto \tau(o)$, and for $x \in \ell$, we write τ_x for the unique element of $\mathbf{T}$ mapped on x by this bijection, that is $\tau_x(o) = x$; then τ_x is called the *translation by* x. This bijection provides ℓ with the structure of abelian group (for an addition operation $+$) isomorphic to $\mathbf{T}$, namely $x + y$ is defined by $\tau_{x+y} = \tau_x\tau_y$, in other words $x + y = \tau_x(y) = \tau_y(x) = \tau_x\tau_y(o) = \tau_y\tau_x(o)$, the neutral element is o, and the opposite $-x$ of x is defined by $\tau_{-x} = \tau_x^{-1}$. For $a \in \ell$ and $X \in \mathscr{L}$, we write X_a for $\tau_a(X)$, and we call it the *translate of* X *by* a.

For example for $E = \mathbb{Z}^d$ or $\mathbb{R}^d$ (in fact, whenever E has a structure of an abelian group), $\mathscr{P}(E)$ is a Minkowski lattice by taking for ℓ the family of singletons, and for $\mathbf{T}$ the set of all translations. Also $\mathrm{Fun}(E, \mathscr{T})$, where E is an abelian group and $\mathscr{T} = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$, is a Minkowski lattice by taking as members of ℓ the "impulse" functions $f_{(h,v)}$ for $h \in E$ and $v \in \mathbb{R}$ (resp., $v \in \mathbb{Z}$), defined by setting for every point $p \in E$:

$$f_{(h,v)}(p) = \begin{cases} v & \text{if } p = h, \\ -\infty & \text{if } p \neq h, \end{cases} \tag{3.1}$$

of $\mathscr{L}$ such that for every

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(3.1)

and as members of **T** all translations $\tau_{(h,v)}$ for $h \in E$ and $v \in \mathbb{R}$ (resp., $v \in \mathbb{Z}$), defined by setting for every function F and a point $p \in E$:

$$\tau_{(h,v)}(F)(p) = F(p - h) + v. \tag{3.2}$$

Note that for a grey-level function F, the set of (h, v) such that $f_{(h,v)} \in \ell(F)$ is the *umbra* of F [11].

In [7] we defined on any Minkowski lattice the two Minkowski operations $\oplus$ and $\ominus$ by

$$X \oplus Y = \bigvee_{y \in \ell(Y)} X_y \qquad \text{and} \qquad X \ominus Y = \bigwedge_{y \in \ell(Y)} X_{-y}, \tag{3.3}$$

and we showed that they have essentially the same properties as in the case of sets or grey-level functions. From these we derived the dilation $\delta_A : X \mapsto X \oplus A$ and the erosion $\varepsilon_A : X \mapsto X \ominus A$ by an arbitrary element A of $\mathscr{L}$.

Remark The conditions (a)–(c) in Definition 3.2 were introduced in [7], and were called the "Basic Assumption" there. The terminology "Minkowski lattice" was introduced in [12]. In [6], the terminology "convolution lattice" was used to designate a complete lattice satisfying the Basic Assumption (in other words, a Minkowski lattice), with the further condition that all invertible elements of $\mathscr{L}$ belong to ℓ: if $X, Y \in \mathscr{L}$ and $X \oplus Y = o$, then $X \in \ell$. It can be shown that a Minkowski lattice is lower if and only if for $X, Y \in \mathscr{L}$ satisfying $X \oplus Y \leq o$, we must have $X \in \ell$. In particular, a lower Minkowski lattice is a convolution lattice.

Example 3.4 (a) The family $\mathrm{Con}(\mathbb{R}^d)$ of convex subsets of $\mathbb{R}^d$, ordered by inclusion, is a Minkowski lattice for the group **T** of translations, with ℓ consisting of singletons; the latter are *atoms*, in other words minimal elements of the lattice, and as this lattice is generated by atoms, it is said to be *atomic*. In $\mathrm{Con}(\mathbb{R}^d)$ the infimum and supremum of a family of convex sets are respectively their intersection and the convex hull of their union. This lattice is not modular, and hence not distributive.

(b) Let B be any nonvoid subset of $E = \mathbb{R}^d$ or $\mathbb{Z}^d$. Consider the family $\mathscr{L}$ of all subsets of E which are unions of translates of B, in other words which take the form $X \oplus B$ for some $X \subseteq E$. Ordered by inclusion, $\mathscr{L}$ is a Minkowski lattice for the group **T** of translations, with ℓ consisting of all translates of B. A family X_i $(i \in \mathscr{I})$ of elements of $\mathscr{L}$ has its union $\bigcup_{i \in \mathscr{I}} X_i$ as supremum, while its infimum is the set

$$\Big(\bigcap_{i \in \mathscr{I}} X_i\Big) \circ B,$$

where $\circ$ denotes the opening operation defined by setting $X \circ B$ equal to the union of all translates of B included in X. When B is bounded, the elements of ℓ

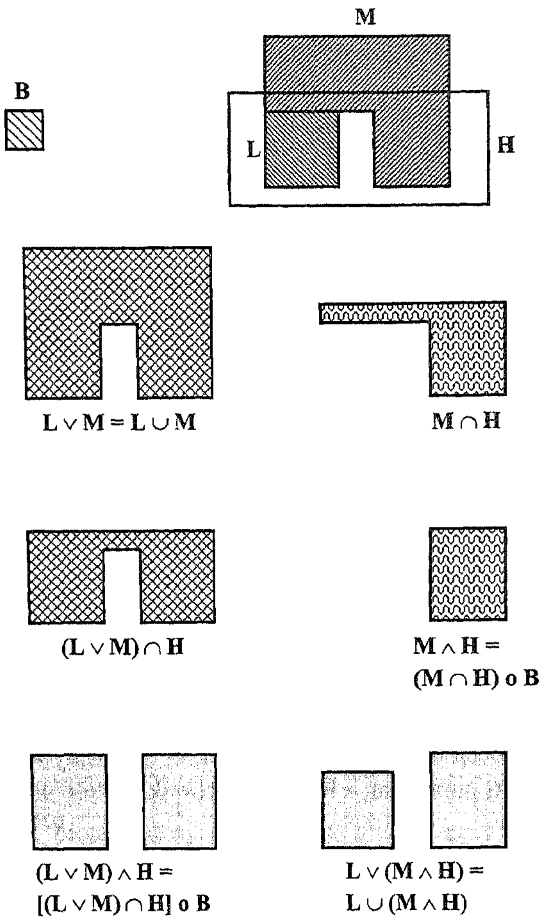

Fig. 3. The Minkowski lattice $\mathscr{L}$ consists of all subsets of $\mathbb{Z}^2$ which can be decomposed as a union of translates of B (the latter are the sup-generators of $\mathscr{L}$). Three elements L, M, and H of $\mathscr{L}$, where L $\leq$ H (L and M are shown hatched, H is transparent). We see that L $\vee$ (M $\wedge$ H) is a strict subset of (L $\vee$ M) $\wedge$ H. Hence the lattice $\mathscr{L}$ is not modular

(the translates of B) are atoms, and so $\mathscr{L}$ is atomic. This lattice is generally not modular, and hence not distributive; we illustrate this fact in Fig. 3 for $d = 2$ and B consisting of a square.

(c) The family $\mathrm{Lip}(\mathbb{R}^d)$ of Lipschitz functions $\mathbb{R}^d \to \mathbb{R}$ with Lipschitz constant 1 (including the constantly infinite functions $+\infty$ and $-\infty$) is a Minkowski lattice for the same group of translations as in $\mathrm{Fun}(\mathbb{R}^d)$ (see (3.2)); here the members of ℓ, instead of being the "impulse" functions $f_{(h,v)}$ defined in (3.1), are the "cone" functions $c_{(h,v)}$ (for $h \in \mathbb{R}^d$ and $v \in \mathbb{R}$) defined by:

$$c_{(h,v)}(p) = v - \|p - h\| \qquad (p \in \mathbb{R}^d).$$

Note that in $\mathrm{Lip}(\mathbb{R}^d)$, the infimum and supremum of a family of Lipschitz functions is their usual infimum and supremum in $\mathrm{Fun}(\mathbb{R}^d)$; thus $\mathrm{Lip}(\mathbb{R}^d)$ inherits the IID and ISD properties of Fun($\mathbb{R}^d$). See [10] for a detailed study of this lattice.

From now on, we will freely use the techniques introduced in [7, Section 3] for manipulating Minkowski operations in conjunction with elements of ℓ. Readers who want to study our proofs in depth are referred to this fundamental paper.

We will consider the generalization to Minkowski lattices of annular filters and symmetric structuring elements, but this requires first a few technical results:

Lemma 3.5 *Let $(\mathcal{L}, \ell, \mathbf{T})$ be a Minkowski lattice.*

(i) For $x, y \in \ell$, we have

$$x \leq y \iff \tau_x \leq \tau_y \iff -y \leq -x \iff \tau_{-y} \leq \tau_{-x}.$$

(ii) For $a, b \in \ell$, we have $a \vee b \in \ell \Leftrightarrow (-a) \vee (-b) \in \ell$ and $a \wedge b \in \ell \Leftrightarrow (-a) \wedge (-b) \in \ell$. Furthermore, if ℓ is lower, then the four statements are equivalent:

$$a \vee b \in \ell \iff (-a) \vee (-b) \in \ell \iff a \wedge b \in \ell \iff (-a) \wedge (-b) \in \ell.$$

Proof. (*i*) If $x \leq y$, then for every $h \in \ell$ we have $\tau_h(x) \leq \tau_h(y)$, and so by [7, Lemma 3.3], for every $Z \in \mathcal{L}$ we get:

$$\tau_x(Z) = \bigvee_{z \in \ell(Z)} \tau_z(x) \leq \bigvee_{z \in \ell(Z)} \tau_z(y) = \tau_y(Z),$$

and hence $\tau_x \leq \tau_y$. Conversely, if $\tau_x \leq \tau_y$, then $x = \tau_x(o) \leq \tau_y(o) = y$, and the two are equivalent. We get similarly the equivalence between $-y \leq -x$ and $\tau_{-y} \leq \tau_{-x}$. Finally, since $\tau_{-x-y}(x) = -y$ and $\tau_{-x-y}(y) = -x$, we get $x \leq y \Leftrightarrow \tau_{-x-y}(x) \leq \tau_{-x-y}(y) \Leftrightarrow -y \leq -x$.

(*ii*) We have $\tau_{-a-b}(a) = -b$ and $\tau_{-a-b}(b) = -a$. Thus $\tau_{-a-b}(a \vee b) = \tau_{-a-b}(a) \vee \tau_{-a-b}(b) = (-b) \vee (-a)$ and $\tau_{-a-b}(a \wedge b) = \tau_{-a-b}(a) \wedge \tau_{-a-b}(b) = (-b) \wedge (-a)$, so that we get

$$a \vee b \in \ell \iff \tau_{-a-b}(a \vee b) \in \ell \iff (-a) \vee (-b) \in \ell$$

and

$$a \wedge b \in \ell \iff \tau_{-a-b}(a \wedge b) \in \ell \iff (-a) \wedge (-b) \in \ell.$$

Suppose now that ℓ is lower, in other words $O < y \leq x \in \ell$ implies $y \in \ell$. If $a \vee b \in \ell$, then writing $c = a \vee b$, we have $a \leq c$ and $b \leq c$, so that by (*i*) we get $-c \leq -a$ and $-c \leq -b$, that is $-c \leq (-a) \wedge (-b)$; thus $-a \geq (-a) \wedge (-b) \geq -c > O$, and as ℓ is lower, we get $(-a) \wedge (-b) \in \ell$. Conversely, if $(-a) \wedge (-b) \in \ell$, then writing $d = (-a) \wedge (-b)$, we have $d \leq -a$ and $d \leq -b$, so that by (*i*) we get $a \leq -d$ and $b \leq -d$, that is $a \vee b \leq -d$; thus $O < a \leq a \vee b \leq -d$, and as ℓ is lower, we get $a \vee b \in \ell$. Therefore $a \vee b \in \ell \Leftrightarrow (-a) \wedge (-b) \in \ell$, and the four statements are equivalent. ∎

Let us now generalize to the framework of Minkowski lattices the notion of a symmetric structuring element (used for annular filters on sets):

Definition 3.6 Let $(\mathscr{L}, \ell, \mathbf{T})$ be a Minkowski lattice, and let $A \in \mathscr{L}$.

(i) Let $-\ell(A) = \{-a \mid a \in \ell(A)\}$. The *symmetric part* of A is the set

$$\ell^*(A) = \ell(A) \cap -\ell(A) = \{a \in \ell(A) \mid -a \in \ell(A)\}.$$

(ii) We say that A is *symmetric* if $A = \bigvee \ell^*(A)$.

(iii) We say that A is *annular* if for every $a \in \ell(A)$ there is some $a' \in \ell^*(A)$ such that $a' \geq a$.

For $\mathscr{L} = \mathscr{P}(E)$, these definitions of a symmetric set and of an annular set reduce to the usual notion of a set which is symmetric w.r.t. the origin. In the case where $\mathscr{L} = \mathrm{Fun}(E)$, a function F is symmetric if and only if it is annular, and this means in fact that the support of F (the set of $x \in E$ such that $F(x) > -\infty$) is symmetric and $F(x) + F(-x) \geq 0$ for every x in that support. For example if $E = \mathbb{R}$, the function F given by $F(x) = x$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$ is symmetric, and also annular. Thus for functions, unlike sets, the word "symmetric" does not take the usual meaning of "invariant under the central symmetry of E".

In [13] we restricted ourselves to annular openings and to annular structuring elements (see equation (3.4) and Theorems 3.3 and 3.4 there). Here we will study annular openings $\mathbf{id} \wedge \delta_A$, annular closings $\mathbf{id} \vee \varepsilon_A$, and finally annular filters $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$, and we will also consider both conditions on the structuring elements, namely being symmetric or annular, the former being in fact slightly more general, but also easier to use. The following result gives the relation between these two properties:

Proposition 3.7 *Let $(\mathscr{L}, \ell, \mathbf{T})$ be a Minkowski lattice. An annular element of $\mathscr{L}$ is symmetric; if $\mathscr{L}$ is ISD and ℓ is lower, then conversely a symmetric element of $\mathscr{L}$ is annular.*

Proof. Let A be annular. For every $a \in \ell(A)$ we have $a' \in \ell^*(A)$ such that $a \leq a'$, so that $a \leq \bigvee \ell^*(A)$; we deduce that $\bigvee \ell(A) \leq \bigvee \ell^*(A)$. As $\ell^*(A) \subseteq \ell(A)$, we get the equality $\bigvee \ell^*(A) = \bigvee \ell(A) = A$, meaning that A is symmetric.

Suppose now that $\mathscr{L}$ is ISD and ℓ is lower, and let A be symmetric. For $a \in \ell(A)$ we have $a \leq A = \bigvee \ell^*(A)$ and ISD gives

$$a = a \wedge \left(\bigvee \ell^*(A)\right) = \bigvee_{b \in \ell^*(A)} (a \wedge b);$$

as $a > O$ we deduce that there is some $b \in \ell^*(A)$ such that $a \wedge b > O$. As $a \geq a \wedge b > O$ and ℓ is lower, we get $a \wedge b \in \ell$, and by Lemma 3.5 (ii) we have $a \vee b \in \ell$. Let $a' = a \vee b$; as $a, b \in \ell(A)$ we have $a' \leq A$; as $b \leq a'$, by Lemma 3.5 (i) we have $-a' \leq -b$, and as $b \in \ell^*(A)$, we have $-b \leq A$, so that $-a' \leq A$. Thus $a' \in \ell$ with both $a' \leq A$ and $-a' \leq A$, so that $a' \in \ell^*(A)$, and of courses $a \leq a'$. Therefore A is annular. ∎

The following result generalizes well-known properties of symmetric parts of $\mathbb{R}^d$ or $\mathbb{Z}^d$:

Proposition 3.8 *Given a Minkowski lattice $(\mathcal{L}, \ell, \mathrm{T})$, the family of symmetric elements of $\mathcal{L}$ is closed under the supremum operation, and it contains the universal bounds O and I. Furthermore, the Minkowski sum $A \oplus B$ of two symmetric elements A and B is symmetric.*

Proof. As $\ell(I) = \ell$ and $\ell(O) = \emptyset$, O and I are symmetric. Consider a family X_i, $i \in \mathcal{I}$, of symmetric elements of $\mathcal{L}$, and let $Y = \bigvee_{i \in \mathcal{I}} X_i$. If $\mathcal{I}$ is empty, then $Y = O$, which is symmetric. Otherwise we set

$$\ell_{\mathcal{I}} = \bigcup_{i \in \mathcal{I}} \ell^*(X_i);$$

we have $\ell_{\mathcal{I}} \subseteq \ell^*(Y)$, and for every $i \in \mathcal{I}$, $X_i = \bigvee \ell^*(X_i) \leq \bigvee \ell_{\mathcal{I}}$. We deduce that

$$Y \leq \bigvee \ell_{\mathcal{I}} \leq \bigvee \ell^*(Y) \leq \bigvee \ell(Y) = Y,$$

so that $Y = \bigvee \ell^*(Y)$, in other words, Y is symmetric.

Let A and B be symmetric. As the Minkowski sum distributes the supremum operation (see [7, Subsection 3.2]), we have

$$A \oplus B = \Big(\bigvee \ell^*(A)\Big) \oplus \Big(\bigvee \ell^*(B)\Big) = \bigvee_{a \in \ell^*(A)} \bigvee_{b \in \ell^*(B)} (a \oplus b) = \bigvee_{a \in \ell^*(A)} \bigvee_{b \in \ell^*(B)} (a + b)$$

now for $a \in \ell^*(A)$ and $b \in \ell^*(B)$ we have $-a \in \ell^*(A)$ and $-b \in \ell^*(B)$, so that both $a + b$ and $-(a + b)$ belong to $\ell(A \oplus B)$ (we used the fact that for $x \leq A$ and $y \leq B$, we have $x + y \leq A \oplus B$, see [7, Section 3]). Therefore

$$A \oplus B = \bigvee_{a \in \ell^*(A)} \bigvee_{b \in \ell^*(B)} (a + b) \leq \bigvee \ell^*(A \oplus B) \leq \bigvee \ell(A \oplus B) = A \oplus B,$$

and $A \oplus B$ is symmetric. ∎

Note that this result does not extend to the infimum and Minkowski difference. For example in the Minkowski lattice Fun($\mathbb{R}$), the functions C and D defined by

$$C(x) = x \qquad \text{and} \qquad D(x) = -x$$

are symmetric, but their meet $C \wedge D$ satisfies

$$(C \wedge D)(x) = -|x|,$$

and it is not symmetric. Similarly the functions A and B defined by

$$A(x) = \begin{cases} x & \text{if } |x| \leq 2, \\ -\infty & \text{if } |x| > 2, \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ -\infty & \text{if } |x| > 1, \end{cases}$$

are symmetric, but $A \ominus B$ satisfies

$$(A \ominus B)(x) = \begin{cases} x - 1 & \text{if } |x| \leq 1, \\ -\infty & \text{if } |x| > 1, \end{cases}$$

and it is not symmetric.

From Proposition 3.8 we deduce that the set $\mathcal{S}$ of symmetric elements of $\mathcal{L}$ is itself a complete lattice for the ordering by $\leq$. Here the supremum operation coincides with the one in $\mathcal{L}$ (for $X_i \in \mathcal{S}$, $i \in \mathcal{I}$, $\bigvee_{i \in \mathcal{I}} X_i \in \mathcal{S}$), but the infimum is different: the infimum in $\mathcal{S}$ of a family $X_i \in \mathcal{S}$, $i \in \mathcal{I}$, is the greatest element of $\mathcal{S}$ which is $\leq \bigwedge_{i \in \mathcal{I}} X_i$; we write it $\prod_{i \in \mathcal{I}} X_i$; thus $\prod_{i \in \mathcal{I}} X_i \in \mathcal{S}$ and $\prod_{i \in \mathcal{I}} X_i \leq \bigwedge_{i \in \mathcal{I}} X_i$. For $A, B \in \mathcal{S}$, we write $A \sqcap B$ for their meet in the lattice $\mathcal{S}$; in other words $A \sqcap B$ is the greatest symmetric C such that $C \leq A \wedge B$.

In [13, Theorem 3.3] we showed that given an ISD Minkowski lattice $(\mathcal{L}, \ell, \mathbf{T})$ and an annular $A \in \mathcal{L}$, then $\mathbf{id} \wedge \delta_A$ is idempotent, and the elements of $\mathcal{L}$ fixed by $\mathbf{id} \wedge \delta_A$ are suprema of terms of the form $x \vee (x + a)$ for $x \in \ell$ and $a \in \ell^*(A)$. We will extend here this result to the case where A is symmetric, and obtain the dual result for $\varepsilon_A \vee \mathbf{id}$ when $\mathcal{L}$ is IID instead; we rely on the following general result, which should be compared to [13, Proposition 3.2]:

Proposition 3.9 *Let $\mathcal{L}$ be a complete lattice, and let $\mathcal{S}$ be a family of automorphisms of $\mathcal{L}$ such that for every $\sigma \in \mathcal{S}$, $\sigma^{-1} \in \mathcal{S}$; set $\delta = \bigvee \mathcal{S}$ and $\varepsilon = \bigwedge \mathcal{S}$, in other words $\delta(X) = \bigvee_{\sigma \in \mathcal{S}} \sigma(X)$ and $\varepsilon(X) = \bigwedge_{\sigma \in \mathcal{S}} \sigma(X)$.*

- (i) *If $\mathcal{L}$ is ISD, then $\mathbf{id} \wedge \delta$ is idempotent, and an element of $\mathcal{L}$ is fixed by $\mathbf{id} \wedge \delta$ if and only if it is a supremum of terms of the form $X \vee \sigma(X)$ for $X \in \mathcal{L}$ and $\sigma \in \mathcal{S}$.*
- (ii) *If $\mathcal{L}$ is IID, then $\varepsilon \vee \mathbf{id}$ is idempotent, and an element of $\mathcal{L}$ is fixed by $\varepsilon \vee \mathbf{id}$ if and only if it is an infimum of terms of the form $X \wedge \sigma(X)$ for $X \in \mathcal{L}$ and $\sigma \in \mathcal{S}$.*

Proof. (i) For $\sigma \in \mathcal{S}$ and $X \in \mathcal{L}$, we have

$$\begin{aligned} \delta\big[X \vee \sigma(X)\big] &\geq \sigma\big[X \vee \sigma(X)\big] \vee \sigma^{-1}\big[X \vee \sigma(X)\big] \\ &= \big[\sigma(X) \vee \sigma^2(X)\big] \vee \big[\sigma^{-1}(X) \vee X\big] \geq X \vee \sigma(X), \end{aligned}$$

from which we derive that

$$(\mathbf{id} \wedge \delta)\big[X \vee \sigma(X)\big] = \big[X \vee \sigma(X)\big] \wedge \delta\big[X \vee \sigma(X)\big] = X \vee \sigma(X).$$

d B defined by

$\begin{cases} 0 & \text{if } |x| \leq 1, \\ -\infty & \text{if } |x| > 1, \end{cases}$

1,
1,

of symmetric elements
$\leq$. Here the supremum
$\in \mathscr{I}$, $\bigvee_{i\in\mathscr{I}} X_i \in \mathscr{S}$),
amily $X_i \in \mathscr{S}$, $i \in \mathscr{I}$,
write it $\prod_{i\in\mathscr{I}} X_i$: thus
$\mathscr{S}$, we write $A \sqcap B$ for
greatest symmetric C

SD Minkowski lattice
empotent, and the ele-
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sult to the case where
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e a family of automor-
$= \bigvee \mathscr{S}$ *and* $\varepsilon = \bigwedge \mathscr{S}$,
$\sigma(X)$.

nent of $\mathscr{L}$ *is fixed by*
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of $\mathscr{L}$ *is fixed by* $\varepsilon \vee$ **id**
$\wedge\, \sigma(X)$ *for* $X \in \mathscr{L}$

$\geq X \vee \sigma(X)$,

$= X \vee \sigma(X)$.

Thus $X \vee \sigma(X)$ is fixed by **id** $\wedge\, \delta$; by [13, Proposition 2.2], a supremum of terms having that form $X \vee \sigma(X)$ (with $\sigma \in \mathscr{S}$) is fixed by **id** $\wedge\, \delta$.

Take any $Y \in \mathscr{L}$. As $\mathscr{S}$ is invariant under the permutation $\sigma \mapsto \sigma^{-1}$, we have

$$\delta(Y) = \bigvee_{\sigma\in\mathscr{S}} \sigma(Y) = \bigvee_{\sigma\in\mathscr{S}} \sigma^{-1}(Y) = \bigvee_{\sigma\in\mathscr{S}} \left[\sigma^{-1}(Y) \vee \sigma(Y)\right].$$

As $\mathscr{L}$ is ISD, we get

$$Y \wedge \delta(Y) = Y \wedge \left(\bigvee_{\sigma\in\mathscr{S}} \left[\sigma^{-1}(Y) \vee \sigma(Y)\right]\right) = \bigvee_{\sigma\in\mathscr{S}} \left(\left[Y \wedge \sigma^{-1}(Y)\right] \vee \left[Y \wedge \sigma(Y)\right]\right)$$
$$= \bigvee_{\sigma\in\mathscr{S}} \left(\left[Y \wedge \sigma^{-1}(Y)\right] \vee \sigma\left[Y \wedge \sigma^{-1}(Y)\right]\right).$$

Hence $(\mathbf{id} \wedge \delta)(Y) = Y \wedge \delta(Y)$ is a supremum of terms of the form $X \vee \sigma(X)$ (for $X \in \mathscr{L}$ and $\sigma \in \mathscr{S}$), and so it must be fixed by **id** $\wedge\, \delta$. This means that $(\mathbf{id} \wedge \delta)^2(Y) = (\mathbf{id} \wedge \delta)(Y)$, and **id** $\wedge\, \delta$ is idempotent.

Finally, for every $Y \in \mathscr{L}$ fixed by **id** $\wedge\, \delta$, we have $Y = Y \wedge \delta(Y)$, which takes the above form of a supremum of terms $X \vee \sigma(X)$ where $\sigma \in \mathscr{S}$. Therefore this form characterizes elements of $\mathscr{L}$ invariant under **id** $\wedge\, \delta$.

(ii) is the dual of (i) under the inversion of the order relations $\leq$ and $\geq$. ∎

Lemma 3.10 *Let* $(\mathscr{L}, \ell, \mathbf{T})$ *be a Minkowski lattice and let* $A \in \mathscr{L}$ *be symmetric. Then*

$$\delta_A = \bigvee_{a\in\ell^*(A)} \tau_a = \bigvee_{a\in\ell^*(A)} \tau_{-a} \quad \text{and} \quad \varepsilon_A = \bigwedge_{a\in\ell^*(A)} \tau_a = \bigwedge_{a\in\ell^*(A)} \tau_{-a}, \tag{3.4}$$

in other words for every $X \in \mathscr{L}$ *we have*

$$X \oplus A = \bigvee_{a\in\ell^*(A)} X_a = \bigvee_{a\in\ell^*(A)} X_{-a} \quad \text{and} \quad X \ominus A = \bigwedge_{a\in\ell^*(A)} X_a = \bigwedge_{a\in\ell^*(A)} X_{-a}. \tag{3.5}$$

Proof. Let $X \in \mathscr{L}$. As A is symmetric, we have $A = \bigvee \ell^*(A)$, and the fact that the Minkowski sum distributes the supremum operation implies that

$$X \oplus A = X \oplus \left(\bigvee \ell^*(A)\right) = \bigvee_{a\in\ell^*(A)} (X \oplus a) = \bigvee_{a\in\ell^*(A)} X_a.$$

As $\ell^*(A)$ is invariant under the permutation $a \mapsto -a$, we get

$$X \oplus A = \bigvee_{a\in\ell^*(A)} X_a = \bigvee_{a\in\ell^*(A)} X_{-a}.$$

in other words the left half of (3.5), and we deduce the formula

$$\delta_A = \bigvee_{a \in \ell(A)} \tau_a = \bigvee_{a \in \ell^*(A)} \tau_a = \bigvee_{a \in \ell^*(A)} \tau_{-a},$$

that is the left half of (3.4). Thanks to [7, Lemma 3.4] we deduce from this equality the following one:

$$\varepsilon_A = \bigwedge_{a \in \ell(A)} \tau_{-a} = \bigwedge_{a \in \ell^*(A)} \tau_{-a} = \bigwedge_{a \in \ell^*(A)} \tau_a,$$

in other words the right half of (3.4), which gives

$$X \ominus A = \bigwedge_{a \in \ell^*(A)} X_a = \bigwedge_{a \in \ell^*(A)} X_{-a},$$

that is the right half of (3.5). ∎

Combining the above two results for $\mathcal{S} = \{\tau_a \mid a \in \ell^*(A)\}$, we obtain the following:

Corollary 3.11 *Let $(\mathcal{L}, \ell, \mathbf{T})$ be a Minkowski lattice and let $A \in \mathcal{L}$ be symmetric.*

(*i*) *If $\mathcal{L}$ is ISD, then* $\mathbf{id} \wedge \delta_A$ *is idempotent, and an element of $\mathcal{L}$ is fixed by* $\mathbf{id} \wedge \delta_A$ *if and only if it is a supremum of terms of the form $X \vee X_a$ for $X \in \mathcal{L}$ and $a \in \ell^*(A)$.*

(*ii*) *If $\mathcal{L}$ is IID, then $\varepsilon_A \vee$* $\mathbf{id}$ *is idempotent, and an element of $\mathcal{L}$ is fixed by $\varepsilon_A \vee$* $\mathbf{id}$ *if and only if it is an infimum of terms of the form $X \wedge X_a$ for $X \in \mathcal{L}$ and $a \in \ell^*(A)$.*

The following example shows that we cannot avoid the ISD condition for proving the idempotence of $\mathbf{id} \wedge \delta_A$:

Example 3.12 Let $\mathcal{L}$ be the family of all closed subsets of $\mathbb{R}^2$, ordered by inclusion. Then it is an IID Minkowski lattice, where the infimum and supremum of a family of closed set are given respectively by their intersection and the closure of their union, ℓ consists of all singletons, and $\mathbf{T}$ is the family of all translations of $\mathbb{R}^2$. Moreover $\mathcal{L}$ is atomic, the elements of ℓ forming the atoms, but $\mathcal{L}$ is not ISD. There is nevertheless a symmetric structuring element A for which $\mathbf{id} \wedge \delta_A$ is not idempotent: we take

$$A = \{(z, 0) \mid z \in \mathbb{Z} \text{ and } z \neq 0\}.$$

Indeed, let X be the closed set consisting of the two points $(0, 1)$ and $(0, -1)$, and of the curve made of all points $(x, f(x))$ for $x \in \mathbb{R}$, where f is a strictly increasing continuous function satisfying $\lim_{x\to+\infty} = +1$ and $\lim_{x\to-\infty} = -1$ (for example $f(x) = \frac{2}{\pi}\arctan(x)$). We illustrate X and A in Fig. 4 (a)).

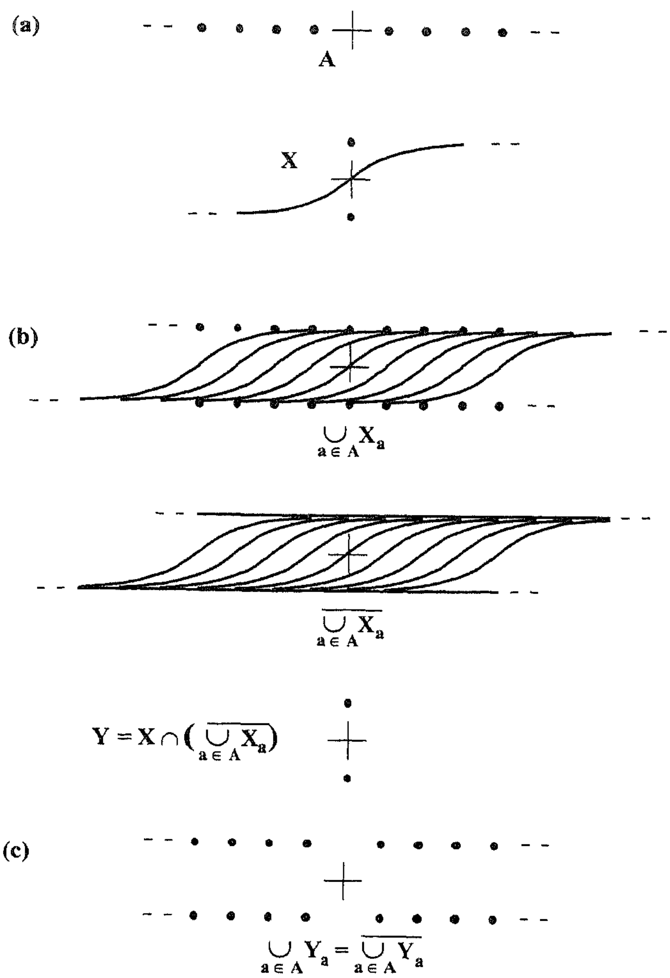

Fig. 4. The cross indicates the position of the origin in $\mathbb{R}^2$. (**a**) We show two closed sets A and X; A is symmetric (in the sense of Definition 3.6). (**b**) In the Minkowski lattice of closed sets of $\mathbb{R}^2$, the dilation $\delta_A(X)$ of X by A is the supremum of translates X_a of X by points $a \in A$, in other words the closure of their union; now $Y = (\mathbf{id} \wedge \delta_A)(X)$ is obtained by intersecting X with $\delta_A(X)$. (**c**) The union of translates Y_a of Y by points $a \in A$ is closed; it is thus the dilation $\delta_A(Y)$ of Y by A; clearly the intersection of Y and $\delta_A(Y)$ is empty, so that $Y \neq (\mathbf{id} \wedge \delta_A)(Y) = \emptyset$. Hence $\mathbf{id} \wedge \delta_A$ is not idempotent. Thus Corollary 3.11 fails here because the Minkowski lattice of closed sets of $\mathbb{R}^2$ is only distributive, but not ISD

As explained in [7, Subsection 4.2], $\delta_A(X)$ is the closure of the union of all X_a for $a \in A$. Now all X_a (for $a \in A$) are disjoint from X, but the points $(x, 1)$ and $(x, -1)$ (for $x \in \mathbb{R}$) are adherent to the union of all X_a; thus $(0, 1)$ and $(0, -1)$ are the only points of X inside the closure of the union of all X_a, and so $Y = (\mathbf{id} \wedge \delta_A)(X)$ consists of the two points $(0, 1)$ and $(0, -1)$: see Fig. 4 (b). Now it is easily seen that $(\mathbf{id} \wedge \delta_A)(Y) = \emptyset \neq Y$, that is $(\mathbf{id} \wedge \delta_A)^2(X) \neq (\mathbf{id} \wedge \delta_A)(X)$: see Fig. 4 (c).

Our next example shows that we cannot avoid distributivity for proving the idempotence of $\varepsilon_A \vee \mathbf{id}$:

Example 3.13 Let $\mathscr{L}$ be the family of all "horizontally convex" subsets of $\mathbb{R}^2$, in other words sets $S \subseteq \mathbb{R}^2$ such that for every $t \in \mathbb{R}$, the horizontal cross-section

$$X_t(S) = \{x \in \mathbb{R} \mid (x, t) \in S\}$$

is convex. Ordered by inclusion, it is a complete lattice: the infimum of a family of "horizontally convex" sets S_i is their intersection, while the supremum of that family is $HCH\big(\bigcup_i S_i\big)$, the "horizontally convex hull" of their union; here the "horizontally convex hull" $HCH(S)$ of a set S is the set whose horizontal cross-sections are the convex hulls of the corresponding horizontal cross-sections of S:

$$X_t\big(HCH(S)\big) = CH\big(X_t(S)\big),$$

where $CH(X)$ denotes the convex hull of X. Taking ℓ consisting of all singletons, and $\mathbf{T}$ the family of all translations of $\mathbb{R}^2$, $\mathscr{L}$ is a Minkowski lattice. Note that $\mathscr{L}$ is not distributive. We take the symmetric structuring element

$$A = [-1, 1] \times \{-1, 1\}$$

and the horizontally convex set

$$X = \big([0, 3] \times \{-2, 0, 2\}\big) \cup \{(-2, -1), (-2, 1)\}$$

shown in Fig. **5.** (a). Here we have

$$(\varepsilon_A \vee \mathbf{id})(X) = HCH\big((X \ominus A) \cup X\big),$$

where $X \ominus A$ is taken in the usual sense (of the lattice $\mathscr{P}(\mathbb{R}^2)$), and this gives (see Fig. **5.** (b)):

$$(\varepsilon_A \vee \mathbf{id})(X) = \big([0, 3] \times \{-2, 0, 2\}\big) \cup \big([-2, 2] \times \{-1, 1\}\big).$$

We get then (see Figure 5 (c))

$$(\varepsilon_A \vee \mathbf{id})^2(X) = (\varepsilon_A \vee \mathbf{id})(X) \cup \big([-1, 0[\times\{0\}\big) \supset (\varepsilon_A \vee \mathbf{id})(X).$$

Thus $\varepsilon_A \vee \mathbf{id}$ is not idempotent.

The following result represents in some way a converse of Corollary 3.11 (i). It generalizes [13, Theorem 3.4] in the sense that it uses a weaker hypothesis, namely that $\mathscr{L}$ is distributive, where in [13] we required the elements of ℓ to be "co-prime" (something which implies the distributivity of $\mathscr{L}$, see [13, Example 3.2]; note also that the original statement of [13, Theorem 3.4] mentions the condition that $\mathscr{L}$ is ISD, but the proof uses only the fact that $\mathscr{L}$ is distributive). Recall that the definition of the translation by an element p of ℓ, of the Minkowski operations $\oplus$ and $\ominus$, and of symmetric elements of $\mathscr{L}$, depend on

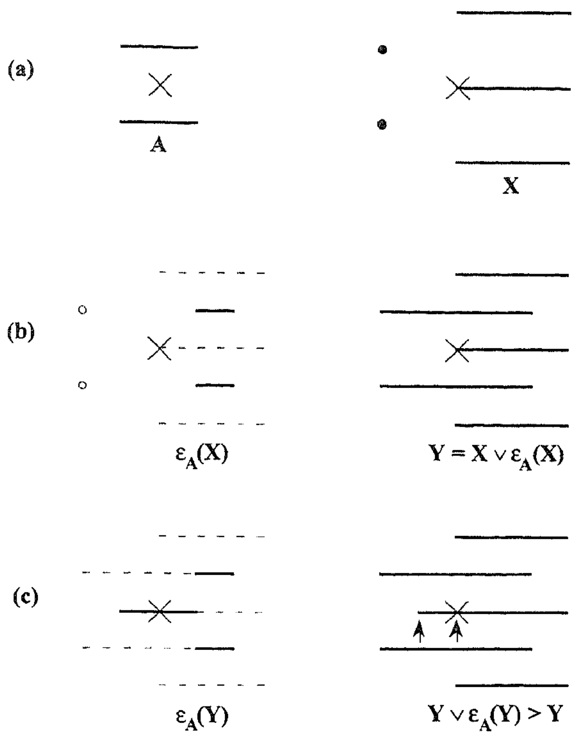

Fig. 5. The cross indicates the position of the origin in $\mathbb{R}^2$. (**a**) We show two horizontally convex sets A and X; A is symmetric (in the sense of Definition 3.6). (**b**) In the left figure, hollow circles and dashed lines indicate X; in the Minkowski lattice of horizontally convex sets of $\mathbb{R}^2$, $\varepsilon_A(X)$ is the horizontal convex hull of the usual erosion of X by A, and $X \vee \varepsilon_A(X)$ is the horizontal convex hull of $X \cup \varepsilon_A(X)$. (**c**) We construct similarly $\varepsilon_A(Y)$ and $Y \vee \varepsilon_A(Y)$ for $Y = X \vee \varepsilon_A(X)$; here in the left figure dashed lines indicate Y. The two arrows point to a segment included in $Y \vee \varepsilon_A(Y)$ but not in Y. Thus Corollary 3.11 fails here because the Minkowski lattice of horizontally convex subsets of $\mathbb{R}^2$ is not distributive

the choice of the "origin" o in ℓ (the same happen in Euclidean space); thus it is not astonishing that this element o will appear explicitly in the conditions imposed on the elements of $\mathcal{L}$ from which dilations or erosions are made.

Proposition 3.14 *Let $(\mathcal{L}, \ell, \mathrm{T})$ be a distributive lower Minkowski lattice. Let $A \in \mathcal{L}$ such that $o \notin \ell(A)$ and $\mathbf{id} \wedge \delta_A$ is idempotent; then A is annular.*

Proof. We have $(\mathbf{id} \wedge \delta_A)(o) = o \wedge A$. Suppose that $o \wedge A \neq O$. Then the fact that ℓ is lower implies that $o \wedge A \in \ell$; we set $o \wedge A = x$. As $\mathbf{id} \wedge \delta_A$ is idempotent, we have

$$x \wedge A_x = (\mathbf{id} \wedge \delta_A)(x) = (\mathbf{id} \wedge \delta_A)(\mathbf{id} \wedge \delta_A)(o) = (\mathbf{id} \wedge \delta_A)(o) = x,$$

that is $x \leq A_x$, and by translating by $-x$, we get $o \leq A$, contradicting the hypothesis. Therefore $o \wedge A = O$. For every $z \in \ell$ we have thus

$$z \wedge A_z = (o \wedge A)_z = O_z = O. \tag{3.6}$$

Take $a \in \ell(A)$, and let $X = o \vee a$ and $Y = (\mathbf{id} \wedge \delta_A)(X)$. We have $\delta_A(X) = A \vee A_a$ and $(\mathbf{id} \wedge \delta_A)(X) = X \wedge \delta_A(X) = (o \vee a) \wedge (A \vee A_a)$. By distributivity, this gives

$$Y = (o \vee a) \wedge (A \vee A_a) = (o \wedge A) \vee (a \wedge A) \vee (o \wedge A_a) \vee (a \wedge A_a).$$

Now $o \wedge A = a \wedge A_a = O$ by (3.6); also $a \in \ell(A)$ implies that $a \leq A$, in other words $a \wedge A = a$. Hence $Y = a \vee (o \wedge A_a)$. If we had $o \wedge A_a = O$, then we would get $Y = a$, and as $Y = (\mathbf{id} \wedge \delta_A)(X)$ and $\mathbf{id} \wedge \delta_A$ is idempotent, this would imply that $a = (\mathbf{id} \wedge \delta_A)(a) = a \wedge A_a$, but $a \wedge A_a = O$ by (3.6), a contradiction. Hence $o \wedge A_a \neq O$; as ℓ is lower, $o \wedge A_a \in \ell$; we set $u = o \wedge A_a$. Thus

$$u \leq o, \quad u \leq A_a, \tag{3.7}$$

and $Y = a \vee u$. As $Y = (\mathbf{id} \wedge \delta_A)(X)$ and $\mathbf{id} \wedge \delta_A$ is idempotent, we have $(\mathbf{id} \wedge \delta_A)(Y) = Y$; now $\delta_A(Y) = A_a \vee A_u$ and so $(\mathbf{id} \wedge \delta_A)(Y) = (a \vee u) \wedge (A_a \vee A_u)$; distributivity gives thus:

$$a \vee u = (a \vee u) \wedge (A_a \vee A_u) = (a \wedge A_a) \vee (u \wedge A_a) \vee (a \wedge A_u) \vee (u \wedge A_u).$$

Now (3.6) gives $a \wedge A_a = u \wedge A_u = O$; also $u \leq A_a$ by (3.7), in other words $u \wedge A_a = u$. Therefore the above equation reduces to

$$a \vee u = u \vee (a \wedge A_u),$$

and applying again distributivity together with the fact that $a \leq a \vee u$, we obtain:

$$a = a \wedge (a \vee u) = a \wedge [u \vee (a \wedge A_u)] = (a \wedge u) \vee [a \wedge (a \wedge A_u)] = (a \wedge u) \vee (a \wedge A_u).$$

But $u \leq A_a$ by (3.7), so that $a \wedge u \leq a \wedge A_a = O$ by (3.6), and the above equation leads to $a = a \wedge A_u$, that is

$$a \leq A_u. \tag{3.8}$$

Let $a' = a - u$. As $u \leq o$ by (3.7), we have $a' = (a-u)+o \geq (a-u)+u = a$. As $u \leq A_a$ by (3.7), we have $-a' = u - a \leq (A_a)_{-a} = A$, that is $-a' \in \ell(A)$. As $a \leq A_u$ by (3.8), we get $a' = a - u \leq (A_u)_{-u} = A$, that is $a' \in \ell(A)$.

Therefore we have shown that for every $a \in \ell(A)$ there is $a' \in \ell(A)$ such that $a' \geq a$ and $-a' \in \ell(A)$, that is A is annular. ∎

Note that for every $A \in \mathscr{L}$ such that $o \in \ell(A)$, we have $\mathbf{id} \wedge \delta_A = \mathbf{id}$, which is obviously idempotent. It would be interesting to obtain such a result in the case of the idempotence of $\varepsilon_A \vee \mathbf{id}$.

In order to build an annular filter we will require two symmetric elements A and B of $\mathscr{L}$ such that $\varepsilon_B \leq \delta_A$, and verify the idempotence of $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$. We can state our main result:

Proposition 3.15 *Let* $(\mathscr{L}, \ell, \mathbf{T})$ *be a Minkowski lattice such that* $\mathscr{L}$ *is both ISD and IID. Let* $A, B \in \mathscr{L}$ *be symmetric and such that* $o \leq A \oplus B \oplus (A \sqcap B)$. *Then* $\varepsilon_B \leq \delta_A$ *and the three operators* $\mathbf{id} \wedge \delta_A$, $\varepsilon_B \vee \mathbf{id}$, *and* $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$, *are idempotent.*

Proof. Let $C = A \sqcap B$. As $o \leq A \oplus B \oplus C$, we have $C \neq O$ (because $X \oplus O = O$ for every $X \in \mathscr{L}$, [7]), and as C is symmetric, $\ell^*(C)$ is not empty; taking $c \in \ell^*(C)$, we have $c \leq C \leq B$ and $-c \leq C \leq A$. Let $X \in \mathscr{L}$. By definition (3.3) of $X \ominus B$, for every $b \in \ell(B)$ we have $X \ominus B \leq X_{-b}$; as $c \in \ell(B)$, this gives $X \ominus B \leq X_{-c}$. By definition (3.3) of $X \oplus A$, for every $a \in \ell(A)$ we have $X_a \leq X \oplus A$; as $-c \in \ell(A)$, this gives $X_{-c} \leq X \oplus A$. Hence

$$\varepsilon_B(X) = X \ominus B \leq X_{-c} \leq X \oplus A = \delta_A(X),$$

and we deduce that $\varepsilon_B \leq \delta_A$.

Since $\mathscr{L}$ is ISD and IID, while A and B are symmetric, Corollary 3.11 implies that $\mathbf{id} \wedge \delta_A$ and $\varepsilon_B \vee \mathbf{id}$ are idempotent.

Let $a \in \ell(A)$, $b \in \ell(B)$, and $c \in \ell^*(C)$ such that $o \wedge (a + b + c) \neq O$, and $x \in \ell\big(o \wedge (b + a + c)\big)$. We set $a' = x - b - c$; thus $x = b + a' + c$. As $x \leq b + a + c$, we have $a' \leq a$, so that $a' \in \ell(A)$; as $x \leq o$, we have $b + a' + c \leq o$ and hence $b + a' \leq -c$; also $c, -c \in \ell(A) \cap \ell(B)$, so that $b + a' \in \ell(A) \cap \ell(B)$.

Let $X \in \mathscr{L}$ and $y \in \ell(X \ominus B)$. As $b \in \ell(B)$, we have $y \leq X \ominus B \leq X_{-b}$, so that $y + b \leq X$; similarly, as $b + a' \in \ell(B)$, we have $y + b + a' \leq X$. As $a' \in \ell(A)$ and $y + b \leq X$, we get $y + b + a' \leq X_{a'} \leq X \oplus A$, and since $y + b + a' \leq X$, we get $y + b + a' \leq X \wedge (X \oplus A)$. Now $c \leq A$, so that

$$\tau_{b+a'+c}(y) = (y + b + a') + c \leq \big(X \wedge (X \oplus A)\big)_c \leq \big(X \wedge (X \oplus A)\big) \oplus A.$$

As $\tau_{b+a'+c}(y) \leq \big(X \wedge (X \oplus A)\big) \oplus A$ for every $y \in \ell(X \ominus B)$, we deduce that

$$\tau_{b+a'+c}(X \ominus B) = \bigvee_{y \in \ell(X \ominus B)} \tau_{b+a'+c}(y) \leq \big(X \wedge (X \oplus A)\big) \oplus A$$

for every $X \in \mathscr{L}$, in other words $\tau_x \varepsilon_B = \tau_{b+a'+c} \varepsilon_B \leq \delta_A(\mathbf{id} \wedge \delta_A)$.

Let $X \in \mathscr{L}$, $Y = (X \ominus B) \vee X$, and $z \in \ell(Y \ominus B)$. As $c \in \ell(B)$, we have then $z \leq Y \ominus B \leq Y_{-c}$, so that $z + c \leq Y$. Let $V = (z + c) \wedge (X \ominus B)$ and $W = (z + c) \wedge X$; as $z + c \leq Y = (X \ominus B) \vee X$ and $\mathscr{L}$ is distributive, we have

$$\begin{aligned} z + c = (z + c) \wedge Y &= (z + c) \wedge \big[(X \ominus B) \vee X\big] \\ &= \big[(z + c) \wedge (X \ominus B)\big] \vee \big[(z + c) \wedge X\big] = V \vee W, \end{aligned}$$

so that

$$z + c = \Big(\bigvee \ell(V)\Big) \vee \Big(\bigvee \ell(W)\Big) = \bigvee\big(\ell(V) \cup \ell(W)\big).$$

For $v \in \ell(V)$, as $v \leq X \ominus B$ and $b \in \ell(B)$, we get $v \leq X_{-b}$, and so $v+b \leq X$, and as $a' \in \ell(A)$, we get $v + b + a' \leq X_{a'} \leq X \oplus A$. On the other hand, for $w \in \ell(W)$, as $w \leq X$ and $b+a' \in \ell(A)$, we get $w+b+a' \leq X_{b+a'} \leq X \oplus A$. Hence for every $u \in \ell(V) \cup \ell(W)$, $u + b + a' \leq X \oplus A$; now

$$\begin{aligned} z + c + b + a' = \tau_{b+a'}(z + c) &= \tau_{b+a'}\Big(\bigvee\big(\ell(V) \cup \ell(W)\big)\Big) \\ &= \bigvee\big\{\tau_{b+a'}(u) \mid u \in \ell(V) \cup \ell(W)\big\} \\ &= \bigvee\big\{u + b + a' \mid u \in \ell(V) \cup \ell(W)\big\}, \end{aligned}$$

from which we deduce that $\tau_{b+a'+c}(z) = z+c+b+a' \leq X \oplus A$. As this holds for every $z \in \ell(Y \ominus B)$, it follows that

$$\tau_{b+a'+c}\Big(\big((X \ominus B) \vee X\big) \ominus B\Big) = \tau_{b+a'+c}(Y \ominus B) = \bigvee_{z \in \ell(Y \ominus B)} \tau_{b+a'+c}(z) \leq X \oplus A$$

for every $X \in \mathcal{L}$, in other words $\tau_x \varepsilon_B(\varepsilon_B \vee \mathbf{id}) = \tau_{b+a'+c}\varepsilon_B(\varepsilon_B \vee \mathbf{id}) \leq \delta_A$.

Now $A \oplus B = \bigvee_{a \in \ell(A)} \bigvee_{b \in \ell(B)} (a+b)$ and Lemma 3.10 gives $(A \oplus B) \oplus C = \bigvee_{c \in \ell^*(C)} (A \oplus B)_c$, so that

$$A \oplus B \oplus C = \bigvee_{a \in \ell(A)} \bigvee_{b \in \ell(B)} \bigvee_{c \in \ell^*(C)} (a + b + c);$$

as $o \leq A \oplus B \oplus C$, by ISD we have

$$\begin{aligned} o = o \wedge (A \oplus B \oplus C) &= \bigvee_{a \in \ell(A)} \bigvee_{b \in \ell(B)} \bigvee_{c \in \ell^*(C)} \big(o \wedge (a + b + c)\big) \\ &= \bigvee_{a \in \ell(A)} \bigvee_{b \in \ell(B)} \bigvee_{c \in \ell^*(C)} \bigvee \ell\big(o \wedge (a + b + c)\big) = \bigvee \mathcal{X}, \end{aligned}$$

where

$$\mathcal{X} = \bigcup\big\{\ell\big(o \wedge (a + b + c)\big) \mid a \in \ell(A),\ b \in \ell(B),\ c \in \ell^*(C) \text{ and } o \wedge (a + b + c) \neq O\big\}.$$

Then [7, Lemma 3.4] gives

$$\mathbf{id} = \tau_o = \bigvee_{x \in \mathcal{X}} \tau_x.$$

Now we showed above that for every such $x \in \mathcal{X}$ we have $\tau_x \varepsilon_B \leq \delta_A(\mathbf{id} \wedge \delta_A)$ and $\tau_x \varepsilon_B(\varepsilon_B \vee \mathbf{id}) \leq \delta_A$, so that we get

$$\varepsilon_B = \Big(\bigvee_{x \in \mathcal{X}} \tau_x\Big)\varepsilon_B = \bigvee_{x \in \mathcal{X}} (\tau_x \varepsilon_B) \leq \delta_A(\mathbf{id} \wedge \delta_A)$$

and

$$\varepsilon_B(\varepsilon_B \vee \mathbf{id}) = \Big(\bigvee_{x \in \mathscr{X}} \tau_x\Big)\varepsilon_B(\varepsilon_B \vee \mathbf{id}) = \bigvee_{x \in \mathscr{X}} \big(\tau_x \varepsilon_B(\varepsilon_B \vee \mathbf{id})\big) \leq \delta_A.$$

Combining the inequalities $\varepsilon_B \leq \delta_A(\mathbf{id} \wedge \delta_A)$ and $\varepsilon_B(\varepsilon_B \vee \mathbf{id}) \leq \delta_A$ with the idempotence of $\mathbf{id} \wedge \delta_A$ and $\varepsilon_B \vee \mathbf{id}$, it follows from the sufficient condition for idempotence given in Corollary 2.4 that $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$ is idempotent. ∎

The above proofs that $\tau_{b+a'+c}\varepsilon_B \leq \delta_A(\mathbf{id} \wedge \delta_A)$ and $\tau_{b+a'+c}\varepsilon_B(\varepsilon_B \vee \mathbf{id}) \leq \delta_A$ were inspired by the arguments given after (2.10) for 0- and 1-triple points in the case where $\mathscr{L} = \mathscr{P}(E)$.

When $A = B$, we have $A \sqcap A = A$, and Proposition 3.15 becomes:

Corollary 3.16 *Let $(\mathscr{L}, \ell, \mathbf{T})$ be a Minkowski lattice such that $\mathscr{L}$ is both ISD and IID. Let $A \in \mathscr{L}$ be symmetric and such that $o \leq A \oplus A \oplus A$. Then $\varepsilon_A \leq \delta_A$ and the three operators $\mathbf{id} \wedge \delta_A$, $\varepsilon_A \vee \mathbf{id}$, and $\varepsilon_A \vee \mathbf{id} \wedge \delta_A$, are idempotent.*

Let us now consider the meaning of the above results in the lattice $\mathscr{P}(E)$ of subsets of E and the lattice $\mathrm{Fun}(E, \mathscr{T})$ of grey-level functions $E \to \mathscr{T}$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ and $\mathscr{T} = \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ or $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$. Both are Minkowski lattices satisfying all the properties mentioned above, namely they are both ISD and IID, and ℓ is lower. Therefore all our results apply in these two cases.

In the case of $\mathscr{P}(E)$, a structuring element A is symmetric (in the sense of Definition 3.6) if and only if it is annular, and this simply means that $A = \check{A} = \{-a \mid a \in A\}$, in other words that it is symmetric in the ordinary geometric sense. The family of symmetric structuring elements is closed under union and Minkowski sum (cfr. Proposition 3.8), but also under intersection, complementation, and Minkowski difference. By Corollary 3.11 and Proposition 3.14, for any structuring element A, $\mathbf{id} \wedge \delta_A$ is idempotent if and only if A is symmetric or $o \in A$, where o is the origin in E. Thanks to the duality by complementation, the analogue of Proposition 3.14 for erosions is also true, so that $\varepsilon_A \vee \mathbf{id}$ is idempotent if and only if A is symmetric or $o \in A$. Note that for $o \in A$, $\mathbf{id} \wedge \delta_A = \varepsilon_A \vee \mathbf{id} = \mathbf{id}$. In Proposition 3.15, $A \sqcap B$ is symmetric, so that $A \sqcap B = A \cap B$, and hence the hypothesis reduces to the fact that A and B are symmetric and that $o \in (A \cap B) \oplus A \oplus B$; now this condition means that there is $x \in A \oplus B$ and $y \in A \cap B$ with $x + y = o$, that is $x = -y$, and as $A \cap B$ is symmetric, this holds iff $x = -y \in A \cap B$, and so the condition is equivalent to $A \cap B \cap (A \oplus B) \neq \emptyset$; we obtain here what we said in [8, Proposition 6.1].

In the case of $\mathrm{Fun}(E, \mathscr{T})$ (see [7, Section 4]), the "origin" in ℓ is the "impulse" $f_{o,0}$ (cfr. (3.1)) where o is the origin of space E, in other words the function having value 0 on o and $-\infty$ elsewhere; the formulas for the Minkowski operations are:

$$(F \oplus G)(x) = \sup_{h \in \mathbb{R}^d} \big(F(x-h) + G(h)\big)$$

$$\text{and} \quad (F \ominus G)(x) = \inf_{h \in \mathbb{R}^d} \big(F(x+h) - G(h)\big),$$

with the further conventions, in cases of ambiguous expressions of the form $+\infty - \infty$, that $F(x-h)+G(h) = -\infty$ when $F(x-h) = -\infty$ or $G(h) = -\infty$, and that $F(x+h) - G(h) = +\infty$ when $F(x+h) = +\infty$ or $G(h) = -\infty$. Now a structuring function A is symmetric (cfr. Definition 3.6) if and only if it is annular, and as explained in [13], this means that supp(A), the support of A (i.e., the set of points $p \in E$ such that $A(p) > -\infty$), is symmetric and that for all $x \in \text{supp}(A)$ we have $A(x) + A(-x) \geq 0$ (cfr. (1.3)). The family of symmetric structuring functions is closed under supremum and Minkowski sum (cfr. Proposition 3.8), but not under infimum and Minkowski difference (see the examples given after Proposition 3.8). By Corollary 3.11 and Proposition 3.14, for any structuring function A, $\mathbf{id} \wedge \delta_A$ is idempotent if and only if A is symmetric or $A(o) \geq 0$, where o is the origin in E. Thanks to the duality by grey-level inversion, the analogue of Proposition 3.14 for erosions is also true, so that $\varepsilon_A \vee \mathbf{id}$ is idempotent if and only if A is symmetric or $A(o) \geq 0$. Note that for $A(o) \geq 0$, $\mathbf{id} \wedge \delta_A = \varepsilon_A \vee \mathbf{id} = \mathbf{id}$. Given two symmetric structuring functions A and B, $A \sqcap B$ is defined by setting for every point $p \in E$:

$$(A \sqcap B)(p) = \begin{cases} \min\big(A(p), B(p)\big) & \text{if } p \in \text{supp}(A) \cap \text{supp}(B) \text{ and} \\ & \min\big(A(p), B(p)\big) \\ & \quad +\min\big(A(-p), B(-p)\big) \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

In Proposition 3.15 we must check that $\big[A \oplus B \oplus (A \sqcap B)\big](o) \geq 0$.

Let us give a concrete example with space $E = \mathbb{Z}^d$ and grey-level set $\mathcal{T} = \overline{\mathbb{Z}}$. We take two *symmetric* sets S_0 and S_1 not containing the origin o, which represent the neighbourhood of the origin in dark (background) and light (foreground) conditions respectively. We take two grey-levels $t_0, t_1 \geq 0$, which indicate thresholds in grey-level difference with the neighbourhood at which isolated dark and light points respectively must be eliminated. We define the structuring functions A_0 and A_1 having respective supports S_0 and S_1, on which they take constant grey-levels t_0 and t_1 respectively, in other words

$$A_i(p) = \begin{cases} t_i & \text{for } p \in S_i, \\ -\infty & \text{for } p \notin S_i, \end{cases} \qquad i = 0, 1.$$

Since S_i is symmetric and $t_i \geq 0$, it follows that A_i is symmetric (or equivalently, annular) for $i = 0, 1$. Note that $A_0 \sqcap A_1$ has support $S_0 \cap S_1$, and has grey-level $\min(t_0, t_1)$ on it. For a grey-level image $I : E \to \mathcal{T}$, we have $(\mathbf{id} \wedge \delta_{A_1})(I) = I \wedge (I \oplus A_1)$, where $(I \oplus A_1)(p) = [\sup_{a \in S_1} I(p+a)] + t_1$. Thus, whenever the grey-level of a point p is lighter than (superior to) that of all its neighbours (according to S_1) by more than t_1, it is decreased accordingly. Similarly $(\mathbf{id} \vee \varepsilon_{A_0})(I) = I \vee (I \ominus A_0)$ where $(I \ominus A_0)(p) = [\inf_{a \in S_0} I(p+a)] - t_0$. Thus, whenever the grey-level of a point p is darker than (inferior to) that of all its neighbours (according to S_0) by more than t_0, it is increased accordingly. The two operators $\mathbf{id} \wedge \delta_{A_1}$ and $\mathbf{id} \vee \varepsilon_{A_0}$ are idempotent, because A_0 and A_1 are annular; they constitute an annular opening and closing respectively. Now for $S_0 \cap S_1 \neq \emptyset$ we have $\varepsilon_{A_0} \leq \delta_{A_1}$ and so we can consider the annular operator $\varepsilon_{A_0} \vee \mathbf{id} \wedge \delta_{A_1}$. Here $(\varepsilon_{A_0} \vee \mathbf{id} \wedge \delta_{A_1})(I) = (I \ominus A_0) \vee I \wedge (I \oplus A_1)$ is given as follows

$$\big[(I \ominus A_0) \vee I \wedge (I \oplus A_1)\big](p) = (I \ominus A_0)(p) \vee I(p) \wedge (I \oplus A_1)(p)$$

$$= \begin{cases} (I \ominus A_0)(p) \text{ if } I(p) < (I \ominus A_0)(p), \\ I(p) \qquad\qquad \text{if } (I \ominus A_0)(p) \leq I(p) \\ \qquad\qquad\qquad\quad \leq (I \oplus A_1)(p), \\ (I \oplus A_1)(p) \text{ if } (I \oplus A_1)(p) < I(p) \end{cases}$$

where

$$(I \ominus A_0)(p) = \big[\inf_{a \in S_0} I(p+a)\big] - t_0,$$

$$\text{and} \qquad (I \oplus A_1)(p) = \big[\sup_{a \in S_1} I(p+a)\big] + t_1.$$

The behaviour of the annular operator $\varepsilon_{A_0} \vee \mathbf{id} \wedge \delta_{A_1}$ combines that of the annular opening $\mathbf{id} \wedge \delta_{A_1}$ and the annular closing $\mathbf{id} \vee \varepsilon_{A_0}$. When $o \in (S_0 \cap S_1) \oplus S_0 \oplus S_1$, as $t_0, t_1 \geq 0$, we have

$$\big[(A_0 \sqcap A_1) \oplus A_0 \oplus A_1\big](o) = \min(t_0, t_1) + t_0 + t_1 \geq 0.$$

Indeed, $A_0 \sqcap A_1$, A_0, and A_1 are what one calls *flat functions*, that is functions having constant value on their support: $A_0 \sqcap A_1$ has support $S_0 \cap S_1$ with value $\min(t_0, t_1)$ on it, A_0 has support S_0 with value t_0 on it, and A_1 has support S_1 with value t_1 on it; now it is well-known (and easily verified from formulas) that the Minkowski sum of flat functions is the flat function whose support is the Minkowski sum of their supports, and whose value on it is the sum of their respective values. Thus by what we said above, $\varepsilon_{A_0} \vee \mathbf{id} \wedge \delta_{A_1}$ is idempotent; it is an annular filter. We illustrate its behaviour for $d = 1$ in Fig. 6.

4 Composing annular openings and closings, and strong annular filters

In Section 2 we studied sufficient conditions for the idempotence of an annular filter of the form $\varepsilon \vee \mathbf{id} \wedge \delta$ (with $\varepsilon \leq \delta$), defined on an arbitrary modular lattice, where $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$ are themselves idempotent, representing an annular closing and opening respectively. In Section 3, ε and δ were defined in a Minkowski lattice as an erosion and a dilation by symmetric elements; we generalized there previous results on annular closings and openings. In this section we will investigate under which conditions the annular filter $\varepsilon \vee \mathbf{id} \wedge \delta$

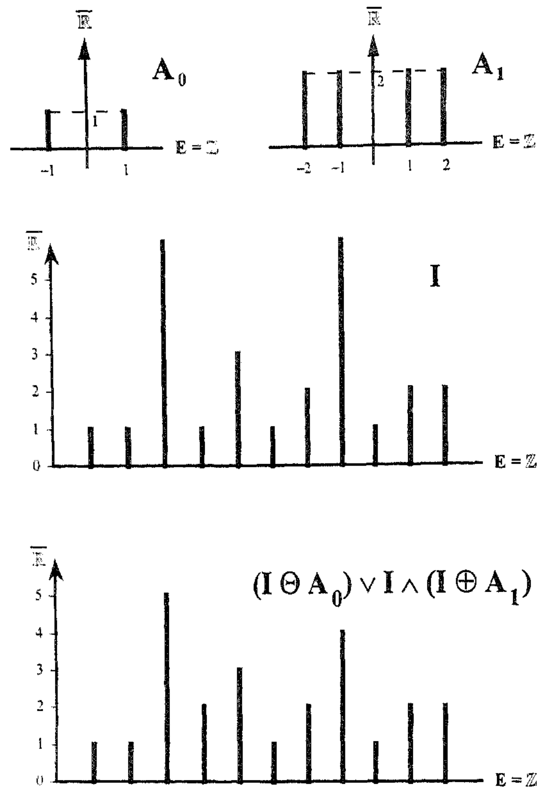

Fig. 6. We consider the one-dimensional discrete space $E = \mathbb{Z}$. *Top*: the two symmetric structuring functions A_0 and A_1 satisfy the condition $(A_0 \prod A_1) \oplus A_0 \oplus A_1 \geq o$, so that for the erosion ε_0 by A_0 and the dilation δ_1 by A_1, the annular operator $\varepsilon_0 \vee \mathbf{id} \wedge \delta_1$ is idempotent. Below we show a function I and the result of applying the filter $\varepsilon_0 \vee \mathbf{id} \wedge \delta_1$ to I; the filtered function is invariant under further filtering

can be obtained by composing the annular opening $\mathbf{id} \wedge \delta$ and the annular closing $\varepsilon \vee \mathbf{id}$, in other words

$$\varepsilon \vee \mathbf{id} \wedge \delta = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta) \tag{4.1}$$

$$\text{or} \qquad \varepsilon \vee \mathbf{id} \wedge \delta = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id}). \tag{4.2}$$

We will see that these equations (4.1) and (4.2) are related to certain properties of the operator $\varepsilon \vee \mathbf{id} \wedge \delta$, namely being an inf-overfilter, a sup-underfilter, and inf-filter, a sup-filter, or a strong filter (cfr. below). We will then show how to obtain such properties when the object space on which the operators δ and ε act is the family of parts of a Euclidean or digital space, and also when they are translation-invariant operators on a Minkowski lattice (such as the one of grey-level functions).

Throughout this section we assume as in Section 2 that the object space is a modular lattice $(\mathscr{L}, \leq)$. Note that we do not assume that $\mathscr{L}$ is complete. We recall a few definitions from [15,13]:

Definition 4.1 Let $\mathscr{L}$ be a lattice and $\psi : \mathscr{L} \to \mathscr{L}$ an operator on $\mathscr{L}$. We say that:

(i) ψ is an *inf-overfilter* if ψ is increasing and $\psi = \psi(\mathbf{id} \wedge \psi)$.
(ii) ψ is a *sup-underfilter* if ψ is increasing and $\psi = \psi(\mathbf{id} \vee \psi)$.
(iii) ψ is a *filter* if ψ is increasing and idempotent.
(iv) ψ is an *inf-filter* if ψ is an idempotent inf-overfilter.
(v) ψ is a *sup-filter* if ψ is an idempotent sup-underfilter.
(vi) ψ is a *strong filter* if ψ is both an inf-overfilter and a sup-underfilter.

The following properties are proved in [15,13]:

Lemma 4.2 *If ψ is an inf-overfilter, then $\mathbf{id} \wedge \psi$ is idempotent and $\psi^2 \geq \psi$. If ψ is a sup-underfilter, then $\mathbf{id} \vee \psi$ is idempotent and $\psi^2 \leq \psi$. If ψ is a strong filter, then ψ, $\mathbf{id} \wedge \psi$, and $\mathbf{id} \vee \psi$ are idempotent.*

We can now consider what this means for the operator $\varepsilon \vee \mathbf{id} \wedge \delta$:

Proposition 4.3 *Let ε, δ be two increasing operators on the modular lattice $\mathscr{L}$, such that $\varepsilon \leq \delta$, and let $\psi = \varepsilon \vee \mathbf{id} \wedge \delta$.*

(i) *ψ is an inf-overfilter if and only if $\mathbf{id} \wedge \delta$ is idempotent and $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$.*
(ii) *If $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent and $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$, then ψ is an inf-filter.*
(iii) *ψ is a sup-underfilter if and only if $\varepsilon \vee \mathbf{id}$ is idempotent and $\psi = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id})$.*
(iv) *If $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent and $\psi = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id})$, then ψ is a sup-filter.*
(v) *ψ is a strong filter if and only if $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent and $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta) = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id})$.*

Proof. (i) By Lemma 2.2 (ii), $\mathbf{id} \wedge \psi = \mathbf{id} \wedge \delta$. If $\mathbf{id} \wedge \delta$ is idempotent, then this and Lemma 2.2 (v) give

$$\psi(\mathbf{id} \wedge \psi) = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta). \tag{4.3}$$

Now, if ψ is an inf-overfilter, then $\mathbf{id} \wedge \psi = \mathbf{id} \wedge \delta$ is idempotent by Lemma 4.2, and as $\psi = \psi(\mathbf{id} \wedge \psi)$, (4.3) gives $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$. Conversely, if $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$ and $\mathbf{id} \wedge \delta = \mathbf{id} \wedge \psi$ is idempotent, then (4.3) gives $\psi(\mathbf{id} \wedge \psi) = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta) = \psi$, and ψ is an inf-overfilter.

(ii) By (i), ψ is an inf-overfilter. Since $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are filters satisfying $\mathbf{id} \wedge \delta \leq \varepsilon \vee \mathbf{id}$, by [15, Criterion 4.6], $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$ is idempotent. It is thus an inf-filter.

(iii) and (iv) are proved in the same way as (i) and (ii) (using item (iv) of Lemma 2.2), or follow by duality. Finally (v) is just the combination of (i) and (iii). ∎

We will now give sufficient conditions for having the decompositions (4.1) and (4.2) of $\varepsilon \vee \mathbf{id} \wedge \delta$. Let us say that the operator ε is *meet-distributive* if $\varepsilon(X \wedge Y) = \varepsilon(X) \wedge \varepsilon(Y)$ for all $X, Y \in \mathscr{L}$; similarly, let us say that the operator δ is *join-distributive* if $\delta(X \vee Y) = \delta(X) \vee \delta(Y)$ for all $X, Y \in \mathscr{L}$.

Lemma 4.4 *Let ε and δ be operators on the modular lattice $\mathscr{L}$, such that $\varepsilon \leq \delta$, and let $\psi = \varepsilon \vee \mathbf{id} \wedge \delta$.*

(i) *If $\varepsilon \leq \varepsilon\delta$ and ε is meet-distributive, then $\psi = (\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta)$.*
(ii) *If $\delta\varepsilon \leq \delta$ and δ is join-distributive, then $\psi = (\mathbf{id} \wedge \delta)(\varepsilon \vee \mathbf{id})$.*

Proof. (i) Since ε is meet-distributive, we have $\varepsilon(\mathbf{id} \wedge \delta) = \varepsilon\mathbf{id} \wedge \varepsilon\delta = \varepsilon \wedge \varepsilon\delta$; now if $\varepsilon \leq \varepsilon\delta$, then this gives $\varepsilon(\mathbf{id} \wedge \delta) = \varepsilon$, and so we get

$$(\varepsilon \vee \mathbf{id})(\mathbf{id} \wedge \delta) = \varepsilon(\mathbf{id} \wedge \delta) \vee \mathbf{id}(\mathbf{id} \wedge \delta) = \varepsilon \vee (\mathbf{id} \wedge \delta) = \psi.$$

(ii) is proved in the same way, or follows by duality. ∎

Corollary 4.5 *Let ε and δ be operators on the modular lattice $\mathscr{L}$, such that ε is meet-distributive, δ is join-distributive, $\varepsilon \leq \delta$, and let $\psi = \varepsilon \vee \mathbf{id} \wedge \delta$.*

(i) *If $\varepsilon \leq \varepsilon\delta$ and $\mathbf{id} \wedge \delta$ is idempotent, then ψ is an inf-overfilter.*
(ii) *If $\varepsilon \leq \varepsilon\delta$ and $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent, then ψ is an inf-filter.*
(iii) *If $\delta\varepsilon \leq \delta$ and $\varepsilon \vee \mathbf{id}$ is idempotent, then ψ is a sup-underfilter.*
(iv) *If $\delta\varepsilon \leq \delta$ and $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent, then ψ is sup-filter.*
(v) *If $\varepsilon \leq \varepsilon\delta$, $\delta\varepsilon \leq \delta$, and $\mathbf{id} \wedge \delta$ and $\varepsilon \vee \mathbf{id}$ are idempotent, then ψ is a strong filter.*

Proof. We combine Proposition 4.3 and Lemma 4.4: items (i) and (ii) of Proposition 4.3 with item (i) of Lemma 4.4, items (iii) and (iv) of Proposition 4.3 with item (ii) of Lemma 4.4, and item (v) of Proposition 4.3 with both items (i) and (ii) of Lemma 4.4, and we get then items (i, ii, iii, iv, v) respectively of the present statement. ∎

We will study equivalent forms of the conditions $\varepsilon \leq \varepsilon\delta$ and $\delta\varepsilon \leq \delta$ of Lemma 4.4.

Let us first recall from [7] that two operators ε and δ form an *adjunction* (ε, δ) if and only if for every $X, Y \in \mathscr{L}$ we have $\delta(X) \leq Y \iff X \leq \varepsilon(Y)$. In [7, Proposition 2.5] we showed that when $\mathscr{L}$ is a complete lattice, in every adjunction (ε, δ), δ distributes the (infinite) supremum operation (and is called a *dilation*), while ε distributes the (infinite) infimum operation (and is called an *erosion*); in the general case where $\mathscr{L}$ is not necessarily complete, the same argument as in [7, Proposition 2.5] shows that δ is join-distributive while ε is meet-distributive, and that both are increasing.

We will use the following general result, which has many useful consequences in addition to our present problem:

Proposition 4.6 *Let $(\varepsilon_a, \delta_a)$ and $(\varepsilon_b, \delta_b)$ be two adjunctions, and η, θ two increasing operators. Then*

$$\eta\varepsilon_a \leq \varepsilon_b\theta \iff \delta_b\eta \leq \theta\delta_a. \tag{4.4}$$

Proof. The adjunction $(\varepsilon_a, \delta_a)$ gives $\varepsilon_a\delta_a \geq \mathbf{id}$ and $\delta_a\varepsilon_a \leq \mathbf{id}$ (see [7, Proposition 2.6]). If $\eta\varepsilon_a \leq \varepsilon_b\theta$, since $\varepsilon_a\delta_a \geq \mathbf{id}$, every $X \in \mathscr{L}$ gives

$$\eta(X) \leq \eta\varepsilon_a\delta_a(X) \leq \varepsilon_b\theta\delta_a(X),$$

and the adjunction $(\varepsilon_b, \delta_b)$ implies then that $\delta_b\eta(X) \leq \theta\delta_a(X)$; thus $\delta_b\eta \leq \theta\delta_a$. Conversely, if $\delta_b\eta \leq \theta\delta_a$, since $\delta_a\varepsilon_a \leq \mathbf{id}$, every $X \in \mathscr{L}$ gives

$$\delta_b\eta\varepsilon_a(X) \leq \theta\delta_a\varepsilon_a(X) \leq \theta(X),$$

and the adjunction $(\varepsilon_b, \delta_b)$ implies then that $\eta\varepsilon_a(X) \leq \varepsilon_b\theta(X)$; thus $\eta\varepsilon_a \leq \varepsilon_b\theta$. ∎

This result has many interesting particular cases, which we will present here. We consider first some inequalities used in the previous section. We have twc adjunctions $(\varepsilon_0, \delta_0)$ and $(\varepsilon_1, \delta_1)$, and we consider the annular filter $\varepsilon \vee \mathbf{id} \wedge \delta$ foı $\varepsilon = \varepsilon_0$ and $\delta = \delta_1$. The first requirement is that $\varepsilon \leq \delta$, in other words, $\varepsilon_0 \leq \delta_1$. Using Proposition 4.6 with $(\varepsilon_a, \delta_a) = (\varepsilon_0, \delta_0)$, $(\varepsilon_b, \delta_b) = (\mathbf{id}, \mathbf{id})$, $\eta = \mathbf{id}$, and $\theta = \delta_1$, we obtain the equivalence

$$\varepsilon_0 \leq \delta_1 \iff \mathbf{id} \leq \delta_1\delta_0. \tag{4.5}$$

In order to obtain the idempotence of $\varepsilon \vee \mathbf{id} \wedge \delta$, we considered the conditions $\varepsilon \leq \delta(\mathbf{id} \wedge \delta)$ and $\varepsilon(\varepsilon \vee \mathbf{id}) \leq \delta$ (cfr. Corollary 2.4). Taking (4.4) with $(\varepsilon_a, \delta_a) = (\varepsilon_0, \delta_0)$, $(\varepsilon_b, \delta_b) = (\mathbf{id}, \mathbf{id})$, $\eta = \mathbf{id}$, and $\theta = \delta_1(\mathbf{id} \wedge \delta_1)$, we get

$$\varepsilon_0 \leq \delta_1(\mathbf{id} \wedge \delta_1) \iff \mathbf{id} \leq \delta_1(\mathbf{id} \wedge \delta_1)\delta_0. \tag{4.6}$$

Interchanging the two sides of (4.4) with $(\varepsilon_a, \delta_a) = (\varepsilon_1, \delta_1)$, $(\varepsilon_b, \delta_b) = (\mathbf{id}, \mathbf{id})$, $\eta = \varepsilon_0(\varepsilon_0 \vee \mathbf{id})$, and $\theta = \mathbf{id}$, gives

$$\varepsilon_0(\varepsilon_0 \vee \mathbf{id}) \leq \delta_1 \iff \varepsilon_0(\varepsilon_0 \vee \mathbf{id})\varepsilon_1 \leq \mathbf{id}. \tag{4.7}$$

In the case where $\mathscr{L}$ is a Minkowski lattice, taking for δ_1 the dilation δ_A : $X \mapsto X \oplus A$ by $A \in \mathscr{L}$ and for ε_0 the erosion $\varepsilon_B : X \mapsto X \ominus B$ by $B \in \mathscr{L}$, then

$\delta_A \delta_B = \delta_{B \oplus A}$, the dilation by $B \oplus A$, while **id** is the dilation by the "origin" o; thus combining (4.5) with the isomorphism $Z \mapsto \delta_Z$ between $\mathscr{L}$ and the lattice of **T**-invariant dilations (see [7, Theorem 3.8]), we get:

$$\varepsilon_B \leq \delta_A \iff \mathbf{id} \leq \delta_A \delta_B \iff o \leq B \oplus A. \tag{4.8}$$

This is a particular case of a result due to Van Droogenbroeck (see [2, Theorem 1]), namely that given the erosions $\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D$ and dilations $\delta_A, \delta_B, \delta_C, \delta_D$ by structuring elements $A, B, C, D \in \mathscr{L}$, we have the equivalences

$$\delta_B \varepsilon_A \leq \varepsilon_D \delta_C \iff \delta_D \delta_B \leq \delta_C \delta_A \iff B \oplus D \leq A \oplus C. \tag{4.9}$$

The first equivalence springs from (4.4) with $(\varepsilon_a, \delta_a) = (\varepsilon_A, \delta_A)$, $(\varepsilon_b, \delta_b) = (\varepsilon_D, \delta_D)$, $\eta = \delta_B$, and $\theta = \delta_C$, while the second one is due to the isomorphism $Z \mapsto \delta_Z$.

Let us now give equivalent forms of the conditions $\varepsilon \leq \varepsilon\delta$ and $\delta\varepsilon \leq \delta$ of Lemma 4.4 using Proposition 4.6. Let $\varepsilon = \varepsilon_0$ and $\delta = \delta_1$, where $(\varepsilon_0, \delta_0)$ and $(\varepsilon_1, \delta_1)$ are adjunctions. Taking $(\varepsilon_a, \delta_a) = (\varepsilon_b, \delta_b) = (\varepsilon_0, \delta_0)$, $\eta = \mathbf{id}$, and $\theta = \delta_1$, (4.4) gives

$$\varepsilon_0 \leq \varepsilon_0 \delta_1 \iff \delta_0 \leq \delta_1 \delta_0. \tag{4.10}$$

Taking next $(\varepsilon_a, \delta_a) = (\varepsilon_0, \delta_0)$, $(\varepsilon_b, \delta_b) = (\mathbf{id}, \mathbf{id})$, and $\eta = \theta = \delta_1$, we get

$$\delta_1 \varepsilon_0 \leq \delta_1 \iff \delta_1 \leq \delta_1 \delta_0. \tag{4.11}$$

When $\mathscr{L}$ is a Minkowski lattice, taking for δ_1 the dilation $\delta_A : X \mapsto X \oplus A$ and for ε_0 the erosion $\varepsilon_B : X \mapsto X \ominus B$, where $A, B \in \mathscr{L}$, then combining the above two equations with the isomorphism $Z \mapsto \delta_Z$, we get:

$$\varepsilon_B \leq \varepsilon_B \delta_A \iff \delta_B \leq \delta_A \delta_B \iff B \leq A \oplus B, \tag{4.12}$$

$$\delta_A \varepsilon_B \leq \delta_A \iff \delta_A \leq \delta_A \delta_B \iff A \leq A \oplus B. \tag{4.13}$$

Suppose now that $\mathscr{L} = \mathscr{P}(E)$, the family of parts of a set E, and that the adjunctions $(\varepsilon_0, \delta_0)$ and $(\varepsilon_1, \delta_1)$ arise from two adjacency relations $\overset{0}{\sim}$ and $\overset{1}{\sim}$ (see [8], and the end of Section 2). Here $\mathbf{id} \wedge \delta_1$ and $\varepsilon_0 \vee \mathbf{id}$ are idempotent (see the discussion at the end of Section 2, especially the paragraphs between (2.10) and (2.11)). By (4.5), we have $\varepsilon_0 \leq \delta_1 \Leftrightarrow \mathbf{id} \leq \delta_1 \delta_0$, and the latter means that for every $x \in E$ there is some $y \in E$ such that $x \overset{0}{\sim} y \overset{1}{\sim} x$: this is [8, Assumption 5.1]. By (4.10) the condition $\varepsilon_0 \leq \varepsilon_0 \delta_1$ is equivalent to $\delta_0 \leq \delta_1 \delta_0$, and the latter means that for every $x, y \in E$

$$x \overset{0}{\sim} y \implies \exists z \in E, \quad x \overset{0}{\sim} z \overset{1}{\sim} y;$$

if it is satisfied, then $\varepsilon_0 \vee \mathbf{id} \wedge \delta_1$ is an inf-filter by Corollary 4.5 (ii). By (4.11) the condition $\delta_1 \varepsilon_0 \leq \delta_1$ is equivalent to $\delta_1 \leq \delta_1 \delta_0$, and the latter means that for every $x, y \in E$,

$$x \overset{1}{\sim} y \implies \exists z \in E, \quad x \overset{0}{\sim} z \overset{1}{\sim} y;$$

if it is satisfied, then $\varepsilon_0 \vee \mathbf{id} \wedge \delta_1$ is a sup-filter by Corollary 4.5 (iv). If both conditions are satisfied, then $\varepsilon_0 \vee \mathbf{id} \wedge \delta_1$ is a strong filter. This result was announced without proof in [8] (see Proposition 5.10 there).

In the case of a Minkowski lattice, combining Corollary 3.11, Lemma 4.4, and Corollary 4.5 with equations (4.8), (4.12), and (4.13), we get the following:

Proposition 4.7 *Let $(\mathscr{L}, \ell, \mathbf{T})$ be a Minkowski lattice such that $\mathscr{L}$ is ISD and IID. Let $A, B \in \mathscr{L}$ such that $o \leq A \oplus B$. Then $\varepsilon_B \leq \delta_A$. Furthermore, for $\psi = \varepsilon_B \vee \mathbf{id} \wedge \delta_A$, we have:*

- (i) *If $B \leq A \oplus B$ and A is symmetric, then $\mathbf{id} \wedge \delta_A$ is idempotent, $\psi = (\varepsilon_B \vee \mathbf{id})(\mathbf{id} \wedge \delta_A)$, and ψ is an inf-overfilter.*
- (ii) *If $B \leq A \oplus B$ and A and B are symmetric, then $\mathbf{id} \wedge \delta_A$ and $\varepsilon_B \vee \mathbf{id}$ are idempotent, $\psi = (\varepsilon_B \vee \mathbf{id})(\mathbf{id} \wedge \delta_A)$, and ψ is an inf-filter.*
- (iii) *If $A \leq A \oplus B$ and B is symmetric, then $\varepsilon_B \vee \mathbf{id}$ is idempotent, $\psi = (\mathbf{id} \wedge \delta_A)(\varepsilon_B \vee \mathbf{id})$, and ψ is a sup-underfilter.*
- (iv) *If $A \leq A \oplus B$ and A and B are symmetric, then $\mathbf{id} \wedge \delta_A$ and $\varepsilon_B \vee \mathbf{id}$ are idempotent, $\psi = (\mathbf{id} \wedge \delta_A)(\varepsilon_B \vee \mathbf{id})$, and ψ is sup-filter.*
- (v) *If $A \vee B \leq A \oplus B$ and A and B are symmetric, then $\mathbf{id} \wedge \delta_A$ and $\varepsilon_B \vee \mathbf{id}$ are idempotent, $\psi = (\mathbf{id} \wedge \delta_A)(\varepsilon_B \vee \mathbf{id}) = (\varepsilon_B \vee \mathbf{id})(\mathbf{id} \wedge \delta_A)$, and ψ is a strong filter.*

Consider now the particular case where $A = B > O$. If A is symmetric, we have some $a \in \ell(A)$ with $-a \in \ell(A)$, and hence $o = a + (-a) \leq A \oplus A$. Therefore Proposition 4.7 reduces to the following:

Corollary 4.8 *Let $(\mathscr{L}, \ell, \mathbf{T})$ be a Minkowski lattice such that $\mathscr{L}$ is ISD and IID. Let $A \in \mathscr{L}$ such that $O < A \leq A \oplus A$ and A is symmetric. Then $\varepsilon_A \leq \delta_A$, $\mathbf{id} \wedge \delta_A$ and $\varepsilon_A \vee \mathbf{id}$ are idempotent,*

$$\varepsilon_A \vee \mathbf{id} \wedge \delta_A = (\mathbf{id} \wedge \delta_A)(\varepsilon_A \vee \mathbf{id}) = (\varepsilon_A \vee \mathbf{id})(\mathbf{id} \wedge \delta_A),$$

and the latter operator is a strong filter.

When $\mathscr{L} = \mathscr{P}(E)$, where $E = \mathbb{Z}^d$ or $\mathbb{R}^d$, the condition "$o \leq A \oplus B$" means here that $A \oplus B$ contains the origin, and this is equivalent to $A \cap \check{B} \neq \emptyset$; for A and B symmetric, this means that $A \cap B \neq \emptyset$. An example where $A \cup B \subseteq A \oplus B$ is given for A and B being respectively the 8-neighbourhood and the 4-neighbourhood of the origin (excluding that origin, see Fig. 7.); thus $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$ is a strong filter. Note that we have then also $A \subseteq A \oplus A$, and $\varepsilon_A \vee \mathbf{id} \wedge \delta_A$ is a strong filter.

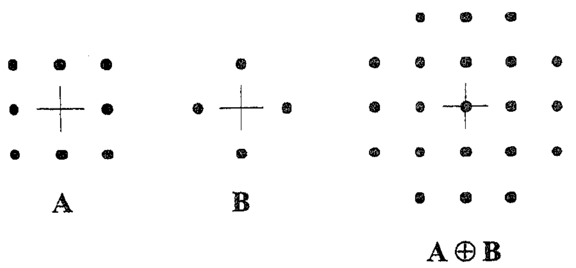

Fig. 7. In $E = \mathbb{Z}^2$, we choose for A and B the 8-neighbourhood and 4-neighbourhood of the origin respectively (the cross indicates the position of the origin, which is excluded from both A and B). Clearly A and B are symmetric and $A \oplus B$ contains $A \cup B$, so that the annular operator $\varepsilon_B \vee \mathbf{id} \wedge \delta_A$ is a strong filter, and $\varepsilon_B \vee \mathbf{id} \wedge \delta_A = (\varepsilon_B \vee \mathbf{id})(\mathbf{id} \wedge \delta_A) = (\mathbf{id} \wedge \delta_A)(\varepsilon_B \vee \mathbf{id})$

Let us apply Proposition 4.7 and Corollary 4.8 when $\mathcal{L} = \mathrm{Fun}(E, \mathcal{T})$, i.e., the lattice of grey-level functions $E \to \mathcal{T}$, where $E = \mathbb{Z}^d$ or $\mathbb{R}^d$ and $\mathcal{T} = \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ or $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$. As said above, a structuring function $A \in \mathcal{L}$ is symmetric if and only the support $supp(A)$ of A (that is, the set of points $p \in E$ such that $A(p) > -\infty$) is a symmetric subset of E, and for every $h \in supp(A)$ we have $A(h) + A(-h) \geq 0$. The condition $o \leq A \oplus B$ translates as follows:

$$\sup_{h \in supp(A) \cap supp(B)} A(h) + B(-h) \geq 0;$$

A sufficient condition is having some $h \in supp(A) \cap supp(B)$ with $A(h) + B(-h) \geq 0$. The condition $A \vee B \leq A \oplus B$ means here that

$$\forall x \in supp(A) \cup supp(B),$$
$$\max\{A(x), B(x)\} \leq \sup\{A(x-h) + B(h) \mid h \in supp(B) \text{ and } x - h \in supp(A)\}.$$

This inequality requires in particular that the support of the smaller function is contained in that of the larger one, that is:

$$supp(A) \cup supp(B) \subseteq supp(A) \oplus supp(B).$$

In order to illustrate Corollary 4.8, we take a structuring function A whose support is a symmetric subset S of E such that $S \subseteq S \oplus S$, and such that for every $x \in S$, $A(x) = \|x\|$, where $\|x\|$ denotes a norm (L^1, L^2, or L^∞). Then A is symmetric; now for every $x \in S$, there is $h \in S$ with $x - h \in S$, we have $\|x\| \leq \|x - h\| + \|h\|$, and this shows that $A \leq A \oplus A$. Thus $\varepsilon_A \vee \mathbf{id} \wedge \delta_A$ is a strong filter. The effect of this filter on an image I is to bring the grey-level $I(p)$ of a point p to the interval

$$\big[J_0(p), J_1(p)\big], \quad \text{where} \quad J_0(p) = \inf_{h \in S}\big(I(p+h) - \|h\|\big) \quad \text{and}$$
$$J_1(p) = \sup_{h \in S}\big(I(p+h) + \|h\|\big),$$

in other words, it transforms I into the new image I' defined by

$$I'(p) = \begin{cases} J_0(p) \text{ if } I(p) < J_0(p), \\ I(p) \text{ if } J_0(p) \leq I(p) \leq J_1(p), \\ J_1(p) \text{ if } J_1(p) < I(p). \end{cases} \tag{4.14}$$

Here noisy isolated extrema (either dark or bright) are reduced to a value comparable to their surrounding, in a sense that we will explain now. Given $p \in E$, we have $I(p) \leq J_1(p) = \sup_{h \in S}\big(I(p+h) + \|h\|\big)$ if and only if for every real $\varepsilon > 0$ there is some $h \in S$ such that $I(p) - \varepsilon < I(p+h) + \|h\|$, that is $\big(I(p+h) - I(p)\big)/\|h\| > -1 - \varepsilon/\|h\|$, in other words it is equivalent to

$$\sup_{h \in S} \frac{I(p+h) - I(p)}{\|h\|} \geq -1. \tag{4.15}$$

Similarly, $I(p) \geq J_0(p) = \inf_{h \in S}\big(I(p+h) - \|h\|\big)$ if and only if for every real $\varepsilon > 0$ there is some $h \in S$ such that $I(p) + \varepsilon > I(p+h) - \|h\|$, that is $\big(I(p+h) - I(p)\big)/\|h\| < 1 + \varepsilon/\|h\|$, in other words it is equivalent to

$$\inf_{h \in S} \frac{I(p+h) - I(p)}{\|h\|} \leq 1. \tag{4.16}$$

Thus from (4.14) we get that $I'(p) = I(p)$ (the image does not change grey-level at p) if and only if both (4.15,4.16) hold. Now since the filter is idempotent, I' does not change from applying again the filter $\varepsilon_A \vee \mathbf{id} \wedge \delta_A$; thus by (4.15,4.16) for every point $p \in E$ we have

$$\sup_{h \in S} \frac{I'(p+h) - I'(p)}{\|h\|} \geq -1 \qquad \text{and} \qquad \inf_{h \in S} \frac{I'(p+h) - I'(p)}{\|h\|} \leq 1.$$

This property of I' is weaker than the Lipschitz condition studied in [10] (see also Example 3.4 (c)), which would imply here that

$$\forall h \in S, \qquad -1 \leq \frac{I'(p+h) - I'(p)}{\|h\|} \leq 1.$$

For example, if $E = \mathbb{R}$ and $\mathcal{T} = \overline{\mathbb{R}}$, every monotonic increasing function $I : \mathbb{R} \to \overline{\mathbb{R}}$ will be invariant under the filter $\varepsilon_A \vee \mathbf{id} \wedge \delta_A$, even if its slope does not belong to the interval $[-1, 1]$ required by the Lipschitz condition.

5 Conclusion

We have investigated the idempotence of operators of the form $\varepsilon \vee \mathbf{id} \wedge \delta$ (where $\varepsilon \leq \delta$ and both ε and δ are increasing) on a modular lattice $\mathcal{L}$, in relation to the idempotence of the operators $\varepsilon \vee \mathbf{id}$ and $\mathbf{id} \wedge \delta$. Our motivation, following [8], lies in the application of our theory to a particular branch of image processing, called *mathematical morphology*, where many operations have been formalized in the framework of lattice theory [4]. In this respect, the idempotent operator

id $\wedge\, \delta$ is called an *annular opening*, and it removes isolated light spots in a picture, while the idempotent operator $\varepsilon \vee$ **id** is called an *annular closing*, and it removes isolated dark spots in a picture; the idempotent compound operator $\varepsilon \vee$ **id** $\wedge\, \delta$, called *annular filter*, combines the behaviour of the above two (removing isolated spots, either dark or light).

Besides general results given in Section 2, there are two instances where the idempotence of these operators can be analysed:

(1°) When $\mathscr{L} = \mathscr{P}(E)$, the lattice of parts of an arbitrary space E. We dealt with this question in [8], using adjacency relations to characterize the "isolation" of points in background or foreground conditions.

(2°) When $\mathscr{L}$ is a Minkowski lattice, in other words, when it is a complete lattice having a sup-generating family ℓ and an abelian group $\mathbf{T}$ of automorphisms, such that $\mathbf{T}$ preserves ℓ and acts transitively on it. This was studied in Section 3. Here ε and δ are assumed to be an erosion and a dilation arising through Minkowski operations on $\mathscr{L}$, and they are $\mathbf{T}$-invariant. This framework applies in particular when $\mathscr{L} = \mathrm{Fun}(E, \mathscr{T})$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ and $\mathscr{T} = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$; here $\mathbf{T}$ consists of translations combining a spatial component in E and a numerical component in $\mathscr{T} \setminus \{\pm\infty\}$, while ℓ consists of "impulse" functions. The practical application of this theory is the processing of grey-level pictures.

It might be possible to extend both (1°) and (2°) to the situation of a complete lattice $\mathscr{L}$ with a sup-generating family ℓ on which we would define adjacency relations, without any constraint of invariance under a given group of automorphisms. Possible fields of applications of such a theory include the processing of pictures for which the group of "translations" is either non-existent or non-transitive on the generators in ℓ (which represent coloured points); indeed this is the case when the space E is a bounded part of $\mathbb{Z}^2$, or when the grey-level set $\mathscr{T}$ is a bounded interval in $\mathbb{Z}$, or else with colour pictures having colours in a 3-dimensional RGB space, for which translations cannot permute transitively the colours. We have not (yet) studied this possible extension of our theory.

The second problem that we have studied is whether the annular filter $\varepsilon \vee$ **id** $\wedge\, \delta$ can be obtained by composing the annular closing $\varepsilon \vee$ **id** and opening **id** $\wedge\, \delta$. As we saw in Section 4, this question is related to ψ being an inf-overfilter, a sup-underfilter, or a strong filter (three notions coming again from mathematical morphology). Surprisingly, we can answer these questions in an arbitrary modular lattice, without recourse to adjacency relations (for sets) or translation-invariance and Minkowski lattices.

Another question concerns the practical applications of annular filters in image processing. In [8] we gave examples of the use of annular filters for removing "salt-and-pepper" impulsive noise in grey-level images; we chose there for A and B two flat structuring functions (in other words, their value is constantly zero on their support), because in this case the behaviour of the filter can be described in terms of a filter for sets acting on each level set

$I_t = \{p \in E \mid I(p) \geq t\}$, where t ranges over the grey-level set $\mathcal{T}$ [3,14]. We have not yet investigated specific applications of annular filters using non-flat structuring functions. Note however that when the structuring functions A and B have positive values, isolated grey-level values (corresponding to light or dark spots in the image) are not completely removed, but rather modified to a level closer to their surrounding (see for example Fig. 6). Hence such filters with non-flat structuring functions would not be interesting for removing impulsive noise. However they could perhaps find applications in feature detection: the arithmetic difference between the original image and the filtered one could reveal some types of local grey-level patterns.

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