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A LAW OF LARGE NUMBERS FOR RANDOM WALKS IN RANDOM ENVIRONMENT

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We derive a law of large numbers for a class of multidimensional random walks in random environment satisfying a condition which first appeared in the work of Kalikow. The approach is based on the existence of a renewal structure under an assumption of "transience in the direction l." This extends, to a multidimensional context, previous work of Kesten. Our results also enable proving the convergence of the law of the environment viewed from the particle toward a limiting distribution.

0. Introduction. The main purpose of this article is the derivation of a law of large numbers for a class of random walks in random environment. The law of large numbers for one-dimensional random walks in random environment is well known and goes back to Solomon [10]. In contrast, in higher dimension, asymptotic properties of random walks in random environment in general, and the law of large numbers in particular, are rather poorly understood. In fact, embarrassingly simple questions are yet unanswered. One reason for this situation is the truly nonreversible character of the model and the absence of obviously applicable ergodic theorems. Some of the known results in this higher dimensional situation can, for instance, be found in [2], [5], [11], and [12].

Let us now describe the model more precisely. The random environment is given by i.i.d. $(2d)$-dimensional vectors $(\omega(x,e))_{e\in\mathbb{Z}^d,|e|=1}$, $x\in\mathbb{Z}^d$, with nonnegative components adding to 1 and common distribution μ. Throughout this work we assume $d\geq 1$, and the following ellipticity condition:

(0.1) There exists $\kappa(\mu)\in(0,1)$ such that μ is supported by the set $\mathscr{P}_\kappa$ of $(2d)$-vectors $p(e)$, $e\in\mathbb{Z}^d$, $|e|=1$, with $p(e)\in[\kappa,1]$ for all e, and $\sum_e p(e)=1$.

More precisely, the random variables $\omega(x,e)$, $x\in\mathbb{Z}^d$, $|e|=1$, will be the canonical coordinates on the product space $\Omega=\mathscr{P}_\kappa^{\mathbb{Z}^d}$ endowed with the canonical product σ-algebra and the product measure $\mathbb{P}=\mu^{\otimes\mathbb{Z}^d}$. The random walk in the random environment ω is then the canonical Markov chain $(X_n)_{n\geq 0}$, on $(\mathbb{Z}^d)^{\mathbb{N}}$, with state space $\mathbb{Z}^d$ and "quenched law" $P_{x,\omega}$ starting from x, for which

$$
\begin{aligned}
&P_{x,\omega}[X_{n+1}=X_n+e\mid X_0,\ldots,X_n]\\
&\overset{P_{x,\omega}\text{-a.s.}}{=}\omega(X_n,e),\ n\geq 0,\ e\in\mathbb{Z}^d,\ |e|=1,\ P_{x,\omega}[X_0=x]=1.
\end{aligned}\tag{0.2}
$$

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One also defines the "annealed laws" P_x, as the semidirect products on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$,

$$P_x = \mathbb{P} \times P_{x,\omega}, \qquad x \in \mathbb{Z}^d . \tag{0.3}$$

We shall need the following notation: when $U \subseteq \mathbb{Z}^d$, T_U and H_U will stand for the respective exit and entrance time of $X_{.}$ in U,

$$T_U = \inf\{n \geq 0,\ X_n \notin U\}, \qquad H_U = \inf\{n \geq 0,\ X_n \in U\} . \tag{0.4}$$

The annealed laws are in a sense simpler to investigate than the quenched laws, however, $(X_n)_{n\geq 0}$ is typically not a Markov chain under the annealed laws. This "defect" is in part remedied by constructing certain auxiliary Markov chains. Following [5], we introduce for $U \subsetneq \mathbb{Z}^d$, a connected subset containing 0, the Markov chain on $U \cup \partial U$ (here $\partial U = \{y \in \mathbb{Z}^d \backslash U, \exists\ x \in U, |y - x| = 1\}$), with transition probability

$$\begin{aligned} &\widehat{P}_U(x, x+e) \\ &\quad = \mathbb{E}\Big[E_{0,\omega}\Big[\sum_0^{T_U} 1\{X_n = x\}\Big]\, \omega(x,e)\Big] \Big/ \mathbb{E}\Big[E_{0,\omega}\Big[\sum_0^{T_U} 1\{X_n = x\}\Big]\Big], \end{aligned} \tag{0.5}$$

when $x \in U$ and $e \in \mathbb{Z}^d$, with $|e| = 1$, (the expectations are easily seen to be finite and positive thanks to (0.1) and the connectedness of U), $\widehat{P}_U(x, x) = 1$, when $x \in \partial U$.

The canonical law of this Markov chain starting from $x \in U \cup \partial U$ is denoted by $\widehat{P}_{x,U}$. The interest of these objects stems from the following fact (cf. Proposition 1 of [5]):

(0.6) If $\widehat{P}_{0,U}[T_U < \infty] = 1$, then $P_0[T_U < \infty] = 1$, and X_{T_U} has the same distribution under P_0 and $\widehat{P}_{0,U}$.

We can now formulate what we call Kalikow's condition relative to $l \in \mathbb{R}^d \backslash \{0\}$, namely:

(0.7) There exists $\varepsilon > 0$, such that $\inf_{U,\, x \in U} \sum_{|e|=1} l.e\ \widehat{P}_U(x, x+e) \geq \varepsilon$;

here U runs over all possible connected strict subsets of $\mathbb{Z}^d$, which contain 0. An important result of [5] is that when (0.7) holds,

$$P_0\text{-a.s.}, \qquad \lim_n l.X_n = +\infty . \tag{0.8}$$

The condition (0.7) is hard to check directly, but a more concrete sufficient condition for (0.7) is known; compare [5], pages 759 and 760, and (2.36) below. In particular, as shown at the end of Section 2, the nonnestling walks of [12] [cf. (2.40)], or the random walks in random environment considered in [11], with (0.1) assumed (see above and Remark 2.5) do satisfy (0.7) for a suitable l. Interestingly enough, in the one-dimensional situation, the fulfillment of (0.7) for some nonzero l characterizes the existence of a nondegenerate asymptotic velocity for the random walk in random environment; see Remark 2.5.

One of the main goals of the present article is to show (cf. Theorem 2.3 below) that when (0.7) holds,

$\frac{X_n}{n}$ converges P_0-almost surely to a deterministic nondegenerate velocity.

The principal tool for deriving such a law of large numbers turns out to be the existence of a renewal structure for X. (cf. Section 1). It generalizes to a multidimensional setting the renewal structure which appears in a one-dimensional context, in [6] and implicitly in [7]. An important role is played by a certain renewal time τ_1 [cf. (1.13)]. In the present multidimensional setting, there is a certain arbitrariness in the definition of τ_1. Incidentally, the most straightforward extension of the one-dimensional definition [which roughly corresponds to replacing D in (1.13), with the first time of going strictly below level M_k+a, in the direction l) leads to difficulties, due to the "inhomogeneity" of the discrete boundary of a half-space with a general normal direction.

The main step in the derivation of the strong law of large numbers is then to prove a certain integrability of the variable τ_1 (cf. Theorem 2.3). The renewal time τ_1 is rather complicated, and the extraction of information on its tail is not straightforward. In the one-dimensional setting, the key fact used in [6] and [7] is an identity in law with certain variables expressed in terms of a branching process with immigration in a random environment. In the present multidimensional context, we follow a quite different route. The strategy is roughly to extract information on τ_1, by investigating the "easier variable" $l.X_{\tau_1}$. We first prove the integrability of $l.X_{\tau_1}$ and use it to derive the integrability of τ_1 (cf. Theorem 2.3).

The above results can be applied to study the asymptotic behavior of the law of the environment viewed from the particle

$$\overline{\omega}_n = t_{X_n}\omega = \omega(X_n + \cdot), \qquad n \geq 0\,, \tag{0.9}$$

where t_x, $x \in \mathbb{Z}^d$ denotes the canonical shift on Ω. Under P_0, $(\overline{\omega}_n)_{n\geq 0}$, is a Markov chain with state space Ω, initial distribution $\mathbb{P}$ and transition kernel

$$R(\overline{\omega}, \overline{\omega}') = \begin{cases} \overline{\omega}(0,e), & \text{if } \overline{\omega}' = t_e\overline{\omega}, \text{ for } |e| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{0.10}$$

We show in Theorem 3.1 that under (0.7) the law of $\overline{\omega}_n$ under P_0 converges to an invariant distribution of R. It is an interesting question, left untouched in the present article, to determine whether this limiting law is absolutely continuous with respect to $\mathbb{P}$; see the end of Section 3. It is of course also an interesting question to understand how typical (or untypical) condition (0.7) is, within the class of random walks in random environment with ballistic behavior.

Let us describe how the present article is organized. Section 1 develops in the context of "transience in the direction l," a renewal structure for $X_.$ under P_0. Section 2 derives the strong law of large numbers under (0.7) and describes some examples where (0.7) holds. Section 3 applies the results of the previous sections to the analysis of the asymptotic behavior of the law of the environment viewed from the particle. Finally, let us mention that at the time of preparing the final revised version of this article, we learned of the existence of unpublished handwritten notes of H. Kesten, dating back to 1986, about results by S. Kalikow similar to our Theorem 2.3.

1. The renewal structure. The object of this section is to develop a certain renewal structure, under an assumption of "transience in the direction l." We recall that throughout this work we assume (0.1). We still need some notations; we denote by $(\theta_n)_{n\geq 0}$, the canonical shift on $(\mathbb{Z}^d)^{\mathbb{N}}$, and by $\mathcal{F}_n$, $n \geq 0$, the canonical filtration of $X_.$, that is, $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$, for $n \geq 0$. For $l \in \mathbb{R}^d \backslash \{0\}$, we let B_l and C_l, respectively, stand for the events

$$\begin{aligned} B_l &= \{\lim_n \, l.X_n = +\infty\} \cup \{\lim_n \, l.X_n = -\infty\}, \\ C_l &= \{l.X_n \text{ remains of constant sign for large } n\}\,. \end{aligned} \tag{1.1}$$

LEMMA 1.1.

$$\text{For } l \in \mathbb{R}^d \backslash \{0\}, \quad P_0(B_l) = 0 \text{ or } 1\,. \tag{1.2}$$

PROOF. It follows from [5], page 766, that

$$P_0(C_l) = 0 \text{ or } 1\,. \tag{1.3}$$

Let us now prove that for $M > 0$,

$$P_0\text{-a.s., } \{l.X_n \in [0, M], \text{ i.o.}\} \subseteq \{l.X_n < 0, \text{ i.o.}\}\,. \tag{1.4}$$

To this end, with the help of (0.1), choose N large enough and $c > 0$, such that

$$P_{x,\omega}[H_{\{z;\, l.z<0\}} \leq N] \geq c \quad \text{for } \omega \in \Omega \text{ and } x \in \{z;\, 0 \leq l.z \leq M\}\,. \tag{1.5}$$

One can then define the successive return times to $\{z;\, 0 \leq l.z \leq M\}$

$$\begin{aligned} &V_0 = 0,\ V_1 = H_{\{z;\, 0\leq l.z\leq M\}} \leq \infty \quad \text{and by induction} \\ &V_{k+1} = V_1 \circ \theta_{V_k+N} + V_k + N \leq \infty \quad \text{for } k \geq 1\,. \end{aligned}$$

Introduce for $k \geq 1$, the events G_k and H_k such that

$$1_{G_k} = 1_{\{V_k<\infty\}} \quad \text{and} \quad 1_{H_k} = 1_{\{H_{\{z;\, l.z<0\}}\leq N\}} \circ \theta_{V_k} \tag{1.6}$$

(the last expression is understood as 0 on $\{V_k = \infty\}$). It is plain that $G_k \in \mathcal{F}_{V_k}$ and $H_k \in \mathcal{F}_{V_{k+1}}$. Moreover from (1.5), the strong Markov property and $\mathbb{P}$-integration, we see that

$$P_0[H_k \,|\, \mathcal{F}_{V_k}] \geq c\, 1_{G_k}, \qquad k \geq 1\,. \tag{1.7}$$

From Borel-Cantelli's second lemma (cf. [3], page 207),

$$P_0\text{-a.s., } \sum_{k\geq 1} 1_{H_k} = \infty \quad \text{on } \left\{ \sum_{k\geq 1} 1_{G_k} = \infty \right\}, \tag{1.8}$$

which readily implies (1.4). To conclude, observe that $B_l \subset C_l$, so that $P_0(B_l) = 0$, if $P_0(C_l) = 0$. On the other hand [see (1.3)], if $P_0(C_l) = 1$, observe in view of (1.4) that for $M > 0$,

$$P_0[\{l.X_n \text{ is positive for large } n\} \cap \{l.X_n \in [-M, M], \text{ i.o.}\}] = 0 .$$

Of course a similar statement holds with "negative" in place of "positive." Thus, for $M > 0$,

$$P_0[C_l \cap \{l.X_n \in [-M, M], \;\; \text{i.o.}\}] = 0 , \tag{1.9}$$

and as a result $P_0(B_l) = 1$. This completes the proof of (1.2). □

We can now define l-transience (for $l \in \mathbb{R}^d \backslash \{0\}$) of the random walk in random environment as the requirement

$$P_0(B_l) = 1 . \tag{1.10}$$

We now consider for the remainder of this section $l \in \mathbb{R}^d \backslash \{0\}$, such that (1.10) holds. For specificity, we assume that

$$P_0(A_l) > 0 \quad \text{with } A_l = \{\lim_n \; l.X_n = +\infty\} ; \tag{1.11}$$

otherwise we simply replace l with $-l$ in the sequel. It is unknown whether simultaneous occurrence of $P_0(A_l) > 0$ and $P_0(A_{-l}) > 0$ is possible (this is an example of an embarrassingly simple yet unanswered question, as alluded to in the Introduction).

The renewal structure relies on the introduction of a suitable random variable τ_1, P_0-a.s. finite on A_l, after which the path $X_.$ does not "backtrack in the direction l." We need some further notations. We consider the $(\mathscr{F}_n)$-stopping times

$$\begin{aligned} T_u &= \inf\{n \geq 0,\; l.X_n \geq u\} \quad \text{for } u \in \mathbb{R} \quad \text{and} \\ D &= \inf\{n \geq 0,\; l.X_n < lX_0\} . \end{aligned} \tag{1.12}$$

We then introduce a constant $a > 0$, the value of which is for the time being immaterial and will only intervene in Sections 1 and 2. We define two sequences of $(\mathscr{F}_n)$-stopping times, S_k, $k \geq 0$ and R_k, $k \geq 1$, and the sequence of successive maxima, M_k, $k \geq 0$,

$$\begin{aligned} &S_0 = 0,\; M_0 = l.X_0, \\ &S_1 = T_{M_0+a} \leq \infty,\; R_1 = D \circ \theta_{S_1} + S_1 \leq \infty , \\ &M_1 = \sup\{l.X_m,\; 0 \leq m \leq R_1\} \leq \infty , \end{aligned} \tag{1.13}$$

and by induction for $k \geq 1$,

$$\begin{aligned} S_{k+1} &= T_{M_k+a} \leq \infty,\; R_{k+1} = D \circ \theta_{S_{k+1}} + S_{k+1} \leq \infty , \\ M_{k+1} &= \sup\{l.X_m,\; 0 \leq m \leq R_{k+1}\} \leq \infty . \end{aligned}$$

Of course we have

$$0 = S_0 \leq S_1 \leq R_1 \leq S_2 \leq \cdots \leq \infty \tag{1.14}$$

and the inequalities are strict if the left member is finite. We now introduce

$$K = \inf\{k \geq 1,\ S_k < \infty,\ R_k = \infty\} \leq \infty \quad \text{and} \qquad (1.15)$$
$$\tau_1 = S_K \leq \infty \text{ (with the convention } S_\infty = \infty).$$

Thus the random variable τ_1, when finite, is on the one hand the first time at which $l.X_n$ reaches the level $l.X_{\tau_1}$ and on the other hand such that after τ_1, $l.X_n$ never becomes smaller than $l.X_{\tau_1}$. Now comes the first step in the analysis of the renewal structure.

PROPOSITION 1.2. *Assume* (1.11); *then*

$$P_0(D = \infty) > 0\,, \qquad (1.16)$$

$$P_0\text{-}a.s., \quad A_l = \{K < \infty\} = \{\tau_1 < \infty\}\,. \qquad (1.17)$$

PROOF. We begin with the proof of (1.16). Assume by contradiction that $P_0[D = \infty] = 0$. Then for all $x \in \mathbb{Z}^d$, $P_x[D < \infty] = 1$, and thus

$$\mathbb{P}\text{-a.s., for all } x \in \mathbb{Z}^d, \qquad P_{x,\omega}[D < \infty] = 1\,.$$

Using the strong Markov property, this implies that

$$P_0\text{-a.s.,} \qquad \varliminf_n\ l.X_n \leq 0\,,$$

which contradicts (1.11).

We now turn to the proof of (1.17). The second equality is a tautology, and we need only prove the first equality. Observe that

$$\{K < \infty\} \subseteq \{\varliminf_n\ l.X_n = -\infty\}^c\,,$$

and in view of (1.10),

$$P_0\text{-a.s., } \{K < \infty\} \subseteq \{\lim_n\ l.X_n = +\infty\} = A_l\,. \qquad (1.18)$$

As for the reverse inclusion, observe first that for $k \geq 1$,

$$\begin{aligned} P_0[R_k < \infty] &= \mathbb{E}\Big[E_{0,\omega}\Big[S_k < \infty,\ P_{X_{S_k},\omega}[D < \infty]\Big]\Big] \\ &= \sum_{x \in \mathbb{Z}^d} \mathbb{E}[P_{0,\omega}[S_k < \infty,\ X_{S_k} = x]\ P_{x,\omega}[D < \infty]]\,. \end{aligned} \qquad (1.19)$$

Note that $P_{0,\omega}[S_k < \infty,\ X_{S_k} = x]$ and $P_{x,\omega}[D < \infty]$ are, respectively, $\sigma(\omega(y,\cdot);\ l.y < l.x)$ and $\sigma(\omega(y,\cdot);\ l.y \geq l.x)$-measurable, and thus $\mathbb{P}$-independent. The above expression then equals

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} P_0[S_k < \infty,\ X_{S_k} = x]\ P_0[D < \infty] \\ &\quad = P_0[S_k < \infty]\ P_0[D < \infty] \leq P_0[R_{k-1} < \infty]\ P_0[D < \infty]\,. \end{aligned}$$

Using induction, we thus find

$$P_0[R_k < \infty] \leq P_0[D < \infty]^k, \qquad k \geq 1 \qquad (1.20)$$

(it is not hard to see that the above inequality is in fact an equality when $P_0(A_l) = 1$). As a result we see that

$$\inf\{k \geq 1,\ R_k = \infty\} < \infty, \qquad P_0\text{-a.s.} \tag{1.21}$$

On the other hand,

$$P_0\text{-a.s. on } A_l,\ R_k < \infty \Longrightarrow S_{k+1} < \infty \quad \text{for } k \geq 1\,.$$

Thus in view of (1.21),

$$P_0\text{-a.s. on } A_l,\ K = \inf\{k \geq 1,\ S_k < \infty,\ R_k = \infty\} < \infty\,.$$

This completes the proof of (1.21). □

We now introduce the σ-algebra (with the convention $l.X_{\tau_1} = \infty$, on $\{\tau_1 = \infty\}$),

$$\mathscr{G}_1 = \sigma(\tau_1,\ (X_{k\wedge\tau_1})_{k\geq 0},\ (\omega(y,\cdot))_{l.y<l.X_{\tau_1}}) \tag{1.22}$$

(in a slightly more explicit fashion, $\mathscr{G}_1$ is generated by the sets $\{\tau_1 = k\} \cap \{X_{\tau_1} = x\} \cap A$, with $A \in \sigma(\omega(y,\cdot),\ l.y < l.x) \otimes \mathscr{F}_k$, $k \geq 1$, $x \in \mathbb{Z}^d$ and $\{\tau_1 = \infty\} \cap A$ with $A \in \sigma(\omega(y,\cdot),\ y \in \mathbb{Z}^d) \otimes \mathscr{F}_\infty$), and the probability

$$Q_0(\cdot) = P_0(\cdot \mid \tau_1 < \infty) \overset{(1.17)}{=} P_0(\cdot \mid A_l)\,. \tag{1.23}$$

The next step in the study of the renewal structure is the following proposition.

PROPOSITION 1.3.

$$\begin{aligned} &Q_0[(X_{\tau_1+n} - X_{\tau_1})_{n\geq 0} \in \cdot, (\omega(X_{\tau_1} + y,\cdot))_{l.y\geq 0} \in \cdot \mid \mathscr{G}_1] \\ &\quad = P_0[(X_n)_{n\geq 0} \in (\omega(y,\cdot))_{l.y\geq 0} \in \cdot \mid D = \infty]\,. \end{aligned} \tag{1.24}$$

PROOF. We consider f, g, h, bounded functions, respectively, $\sigma(X_n, n \geq 0)$, $\sigma(\omega(y,\cdot),\ l.y \geq 0)$ and $\mathscr{G}_1$-measurable. Recall that t_x, $x \in \mathbb{Z}^d$, denotes the canonical shift on $\mathbb{Z}^d$. Then

$$\begin{aligned} &P_0[\tau_1 < \infty]\, E^{Q_0}[f(X_{\tau_1+\cdot} - X_{\tau_1})\, g \circ t_{X_{\tau_1}} h] \\ &\quad = \sum_{k\geq 1} E_0[f(X_{\tau_1+\cdot} - X_{\tau_1})\, g \circ t_{X_{\tau_1}} h,\ S_k < \infty,\ R_k = \infty] \\ &\quad = \sum_{k\geq 1,\, x} E_0[f(X_{S_k+\cdot} - x)\, g \circ t_x\, h,\ X_{S_k} = x,\ S_k < \infty,\ R_k = \infty]\,. \end{aligned} \tag{1.25}$$

Observe that on the event $\{X_{\tau_1} = x\} \cap \{\tau_1 = S_k\}$, one can find a bounded $\sigma(\omega(y,\,\cdot),\ l.y < l.x) \otimes \mathscr{F}_{S_k}$-measurable random variable $h_{x,\,k}$, which coincides with h [to see this one needs only intersect the above event with $\{S_k = m\}$, for $m \geq 0$, and come back to the definition (1.22) of $\mathscr{G}_1$]. As a result, the rightmost side of (1.25) equals

$$\sum_{k\geq 1,\, x} \mathbb{E}[E_{0,\,\omega}[f(X_{S_k+\cdot} - x)h_{x,\,k},\ S_k < \infty,\ X_{S_k} = x,\ D \circ \theta_{S_k} = \infty]\, g \circ t_x]$$

using the strong Markov property at time S_k; this equals

$$\sum_{k\geq 1,\, x} \mathbb{E}[E_{0,\,\omega}[h_{x,\,k},\ S_k<\infty,\ X_{S_k}=x]\ E_{x,\,\omega}[f(X_{.}-x),\ D=\infty]\ g\circ t_x].$$

Note that $E_{0,\,\omega}[h_{x,\,k},\ S_k<\infty,\ X_{S_k}=x]$ is $\sigma(\omega(y,\cdot),\ l.y<l.x)$-measurable and bounded, whereas $E_{x,\,\omega}[f(X_{.}-x),\ D=\infty]\, g\circ t_x$ is $\sigma(\omega(y,\cdot),\ l.y\geq l.x)$-measurable and bounded. These two random variables are thus $\mathbb{P}$-independent and the last expression equals

$$\begin{aligned}
&\sum_{k\geq 1,x} E_0[h_{x,\,k},\ S_k<\infty,\ X_{S_k}=x]\ E_0[fg,\ D=\infty]\\
&\quad = E_0[fg \mid D=\infty] \sum_{k\geq 1,\, x} E_0[h_{x,k},\ S_k<\infty,\ X_{S_k}=x]\ P_0[D=\infty]\\
&\quad = E_0[fg|D=\infty]\ P_0[\tau_1<\infty]\ E^{Q_0}[h]\,,
\end{aligned}$$

where we used the above calculation in the case $f=1$, and $g=1$, to obtain the last equality. The claim (1.24) now follows. □

In view of (1.10) and (1.17),

$$P_0\text{-a.s.},\ \{D=\infty\}\subset A_l\subset\{\tau_1<\infty\}\,; \tag{1.26}$$

moreover,

$$\{D=\infty\}=\{D\geq\tau_1\}\in\mathscr{G}_1 \tag{1.27}$$

and the above proposition enables defining on $\{\tau_1<\infty\}$ a nondecreasing sequence $\tau_1\leq\tau_2\leq\cdots$, via

$$\tau_2=\tau_1(X_{.})+\tau_1(X_{\tau_1+\cdot}-X_{\tau_1})\ \ (\tau_1 \text{ is viewed as a function of } X_{.}) \tag{1.28}$$

and inductively on $k\geq 1$,

$$\tau_{k+1}=\tau_1(X_{.})+\tau_k(X_{\tau_1+\cdot}-X_{\tau_1})\,,$$

setting $\tau_{k+1}=\infty$ by convention on $\{\tau_k=\infty\}$, for $k\geq 1$. Note that P_0-a.s., $\{\tau_k<\infty\}=\{\tau_1<\infty\}=A_l$, for $k\geq 1$.

We can now introduce the σ-field,

$$\mathscr{G}_k=\sigma(\tau_1,\ldots,\tau_k,\ (X_{n\wedge\tau_k})_{n\geq 0},\ (\omega(y,\cdot))_{l.y<l.X_{\tau_k}})\,. \tag{1.29}$$

The main result displaying the renewal structure under the assumption (1.11) comes in the next theorem showing the independence of the joint variables $(X_{\tau_k+\cdot}-X_{\tau_k})$ and $(\omega(X_{\tau_k}+y,\cdot))_{l.y\geq 0}$, from $\mathscr{G}_k$ under Q_0.

THEOREM 1.4. *For $k\geq 1$,*

$$\begin{aligned}
&Q_0[(X_{\tau_k+n}-X_{\tau_k})_{n\geq 0}\in\cdot,\ (\omega(X_{\tau_k}+y,\cdot))_{l.y\geq 0}\in\cdot|\mathscr{G}_k]\\
&\quad = P_0[(X_n)_{n\geq 0}\in\cdot,\ (\omega(y,\cdot))_{l.y\geq 0}\in\cdot\,|D=\infty]\,.
\end{aligned} \tag{1.30}$$

PROOF. Observe that up to Q_0-null sets, $\mathscr{G}_{k+1}$ is generated by $\mathscr{G}_1$ and $\psi^{-1}(\mathscr{G}_k^+)$, where ψ is the Q_0-a.s. defined map

$$\psi(X_{.}, \omega) = (X_{\tau_1+\cdot} - X_{\tau_1},\ t_{X_{\tau_1}}\omega)$$

and $\mathscr{G}_k^+$ is defined analogously to (1.29), with the additional constraint $0 \le y.l$. Noting that $\{D = \infty\}$ is $\mathscr{G}_1$-measurable [cf. (1.27)], Theorem 1.4 now follows by induction using (1.28) and Proposition 1.3. □

As a direct consequence of Theorem 1.4, we now have the following explicit renewal structure.

COROLLARY 1.5. *Under Q_0, (X_{τ_1}, τ_1), $(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \dots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \dots$, are independent variables. Furthermore, $(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \dots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \dots$, are distributed under Q_0 as (X_{τ_1}, τ_1) under $P_0(\cdot\,|D = \infty)$.*

We now introduce the random variable

$$M = \sup\{l.X_m - l.X_0,\ 0 \le m \le D\} \le \infty\,. \tag{1.31}$$

The sufficient condition on the Q_0-integrability of $X_{\tau_1}l$, stated in the next proposition will be useful in the next section.

PROPOSITION 1.6.

$$E_0[M\,|\,D < \infty] < \infty \quad \Longrightarrow \quad E_0[X_{\tau_1}.l,\ \tau_1 < \infty] < \infty\,. \tag{1.32}$$

PROOF. Observe that P_0-a.s., on $\{\tau_1 < \infty\}$, thanks to (1.13), (1.15),

$$\begin{aligned} l.X_{\tau_1} &= l.X_{S_1} + \sum_{1\le k'<K} l.X_{S_{k'+1}} - l.X_{S_{k'}} \\ &= l.X_{S_1} + \sum_{1\le k'<K} l.X_{S_{k'+1}} - M_{k'} + M_{k'} - l.X_{S_{k'}} \\ &\le c + \sum_{1\le k'<K} c + M_{k'} - l.X_{S_{k'}} \end{aligned} \tag{1.33}$$

with

$$c = a + \sup_{i\in[1,d]} |l_i| \tag{1.34}$$

and where a has been introduced above in (1.13). Therefore,

$$\begin{aligned} &E_0[l.X_{\tau_1},\ \tau_1 < \infty] \\ &\quad\le c\,P_0[\tau_1 < \infty] + \sum_{k'\ge1} E_0[c + M_{k'} - l.X_{S_{k'}}, k' < K < \infty] \\ &\quad= c\,P_0[\tau_1 < \infty] + \sum_{1\le k'<k} E_0[c + M_{k'} - l.X_{S_{k'}},\ S_k < \infty,\ R_k = \infty] \end{aligned} \tag{1.35}$$

and using a similar argument as in (1.19), this equals

$$c\,P_0[\tau_1 < \infty] + \sum_{1\le k'<k} E_0[c + M_{k'} - l.X_{S_{k'}},\ S_k < \infty]\ P_0[D = \infty]\,.$$

Now for $1 \le k' < k$, using repeatedly the argument of (1.19), we find

$$
\begin{aligned}
&E_0[c + M_{k'} - l.X_{S_{k'}},\ S_k < \infty] \\
&\quad \le E_0[c + M_{k'} - l.X_{S_{k'}},\ R_{k'} < \infty]\ P_0[D < \infty]^{k-1-k'} \\
&\quad = P_0[S_{k'} < \infty]\ E_0[c + M,\ D < \infty]\ P_0[D < \infty]^{k-1-k'} \\
&\quad \le P_0[D < \infty]^{k'-1}\ E_0[c + M,\ D < \infty]\ P_0[D < \infty]^{k-1-k'}\,.
\end{aligned} \tag{1.36}
$$

Coming back to (1.35), we find

$$
\begin{aligned}
&E_0[l X_{\tau_1}, \tau_1 < \infty] \le c\, P_0[\tau_1 < \infty] \\
&\qquad + (c + E_0[M \,|\, D < \infty]) \sum_{1 \le k' < k} P_0[D = \infty]\ P_0[D < \infty]^{k-1} \\
&\quad = c\, P_0[\tau_1 < \infty] + (c + E_0[M \,|\, D < \infty])\Big(\frac{1}{P_0[D = \infty]} - 1\Big) < \infty\,,
\end{aligned} \tag{1.37}
$$

since $\sum_{k \ge 1}(k-1)\ P_0[D = \infty]\ P_0[D < \infty]^{k-1} = \frac{1}{P_0[D=\infty]} - 1$.

This completes the proof of (1.32). □

2. The law of large numbers. Similarly to the previous sections, we tacitly assume (0.1) and $d \ge 1$. The main object of the present section is to show that under Kalikow's condition (0.7), one has a strong law of large numbers for (X_n) under P_0. We begin with a preparatory result.

PROPOSITION 2.1. *Assume that* (1.11) *holds for some* $l \in \mathbb{R}^d \backslash \{0\}$, *and*

$$E_0[\tau_1 \,|\, D = \infty] < \infty, \tag{2.1}$$

then

$$E_0[|X_{\tau_1}| \,|\, D = \infty] < \infty, \tag{2.2}$$

$$Q_0\text{-a.s.,} \quad \frac{X_n}{n} \longrightarrow v = \frac{E_0[X_{\tau_1} \,|\, D = \infty]}{E_0[\tau_1 \,|\, D = \infty]} \tag{2.3}$$

and

$$v.l > 0\,. \tag{2.4}$$

PROOF. Since Q_0-a.s., $|X_{\tau_1}| \le \tau_1$, (2.2) follows immediately from (2.1). Let us prove (2.3). As a consequence of Corollary 1.5, and the strong law of large numbers,

$$Q_0\text{-a.s.,} \quad \frac{\tau_k}{k} \longrightarrow E_0[\tau_1 | D = \infty], \quad \frac{X_{\tau_k}}{k} \longrightarrow E_0[X_{\tau_1} | D = \infty] \quad \text{as } k \to \infty\,. \tag{2.5}$$

Let us then define the nondecreasing sequence k_n, $n \ge 0$, Q_0-a.s., tending to $+\infty$, such that

$$\tau_{k_n} \le n < \tau_{k_n + 1} \quad \text{(with the convention } \tau_0 = 0)\,. \tag{2.6}$$

Dividing the above inequalities by k_n, and using (2.5), we find

$$(2.7) \qquad Q_0\text{-a.s.,} \ \frac{k_n}{n} \longrightarrow \frac{1}{E_0[\tau_1 | D = \infty]} \quad \text{as } n \to \infty .$$

As a result,

$$(2.8) \qquad \frac{X_n}{n} = \frac{X_{\tau_{k_n}}}{n} + \frac{X_n - X_{\tau_{k_n}}}{n} ,$$

where in view of (2.5), (2.7),

$$\frac{X_{\tau_{k_n}}}{n} = \frac{X_{\tau_{k_n}}}{k_n} \frac{k_n}{n} \longrightarrow_{n\to\infty} \frac{E_0[X_{\tau_1} | D = \infty]}{E_0[\tau_1 | D = \infty]}, \qquad Q_0\text{-a.s.}$$

and by (2.5), Q_0-a.s.,

$$\frac{|X_n - X_{\tau_{k_n}}|}{n} \le \frac{\tau_{k_n+1} - \tau_{k_n}}{n} = \frac{\tau_{k_n+1}}{k_n+1} \frac{k_n+1}{n} - \frac{\tau_{k_n}}{k_n} \frac{k_n}{n} \to 0 .$$

Coming back to (2.8), this proves (2.9). As for (2.4), it suffices to observe that $E_0[X_{\tau_1}.l \mid D = \infty] > 0$, by construction. □

We shall now bring Kalikow's condition (0.7) into play. We postpone to Proposition 2.4 and Remark 2.5 the discussion of examples of random walks in random environment, where (0.7) is fulfilled. The next lemma will already show that (0.7) implies some ballistic behavior of the walk. For the time being let us recall that in the notation of (1.11) [see Theorem 1 of [5]; (0.1) is tacitly assumed],

$$(2.9) \qquad \text{when (0.7) holds,} \ P_0(A_l) = 1 .$$

In particular when (0.7) holds, P_0 and Q_0 coincide. We shall now derive a useful consequence of (0.7).

LEMMA 2.2. *Under* (0.7), *for any finite connected set U containing* 0,

$$(2.10) \qquad E_0[l.X_{T_U}] \ge \varepsilon\, E_0[T_U] ,$$

where ε is as in (0.7).

PROOF. As a consequence of (0.5) and (0.7), for $x \in U$,

$$E_0\Big[\sum_0^{T_U} 1\{X_n = x\} \sum_{|e|=1} l.e\ \omega(X_n, e)\Big] \ge \varepsilon\, E_0\Big[\sum_0^{T_U} 1\{X_n = x\}\Big]$$

and the expectations are finite thanks to (0.1). Summing over $x \in U$, we find

$$(2.11) \qquad E_0\Big[\sum_0^{T_U-1} \sum_{|e|=1} l.e\ \omega(X_n, e)\Big] \ge \varepsilon\, E_0[T_U] .$$

The left-hand side of (2.11) also equals

$$\mathbb{E}\Big[E_{0,\omega}\Big[\sum_{n\geq 0} 1\{n < T_U\}\, E_{0,\omega}[l.X_{n+1} - l.X_n \,|\, \mathscr{F}_n]\Big]\Big]$$
$$= \mathbb{E}\Big[E_{0,\omega}\Big[\sum_{n\geq 0} 1\{n < T_U\}\, (l.X_{n+1} - l.X_n)\Big]\Big] = E_0[l.X_{T_U}]\,.$$

This completes the proof of (2.10). □

We now come to the main result of this section.

THEOREM 2.3. *Assume* (0.7)*; then* $E_0[\tau_1 \,|\, D = \infty] < \infty$ *and*

$$P_0\text{-a.s.,}\quad \frac{X_n}{n} \longrightarrow v = \frac{E_0[X_{\tau_1} \,|\, D = \infty]}{E_0[\tau_1 \,|\, D = \infty]} \quad \text{where } v.l > 0\,. \tag{2.12}$$

PROOF. In view of Proposition 2.1 and the fact that P_0 and Q_0 coincide under (0.7), the claim (2.12) will follow from

$$E_0[\tau_1 \,|\, D = \infty] < \infty\,. \tag{2.13}$$

The first step in the proof of (2.13) is to show that the assumption of Proposition 1.6 is satisfied, namely,

$$E_0[M \,|\, D < \infty] < \infty\,. \tag{2.14}$$

To this end, in analogy with (1.12), we introduce for $u \in \mathbb{R}$,

$$\widetilde{T}_u = \inf\{n \geq 0,\ l.X_n \leq u\}\,. \tag{2.15}$$

Then for $m \geq 0$ (recall that $P_0(A_l) = 1$),

$$\begin{aligned} P_0[2^m \leq M < 2^{m+1}, D < \infty] &\leq P_0[\widetilde{T}_0 \circ \theta_{T_{2^m}} < T_{2^{m+1}} \circ \theta_{T_{2^m}}] \\ &\leq P_0\Big[\Big|X_{T_{2^m}} - 2^m \frac{l}{|l|^2}\Big| \geq 2\,\frac{2^m}{\varepsilon}\Big] \\ &\quad + P_0\Big[\Big|X_{T_{2^m}} - 2^m \frac{l}{|l|^2}\Big| < 2\,\frac{2^m}{\varepsilon}\,,\, \widetilde{T}_0 \circ \theta_{T_{2^m}} < T_{2^{m+1}} \circ \theta_{T_{2^m}}\Big]\,. \end{aligned} \tag{2.16}$$

Let us first analyze the first term in the rightmost side of (2.16). If U_m denotes the set

$$U_m = \{x \in \mathbb{Z}^d,\ l.x < 2^m\}\,,$$

then $0 \in U_m$ and U_m is easily seen to be connected. As a result of (0.7), it is routine to argue that

$$\widehat{P}_{0,U_m}(T_{U_m} < \infty) = 1\,, \tag{2.17}$$

indeed under $\widehat{P}_{0,U_m}$, $l.X_n$ is the sum of a martingale with bounded increments and an increasing process, which tends to $+\infty$ on the event $\{T_{U_m} = \infty\}$. Thus

$\widehat{P}_{0,U_m}$-a.s., on $\{T_{U_m} = \infty\}$, $\limsup\, l.X_n = +\infty$, which implies (2.17). Observe that thanks to (0.6),

$$P_0\Big[\Big|X_{T_{2^m}} - 2^m\,\frac{l}{|l|^2}\Big| \geq 2\,\frac{2^m}{\varepsilon}\Big] = \widehat{P}_{0,U_m}\Big[\Big|X_{T_{U_m}} - 2^m\,\frac{l}{|l|^2}\Big| \geq 2\,\frac{2^m}{\varepsilon}\Big]. \tag{2.18}$$

Moreover, note that the local drift

$$\hat{d}(x) = \begin{cases} \sum_{|e|=1} \widehat{P}_{U_m}(x,e)e, & x \in U_m, \\ 0, & x \in \partial U_m, \end{cases} \tag{2.19}$$

of the Markov chain with kernel $\widehat{P}_{U_m}$ and law $\widehat{P}_{0,U_m}$, is such that

$$\hat{d}(x).l \geq \varepsilon \quad \text{and} \quad |\hat{d}(x)| \leq 1, \text{ when } x \in U_m\,. \tag{2.20}$$

Thus denoting by C the cone

$$C = \Big\{y \in \mathbb{R}^d,\ \ \Big|y - y.\frac{l}{|l|^2}\,l\Big| \leq \frac{1}{\varepsilon}\,y.l\Big\}, \tag{2.21}$$

we see that

$$\hat{d}(x) \in C \quad \text{for } x \in U_m \cup \partial U_m\,. \tag{2.22}$$

Since under $\widehat{P}_{0,U_m}$, $X_n - \sum_{n'=0}^{n-1} \hat{d}(X_{n'})$ is an $\mathscr{F}_n$-martingale with uniformly bounded increments, it follows from Azuma's inequality (see [1], page 85) that for a suitable $c_1 > 0$, for any number $\eta > 0$ and integer $N \geq 1$,

$$\widehat{P}_{0,U_m}\Big[\sup_{0\leq n\leq N}\Big|X_n - \sum_{n'=0}^{n-1}\hat{d}(X_{n'})\Big| \geq \eta N\Big] \leq 2d\,N\,\exp(-c_1\eta^2 N)\,. \tag{2.23}$$

As a result of (2.20) and (2.23), for a suitable $c_2 > 0$, when m is large,

$$\widehat{P}_{0,U_m}\Big[T_{U_m} \geq 2\,\frac{2^m}{\varepsilon}\Big] \leq \exp(-c_2 2^m)\,.$$

Using the fact that $\sum_0^{T_{U_m}-1}\hat{d}(X_n) \in C$, we see that for suitable $c_3, c_4 > 0$, for large m,

$$\begin{aligned} &\widehat{P}_{0,U_m}\Big[\Big|X_{T_{U_m}} - 2^m\,\frac{l}{|l|^2}\Big| \geq \frac{2^{m+1}}{\varepsilon}\Big] \\ &\leq \exp(-c_2 2^m) \\ &\quad + \widehat{P}_{0,U_m}\Big[T_{U_m} < \frac{2^{m+1}}{\varepsilon},\ \sup_{0\leq n\leq 2^{m+1}/\varepsilon}\Big|X_n - \sum_0^{n-1}\hat{d}(X_{n'})\Big| \geq c_3\,\frac{2^{m+1}}{\varepsilon}\Big] \\ &\leq \exp(-c_4 2^m)\,, \end{aligned} \tag{2.24}$$

using (2.23) in the last step. As for the rightmost side of (2.16), for a suitable $c_5 > 0$, and large m, it is smaller than

$$c_5\,2^{m(d-1)}P_0[\widetilde{T}_{-2^m} < T_{2^m}]\,. \tag{2.25}$$

Defining now the set

$$\widetilde{U}_m = \{y\colon |ly| < 2^m\}\,,$$

which for large m is connected, contains 0 and is such that $\widehat{P}_{0,\widetilde{U}_m}(T_{\widetilde{U}_m} < \infty) = 1$, we obtain as a result of (0.6),

$$P_0[\widetilde{T}_{-2^m} < T_{2^m}] = \widehat{P}_{0,\widetilde{U}_m}[l.X_{T_{\widetilde{U}_m}} < 0] \le \exp(-c\,2^m) \tag{2.26}$$

for large m, using a similar argument to (2.24). Combining (2.24)–(2.26), we see that

$$E_0[M, D < \infty] \le 1 + \sum_{m\ge 0} 2^{m+1} P_0[2^m \le M < 2^{m+1}, D < \infty] < \infty\,,$$

thus proving (2.14). We can now apply Proposition 1.6 and see that the P_0-a.s. positive random variable $l.X_{\tau_1}$ is also P_0-integrable [recall that $P_0(A_l) = P_0(\tau_1 < \infty) = 1$, under (0.7)]. An application of the strong law of large numbers and Corollary 1.5 shows that

$$P_0\text{-a.s.},\ \frac{\tau_k}{k} \longrightarrow E_0[\tau_1 \,|\, D = \infty] \in (0, \infty]\,. \tag{2.27}$$

Introduce the increasing sequence k'_m, $m \ge 0$, P_0-a.s. tending to $+\infty$ (we use the convention $\tau_0 = 0$), such that

$$\tau_{k'_m} \le T_m < \tau_{k'_m+1}\,. \tag{2.28}$$

From the definition of the sequence τ_k, $k \ge 1$, P_0-a.s.,

$$l.X_n < l.X_{\tau_k} \le l.X_{n'} \quad \text{for } 0 \le n < \tau_k \le n'\,; \tag{2.29}$$

therefore for $m \ge 0$,

$$l.X_{\tau_{k'_m}} \le l.X_{T_m} < l.X_{\tau_{k'_m+1}} \tag{2.30}$$

and

$$|l.X_{T_m} - m| \le \sup_{[1,d]} |l_i|\,. \tag{2.31}$$

From the law of large numbers, we know that

$$P_0\text{-a.s.},\ \frac{l.X_{\tau_k}}{k} \longrightarrow E_0[l.X_{\tau_1} \,|D = \infty] \in (0, \infty)\,. \tag{2.32}$$

Dividing both members of (2.30) by k'_m and using (2.31), we find

$$P_0\text{-a.s.}\ \frac{k'_m}{m} \longrightarrow \frac{1}{E_0[l.X_{\tau_1} \,|D = \infty]}\,. \tag{2.33}$$

Coming back to (2.28), we see from (2.27), (2.33) that

$$\frac{T_m}{m} \ge \frac{\tau_{k'_m}}{k'_m}\frac{k'_m}{m} \longrightarrow_{P_0-\text{a.s.}} \frac{E_0[\tau_1 \,|D = \infty]}{E_0[X_{\tau_1}.l \,|D = \infty]} \in (0, \infty]\,. \tag{2.34}$$

However, using (2.10) and the exhaustion of $\{y\colon l.y < m\}$ by an increasing sequence of finite connected sets containing 0, we see that

$$E_0\Big[\lim_m \frac{T_m}{m}\Big] \le \lim_m E_0\Big[\frac{T_m}{m}\Big] \le \frac{1}{\varepsilon}\,. \tag{2.35}$$

This and (2.34) now imply (2.13) and completes the proof of Theorem 2.3. □

We now come back to the discussion of Kalikow's condition (0.7). It is shown in [5], pages 759 and 760, that when $d \ge 1$ and (0.1) hold, a sufficient condition for (0.7) is

$$\inf_{f\in F} \mathbb{E}\bigg[\frac{\sum_{|e|=1}\omega(0,e)l.e}{\sum_{|e|=1}\omega(0,e)f(e)}\bigg]\bigg/\mathbb{E}\bigg[\frac{1}{\sum_{|e|=1}\omega(0,e)f(e)}\bigg] \ge \varepsilon\,, \tag{2.36}$$

where ε is as in (0.7) and F denotes the collection of $f = (f(e))_{|e|=1,\,e\in\mathbb{Z}^d}$, with $f(e) \in [0,1]$, for each e, and $f \ne 0$. Condition (2.36) only involves the law of the random environment at one site and enables concrete applications. For instance [recall (0.1) is implicitly assumed]:

PROPOSITION 2.4. ($d \ge 1$). *Let* $l \in \mathbb{R}^d\backslash\{0\}$, *if*

(2.37) *the support of the law of the local drift* $d(0,\omega) = \sum_{|e|=1}\omega(0,e)\,e$ *is contained in the half-space* $\{z \in \mathbb{R}^d,\ l.z \ge 0\}$, *but not in the hyperplane* $\{z\in\mathbb{R}^d,\ l.z = 0\}$,

then (2.36) *and thus* (0.7) *hold.*

PROOF. From (2.37) it follows that

$$\mathbb{P}\text{-a.s.,}\quad \sum_{|e|=1}\omega(0,e)\,l.e \ge 0 \tag{2.38}$$

and for a suitable $\eta > 0$, on a set E of positive $\mathbb{P}$-probability,

$$\sum_{|e|=1} l.e\ \omega(0,e) \ge \eta\,. \tag{2.39}$$

However, as a result of (0.1), for a suitable $\kappa > 0$, for $f \in F$, $\max_e f(e) \ge \sum_{|e|=1}\omega(0,e)f(e) \ge \kappa\ \max_e\ f(e)$. Therefore, for $f \in F$,

$$\begin{aligned}&\mathbb{E}\Big[\sum_{|e|=1} l.e\omega(0,e)\Big/\sum_{|e|=1} f(e)\omega(0,e)\Big]\Big/\mathbb{E}\Big[1\Big/\sum_{|e|=1} f(e)\,\omega(0,e)\Big]\\ &\quad\ge \eta\ \mathbb{E}[1_E(\max_e\ f(e))^{-1}]\,/\,\mathbb{E}[(\kappa\ \max_e\ f(e))^{-1}] = \kappa\eta\,\mathbb{P}(E) =_{\mathrm{def}} \varepsilon > 0\,.\end{aligned}$$

This proves (2.36). □

In particular, as a result of the above proposition, when (0.1) holds, the nonnestling walks (cf. [12]), where

(2.40) 0 does not belong to the convex hull of the support of the law of $d(0,\omega)$ (which is a compact set) automatically satisfy (2.37) and thus (0.7) for some $l \in \mathbb{R}^d \setminus \{0\}$. Further, when (0.1) holds, the class of random walks in random environment considered in [11], which are "either neutral or with a local drift pointing in the e_1-direction" also satisfy (2.37) and (0.7) with $l = e_1$. We also refer to [5], Corollary to Theorem 1, for further examples where (2.36) hold. The next remark also sheds some light on why (2.36) ensures "a nondegenerate strong law of large numbers" as stated in Theorem 2.3.

REMARK 2.5. (i) In the case of dimension $d = 1$, it is known (cf. [10]) that when (0.1) holds, the random walk in random environment satisfies a strong law of large numbers with a deterministic velocity, which possibly vanishes. This velocity is positive if and only if

$$\mathbb{E}\Big[\frac{q}{p}\Big] < 1 \quad \text{where} \;\; p = \omega(0,1),\;\; q = \omega(0,-1)\,. \tag{2.41}$$

As we shall now see, this last condition is equivalent [under (0.1)] to (2.36), when $d = 1$ and $l = 1$, namely,

$$\inf_{a,\,b>0} \mathbb{E}\Big[\frac{p-q}{ap+bq}\Big] \Big/ \mathbb{E}\Big[\frac{1}{ap+bq}\Big] > 0\,. \tag{2.42}$$

Indeed, choosing $a = 1$ and letting b tend to 0, we see that $\mathbb{E}[(p-q)/p] > 0$, which implies (2.41). Conversely, observe that thanks to (0.1), (2.42) follows from

$$\min_{x\in(0,\,1]} \mathbb{E}\Big[\frac{p-q}{p+xq}\Big] > 0 \quad \text{and} \quad \min_{y\in(0,\,1]} \mathbb{E}\Big[\frac{p-q}{yp+q}\Big] > 0\,. \tag{2.43}$$

However, defining for $x \ge 0$,

$$h(x) = \mathbb{E}\Big[\frac{p-q}{p+xq}\Big] \quad \text{so that} \;\; h'(x) = \mathbb{E}\Big[-\frac{(p-q)q}{(p+xq)^2}\Big]\,,$$

we see that $p \to ((p-q)/p+xq)$ and $p \to (-q/p+xq)$ are nondecreasing functions. Indeed, with $q = 1-p$,

$$\frac{d}{dp}\;\frac{p-q}{p+xq} = \frac{x+1}{(p+xq)^2} > 0, \qquad \frac{d}{dp}\Big(\frac{-q}{p+xq}\Big) = \frac{1}{(p+xq)^2} > 0.$$

Thus they are positively correlated under $\mathbb{P}$ and

$$h'(x) \ge h(x)\,\mathbb{E}\Big[\frac{-q}{p+xq}\Big], \qquad x \ge 0\,. \tag{2.44}$$

Since $h(0) > 0$, by assumption, it easily follows from (2.44) that

$$h(x) \ge h(0)\,\exp\Big\{\int_0^x \mathbb{E}\Big[\frac{-q}{p+uq}\Big]du\Big\} \quad \text{for} \;\; x > 0\,. \tag{2.45}$$

This implies the first inequality of (2.43). By a similar reasoning, using the symmetry of the roles p and q, defining for $y \ge 0$,

$$g(y) = \mathbb{E}\Big[\frac{p-q}{yp+q}\Big]\,,$$

we see that

$$-g'(y) \geq -g(y)\,\mathbb{E}\Big[\frac{-p}{yp+q}\Big]\,, \tag{2.46}$$

since $g(1) = h(1) > 0$, it easily follows from (2.46) that

$$g(y) \geq g(1) = h(1) \quad \text{for} \quad y \in [0, 1]\,.$$

This completes the proof of (2.43) and thus of the equivalence of (2.41) and (2.42) under (0.1).

(ii) Collecting all these results and Theorem 2.3, we see that when $d = 1$ and (0.1) holds, Kalikow's condition with respect to $l \neq 0$, is equivalent to (2.36) and characterizes the situation of walks having a nonvanishing velocity with the same sign as l. □

3. Asymptotic law of the environment viewed from the particle. We recall the standing assumptions $d \geq 2$ and (0.1). We further assume in this section that Kalikow's condition (0.7) relative to $l \in \mathbb{R}^d\backslash\{0\}$ is fulfilled. The main object of this section is to investigate the asymptotic behavior of the law under P_0 of $\overline{\omega}_n$, the environment viewed from the particle [cf. (0.9)]. As mentioned in the introduction, $\overline{\omega}_n$, $n \geq 0$, under P_0 is a Markov chain with state space Ω initial distribution $\mathbb{P}$ and transition kernel R, defined in (0.10). The state space Ω [see after (0.1)] is endowed with the canonical product topology, for which it is compact.

THEOREM 3.1.

(3.1) *The law $\mathbb{P}_n$ of $\overline{\omega}_n$ under P_0 converges weakly to an invariant distribution $\mathbb{P}_\infty$ of R.*

PROOF. Observe that $\mathbb{P}_n = \mathbb{P}\,R^n$, and

$$Rf(\overline{\omega}) = \sum_{|e|=1} \overline{\omega}(0, e)\, f(t_e(\overline{\omega})) \quad \text{for} \quad \overline{\omega} \in \Omega\,,$$

so that R preserves the set of bounded continuous functions on Ω. We therefore simply need to prove that

$$\mathbb{P}_n \text{ is weakly convergent}\,, \tag{3.2}$$

since the limit $\mathbb{P}_\infty$ will automatically be an invariant measure for R.

We then consider a bounded function f on Ω which depends measurably on $\omega(x, \cdot)$ for x such that $|l \cdot x| \leq C$. We also choose $a > 0$ in the definition (1.13), so that

$$a < \sup_{i\in[1,d]} |le_i|, \text{ with } (e_i)_{i\in[1,d]}, \text{ the canonical basis of } \mathbb{R}^d\,. \tag{3.3}$$

As a result,

$$P_0[\tau_1 = 1,\ D = \infty] = P_0[S_1 = 1,\ R_1 = \infty] > 0\,. \tag{3.4}$$

In particular, this shows that the law of τ_1 under $P_0[\cdot\,|D = \infty]$ is aperiodic, a fact which will be useful when later applying the renewal theorem. We choose

$N \geq 1$, large enough so that

$$N\,a \geq C+1\,. \tag{3.5}$$

For $n \geq 1$, we can write

$$E_0[f(\overline{\omega}_n)] = E_0[\tau_{N+1} > n,\ f(\overline{\omega}_n)] + E_0[\tau_{N+1} \leq n,\ f(\overline{\omega}_n)]\,, \tag{3.6}$$

where the first term of the right-hand side tends to 0 as n tends to infinity. Then

$$\begin{aligned} &E_0[\tau_{N+1} \leq n,\ f(\overline{\omega}_n)] \\ &\quad = \sum_{k\geq 1} E_0[\tau_{k+N} \leq n < \tau_{k+N+1},\ f(t_{X_n}\omega)] \\ &\quad = \sum_{\substack{k,\,m\geq 1\\ x,y}} E_0[\tau_{k+N} \leq n < \tau_{k+N+1},\ \tau_k = m,\ X_{\tau_k} = x,\ X_n = y,\ f(t_y\omega)]\,. \end{aligned}$$

Observe that on the event $\{\tau_{k+N} \leq n < \tau_{k+N+1},\ \tau_k = m,\ X_{\tau_k} = x,\ X_n = y\}$, $l.x = l.X_{\tau_k} \leq l.X_{\tau_{k+N}} - N\,a \overset{(3.5)}{\leq} ly - C - 1$, and therefore $f(t_y\omega)$ is a measurable function of $(\omega(z,\cdot);\ l.z \geq l.x)$. Using Theorem 1.4, the above expression equals

$$\sum_{\substack{k\geq 1,\,0\leq m\leq n\\ x,\,y}} \begin{aligned}[t] &P_0[\tau_k = m,\ X_{\tau_k} = x] \\ &E_0[\tau_N \leq n-m < \tau_{N+1},\ X_{n-m} = y-x,\ f(t_{y-x}\omega)|D=\infty], \end{aligned}$$

setting $u = n-m$ and performing the summation over y, it equals

$$\begin{aligned} &\sum_{\substack{k\geq 1,\,0\leq u\leq n\\ x}} \begin{aligned}[t] &P_0[\tau_k = n-u,\ X_{\tau_k} = x], \\ &E_0[\tau_N \leq u < \tau_{N+1},\ f(t_{X_u}\omega)|D=\infty] \end{aligned} \\ &\quad = \sum_{0\leq u\leq n} \begin{aligned}[t] &P_0[n-u \text{ equals } \tau_k \text{ for some } k \geq 1], \\ &\quad E_0[\tau_N \leq u < \tau_{N+1},\ f(t_{X_u}\omega)|D=\infty]\,. \end{aligned} \end{aligned}$$

Note that f is bounded and from Theorem 1.4, (1.27) and (2.13),

$$\sum_{u\geq 0} P_0[\tau_N \leq u < \tau_{N+1}\,|\,D=\infty] = E_0[\tau_1\,|\,D=\infty] < \infty\,. \tag{3.7}$$

Further, from the renewal theorem (cf. [4], page 313) and Corollary 1.5,

$$\begin{aligned} &P_0[n-u \text{ equals } \tau_k \text{ for some } k \geq 1] \\ &\quad = P_0[n-u-\tau_1 = \tau_k - \tau_1 \text{ for some } k \geq 1] \ \longrightarrow_{n\to\infty} 1/E_0[\tau_1|D=\infty] \end{aligned}$$

for any $u \geq 0$. Coming back to the rightmost expression of (3.6), we see that when n tends to infinity, it converges to

$$\sum_{u\geq 0} \frac{E_0[\tau_N \leq u < \tau_{N+1},\ f(t_{X_u}\omega)|D=\infty]}{E_0[\tau_1\,|D=\infty]}\,. \tag{3.8}$$

This proves the convergence of $E_0[f(\overline{\omega}_n)]$ for f an arbitrary bounded measurable function depending on finitely many coordinates. Since Ω is compact, the claim (3.2) follows by routine arguments and this completes the proof of (3.1). □

REMARK 3.2. In the one-dimensional case, a similar theorem can be found in [6], under the assumption [see (2.41)]

$$\mathbb{E}\left[\frac{q}{p}\right] < 1 \,. \tag{3.9}$$

The limiting distribution $\mathbb{P}_\infty$ can in fact be found in [9], page 274 [the situation where $\mathbb{E}[p/q] < 1$, so that the random walk in random environment tends to $-\infty$, is considered in this reference]. Expressed in the present situation (3.9), the measure $\mathbb{P}_\infty$ is absolutely continuous with respect to $\mathbb{P}$ and the relative density depends measurably on the nonnegative coordinates, $p(x, \omega)$, $x \geq 0$.

The existence of an invariant distribution for the "environment viewed from the particle," which is absolutely continuous with respect to the basic law describing the environment, is known to be an important assumption in the investigation of random motions in random environments. We refer on this to [8]. It is an interesting question whether $\mathbb{P}_\infty$ in (3.1) is absolutely continuous with respect to $\mathbb{P}$ or not.

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