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A LAW OF THE ITERATED LOGARITHM FOR DOUBLE ARRAYS OF INDEPENDENT RANDOM VARIABLES WITH APPLICATIONS TO REGRESSION AND TIME SERIES MODELS

BY TZE LEUNG LAI[1] AND CHING ZONG WEI

Columbia University and University of Maryland

Motivated by the problem of establishing laws of the iterated logarithm for least squares estimates in regression models and for partial sums of linear processes, we prove a general log log law for weighted sums of the form $\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i$, where the ε_i are independent random variables with zero means and a common variance σ^2, and $\{a_{ni}: n \geq 1, -\infty < i < \infty\}$ is a double array of constants such that $\sum_{i=-\infty}^{\infty} a_{ni}^2 < \infty$ for every n. Besides applying the general theorem to least squares estimates and linear processes, we also use it to improve earlier results in the literature concerning weighted sums of the form $\sum_{i=1}^{n} f(i/n)\varepsilon_i$.

1. Introduction and summary. Let $\{a_{ni}: n \geq 1, -\infty < i < \infty\}$ be a double array of constants such that

(1.1) $$\sum_{i=-\infty}^{\infty} a_{ni}^2 < \infty \text{ for every } n.$$

Thus, $\mathbf{a}_n = (a_{ni})_{-\infty<i<\infty} \in \ell^2$, and we shall let $\|\mathbf{a}_n\| = (\sum_{i=-\infty}^{\infty} a_{ni}^2)^{1/2}$ denote the ℓ^2 norm of $\mathbf{a}_n$. Let $\cdots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots$ be independent random variables such that

(1.2) $$E\varepsilon_n = 0 \quad \text{and} \quad E\varepsilon_n^2 = \sigma^2 \quad \text{for all } n, \quad \text{and } \sup_n E|\varepsilon_n|^r < \infty$$

for some $r > 2$. Define

(1.3) $$S_n = \sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i.$$

In view of (1.1) and (1.2), the series in (1.3) converges a.s. and therefore S_n is well defined. The following theorem gives conditions involving certain ℓ^2 properties of $\{\mathbf{a}_n\}$ that would entail a log log law of the form

(1.4) $$\limsup_{n\to\infty} |S_n|/\{\|\mathbf{a}_n\|(2\log\log\|\mathbf{a}_n\|)^{1/2}\} = \sigma \quad \text{a.s.}$$

THEOREM 1. *Let $\cdots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots$ be independent random variables such that* (1.2) *holds for some $r > 2$, and let $\{a_{ni}\}$ be a double array of constants satisfying* (1.1). *Define S_n as in* (1.3). *Assume that as $n \to \infty$,*

(1.5) $$A_n = \sum_{-\infty}^{\infty} a_{ni}^2 (= \|\mathbf{a}_n\|^2) \to \infty, \quad \text{and}$$

(1.6) $$\sup_i a_{ni}^2 = o(A_n(\log A_n)^{-\rho}) \text{ for all } \rho > 0.$$

(i) *If there exist constants $c_i \geq 0$ and $d > 2/r$ such that*

(1.7) $$\|\mathbf{a}_n - \mathbf{a}_m\|^2 (= \sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})^2) \leq (\sum_{i=m+1}^{n} c_i)^d \quad \text{for} \quad n > m \geq m_0, \quad \text{and}$$

(1.8) $$(\sum_{i=m_0}^{n} c_i)^d = O(A_n) \quad \text{as} \quad n \to \infty,$$

then

(1.9) $$\limsup_{n\to\infty} |S_n|/(2A_n \log\log A_n)^{1/2} \leq \sigma \quad \text{a.s.}$$

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(ii) *If* $\|\mathbf{a}_n - \mathbf{a}_m\|^2 \le g(n-m)$ *for* $n > m \ge m_0$, *where* g *is a positive function on* $\{1, 2, \cdots\}$ *such that*

(1.10) $$g(n) = O(A_n) \quad as \quad n \to \infty,$$

(1.11) $$\liminf_{n\to\infty} g(Kn)/g(n) > K^{2/r} \quad \text{for some integer} \quad K \ge 2, \quad \text{and}$$

(1.12) $$\forall \gamma > 0, \exists \delta < 1 \quad \text{such that} \quad \limsup_{n\to\infty}\{\max_{\delta n \le i \le n} g(i)/g(n)\} < 1 + \gamma,$$

then (1.9) *still holds.*

(iii) *Suppose that for every* $0 < \gamma < \gamma_0$, *there exist integers* $1 < n_1 < n_2 < \cdots$ *and disjoint subsets* $I_1, I_2, \cdots$ *of the set of integers such that*

(1.13) $$\limsup_{k\to\infty}(\textstyle\sum_{i\notin I_k} a_{n_k, i}^2)/A_{n_k} \le \gamma,$$

(1.14) $$\limsup_{k\to\infty}(\log\log A_{n_k})/(\log k) \le 1 + \gamma, \text{ and}$$

(1.15) $$\liminf_{k\to\infty}(\log\log A_{n_k})/(\log k) > 0,$$

where n_k *and* I_k *may depend on* γ, *then for every* $-\sigma \le q \le \sigma$,

(1.16) $$\liminf_{n\to\infty}|(2A_n \log\log A_n)^{-1/2} S_n - q| = 0 \quad \text{a.s.}$$

For the particular case

(1.17) $$a_{ni} = 1 \quad \text{if} \quad 1 \le i \le n, \qquad a_{ni} = 0 \text{ otherwise},$$

S_n reduces to the partial sum $\sum_1^n \varepsilon_i$, which has been extensively studied in the literature. In this case, $A_n = n$, and (1.5) and (1.6) obviously hold. The assumptions of parts (i) and (ii) of the above theorem are satisfied with $c_i = d = 1$ and $g(n) = n$, while for part (iii) we can choose $n_k = L^k$ and $I_k = \{n : n_{k-1} < n \le n_k\}$, where L is an integer $> \gamma^{-1}$.

More generally, consider the case

(1.18) $$a_{ni} = b_i \text{ if } 1 \le i \le n, \qquad a_{ni} = 0 \text{ otherwise},$$

where $\{b_i\}$ is a sequence of constants such that $A_n = \sum_1^n b_i^2 \to \infty$ and

(1.19) $$b_n^2 = o(A_n(\log A_n)^{-\rho}) \quad \text{for all} \quad \rho > 0.$$

Then the assumptions of Theorem 1 (i) are satisfied with $c_i = b_i^2$ and $d = 1$, while for Theorem 1 (iii) we can let

(1.20) $$n_k = \inf\{n > n_{k-1} : A_n \ge L^k\}, \qquad I_k = \{n : n_{k-1} < n \le n_k\},$$

where $L > \gamma^{-1}$. The condition (1.19) is equivalent to (1.6). In [10], Teicher has provided a detailed analysis of the log log law for weighted sums $\sum_1^n b_i\varepsilon_i$ of i.i.d. random variables with mean 0 and variance $\sigma^2 > 0$. His results show that if nb_n^2 grows faster than $A_n \log\log A_n$, then something beyond a finite second moment is in fact necessary for the log log law (1.4); moreover, if $b_n \sim c^n$ for some $c > 1$ and therefore $b_n^2 \sim (1 - c^{-2})A_n$, then the log log law fails to hold for $\sum_1^n b_i\varepsilon_i$ whenever the common distribution of the ε_i is bounded.

In Section 2, we apply Theorem 1 to establish a log log law for least squares estimates in regression models. For simplicity, first consider the simple linear model

(1.21) $$y_i = \alpha + \beta x_i + \varepsilon_i, \qquad i = 1, 2, \cdots$$

where the ε_i are as in (1.2), x_i are known constants, and α, β are unknown parameters. The least squares estimate of the slope β based on the observations $(x_1, y_1), \cdots, (x_n, y_n)$ is

(1.22) $$b_n = \{\textstyle\sum_1^n (x_i - \bar{x}_n)y_i\}/\{\sum_1^n (x_i - \bar{x}_n)^2\} = \beta + \{\sum_1^n (x_i - \bar{x}_n)\varepsilon_i\}/\{\sum_1^n (x_i - \bar{x}_n)^2\}.$$

Here and in the sequel, we use the notation $\bar{x}_n$ to denote the arithmetic mean $n^{-1}\sum_1^n x_i$ of n numbers $x_1, \cdots, x_n$. In view of (1.22), the limiting behavior of the least squares estimate b_n is determined by that of $\sum_1^n a_{ni}\varepsilon_i$, where

(1.23) $$a_{ni} = x_i - \bar{x}_n \text{ for } 1 \le i \le n, \qquad a_{ni} = 0 \text{ otherwise}.$$

This double array structure makes the problem of a.s. convergence properties of b_n much harder than the well known weak limit theorems for b_n, and only recently has the problem on the strong consistency of b_n, and more generally, of the least squares estimates in the multiple regression model

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i, \qquad i = 1, 2, \cdots, \tag{1.24}$$

been resolved (cf. [3]). In Section 2, we shall make use of Theorem 1(i) and (iii) to solve the problem of iterated logarithm behavior for the least squares estimates in the regression model (1.24). The choice of the n_k and I_k in Theorem 1 (iii) for this problem is of particular interest, and it involves much deeper ideas than the standard choice (1.20).

In time series analysis, an important class of covariance stationary models is defined by

$$z_n = \sum_{i=-\infty}^{\infty} c_{n-i}\varepsilon_i, \tag{1.25}$$

where $\cdots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots$ are independent random variables with $E\varepsilon_n = 0$ and $E\varepsilon_n^2 = \sigma^2$, and c_n are constants such that $\sum_{-\infty}^{\infty} c_n^2 < \infty$. The sequence $\{z_n\}$, called the linear process generated by $\{\varepsilon_n\}$, is a covariance stationary process and therefore

$$E(\sum_{j=m+1}^{n} z_j)^2 = E(\sum_{j=1}^{n-m} z_j)^2 = g(n-m), \quad \text{say, for} \quad n > m. \tag{1.26}$$

Noting that $\sum_{j=1}^{n} z_j = \sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i$ and that $E(\sum_{j=m+1}^{n} z_j)^2 = \sigma^2 \sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})^2$, for $n > m$, where

$$a_{ni} = \sum_{j=1}^{n} c_{j-i}, \tag{1.27}$$

we make use of Theorem 1 (ii) and (iii) to obtain a log log law for the partial sums $\sum_1^n z_i$ of linear processes in Section 3, and extend the recent results of Lai and Stout [4] for the Gaussian case to general linear processes.

In Section 4, we shall prove parts (i) and (ii) of Theorem 1. In this connection, we also extend the upper half of the log log law for random variables of the form (1.3) to a general class of random variables that satisfy certain moment restrictions and exponential inequalities. Part (iii) of Theorem 1 is proved in Section 5, where some further applications of Theorem 1 are also discussed.

2. A log log law for least squares estimates in regression models. Consider the multiple regression model

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i, \qquad i = 1, 2, \cdots,$$

where $p \geq 2$, ε_i are random variables satisfying (1.2) for some $r > 2$, and x_{ij} are constants. Letting

$$\mathbf{T}_i = (x_{i2}, \cdots, x_{ip}), \qquad \mathbf{H}_n = \sum_{i=1}^{n} \mathbf{T}_i'\mathbf{T}_i, \qquad \mathbf{K}_n = \sum_{i=1}^{n} x_{i1}\mathbf{T}_i, \tag{2.1}$$

the least squares estimate b_{n1} of β_1 based on the observations $(x_{11}, \cdots, x_{1p}, y_1), \cdots, (x_{n1}, \cdots, x_{np}, y_n)$ is given by

$$b_{n1} = \{\sum_{i=1}^{n} (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')y_i\}/\{\sum_{i=1}^{n} (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')^2\}, \tag{2.2}$$

provided that $\mathbf{H}_n$ is nonsingular (cf. [3], page 349). Here and throughout this section we let $'$ denote transpose. Note that the vector

$$(\mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_1', \cdots, \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_n') \in R^n \tag{2.3}$$

which appears in the numerator of (2.2) is the projection of the vector $(x_{11}, \cdots, x_{n1})$ into the linear space spanned by

$$(x_{12}, \cdots, x_{n2}), \cdots, (x_{1p}, \cdots, x_{np}), \tag{2.4}$$

and that the matrix $\mathbf{H}_n$ is nonsingular iff the $(p-1)$ vectors in (2.4) are linearly independent. By permuting the column indices $1, \cdots, p$, we obtain a similar expression for the least squares estimate b_{nj} of β_j. Therefore, without loss of generality, we shall only consider b_{n1}. Note also that if $\mathbf{H}_m$ is nonsingular, then $\mathbf{H}_n$ is also nonsingular for all $n \geq m$ (cf. [3]).

From (2.2), it follows that

$$b_{n1} - \beta_1 = \{\textstyle\sum_{i=1}^n (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')\varepsilon_i\}/\{\sum_{i=1}^n (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')^2\}. \tag{2.5}$$

Therefore b_{n1} is an unbiased estimate of β_1 with variance σ^2/A_n, where $A_n = \sum_{i=1}^n (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')^2$. The strong consistency of b_{n1} has recently been established in [3], where it is shown that

$$b_{n1} \to \beta_1 \quad \text{a.s.} \quad \Leftrightarrow A_n \to \infty. \tag{2.6}$$

We now make use of Theorem 1 to obtain a log log law for b_{n1} of the form

$$\limsup_{n\to\infty} \{(A_n/2 \log\log A_n)^{1/2} | b_{n1} - \beta_1 |\} = \sigma \quad \text{a.s.} \tag{2.7}$$

under weak assumptions on the design constants x_{ij}. This is the content of the following theorem.

THEOREM 2. *Let* $\varepsilon_1, \varepsilon_2, \cdots$ *be independent random variables such that* (1.2) *holds for some* $r > 2$, *and let* $\{x_{ij}: i = 1, 2, \cdots; j = 1, \cdots, p\}$ *be a double array of constants* ($p \geq 2$). *Define* $\mathbf{T}_i, \mathbf{H}_n, \mathbf{K}_n$ *by* (2.1), *and assume that* $\mathbf{H}_N$ *is nonsingular for some* $N \geq p - 1$. *Let*

$$A_n = \textstyle\sum_{i=1}^n (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')^2, \qquad n \geq N. \tag{2.8}$$

Suppose that

$$\lim_{n\to\infty} A_n = \infty, \ \limsup_{n\to\infty} (A_{n+1}/A_n) < \infty, \qquad \textit{and} \tag{2.9}$$

$$\max_{1\leq i\leq n} (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')^2 = o(A_n(\log A_n)^{-\rho}) \quad \textit{for all} \quad \rho > 0. \tag{2.10}$$

Then

$$\limsup_{n\to\infty} |\textstyle\sum_{i=1}^n (x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i')\varepsilon_i|/(2A_n \log\log A_n)^{1/2} = \sigma \quad \text{a.s.} \tag{2.11}$$

The proof of Theorem 2 depends on Theorem 1 and the following two lemmas. The proofs of these lemmas are of an algebraic nature and are given at the end of this section after the proof of Theorem 2.

LEMMA 1. *Let* $p \geq 2$, *and let* $\{x_{ij}: i = 1, 2 \cdots; j = 1, \cdots, p\}$ *be a double array of constants. Define* $\mathbf{T}_i, \mathbf{H}_n, \mathbf{K}_n$ *by* (2.1), *and assume that* $\mathbf{H}_N$ *is nonsingular for some* $N \geq p - 1$. *For* $n \geq N$, *let*

$$a_{ni} = x_{i1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_i' \qquad (i = 1, \cdots, n). \tag{2.12}$$

Define A_n *by* (2.8) *so that* $A_n = \sum_{i=1}^n a_{ni}^2$, *and let*

$$d_n = a_{nn} = x_{n1} - \mathbf{K}_n\mathbf{H}_n^{-1}\mathbf{T}_n'. \tag{2.13}$$

Then for $n \geq m > N$,

$$A_n = A_N + \textstyle\sum_{i=N+1}^n d_i^2(1 + \mathbf{T}_i\mathbf{H}_{i-1}^{-1}\mathbf{T}_i'), \tag{2.14}$$

$$\textstyle\sum_{i=1}^m a_{ni}a_{mi} = \sum_{i=1}^m a_{mi}^2. \tag{2.15}$$

Moreover, for $n \geq k \geq m > N$,

$$\textstyle\sum_{i=1}^m a_{ni}^2 \leq 2A_k + 2A_n \operatorname{tr}\{\mathbf{H}_m(\mathbf{H}_k^{-1} - \mathbf{H}_n^{-1})\}, \tag{2.16}$$

where tr $\mathbf{H}$ *denotes the trace of* $\mathbf{H}$.

LEMMA 2. *Let* $\mathbf{T}_i = (t_{i1}, \cdots, t_{ik}) \in R^k$ *and let* $\mathbf{H}_n = \sum_1^n \mathbf{T}_i'\mathbf{T}_i$. *Assume that* $\mathbf{H}_N$ *is nonsingular. Then*

(*i*) $\operatorname{tr}(\mathbf{H}_m\mathbf{H}_n^{-1}) \geq 0$ *for all* $n \geq N$ *and all* m.

(ii) *Let* $N \le m_1 \le m_2 \le \cdots$. *Then for any given positive integers* ν *and* L, *there exists* j *such that* $\nu < j \le \nu + kL$ *and* $\operatorname{tr}\{\mathbf{H}_{m_\nu}(\mathbf{H}_{m_{j-1}}^{-1} - \mathbf{H}_{m_j}^{-1})\} \le 1/L$.

PROOF OF THEOREM 2. With the same notation as in Lemma 1, set $a_{ni} = 0$ if $i \le 0$ or $i > n$. In view of (2.9) and (2.10), (1.5) and (1.6) hold. Moreover, it follows from (2.15) that for $n > m \ge N$,

$$\sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})^2 = \sum_{i=1}^{m} (a_{ni} - a_{mi})^2 + \sum_{i=m+1}^{n} a_{ni}^2 = \sum_{i=1}^{n} a_{ni}^2 - \sum_{i=1}^{m} a_{mi}^2 = A_n - A_m,$$

and therefore in view of (2.14), (1.7) and (1.8) hold with $d = 1$ and $c_i = d_i^2(1 + \mathbf{T}_i\mathbf{H}_{i-1}^{-1}\mathbf{T}_i')$. Hence Theorem 1(i) is applicable and gives the upper half of (2.11).

To prove the lower half of (2.11), we now show that Theorem 1 (iii) is applicable. Given $0 < \gamma < 1$, we can choose in view of (2.9) an integer $L > \max\{A_N^{1/2}, 4/\gamma\}$ such that

$$A_{n+1} \le LA_n \quad \text{for all} \quad n \ge N. \tag{2.17}$$

Define $m_k = \inf\{n \ge N : A_n \ge L^{2k}\}$. Then $m_1 > N$, and by (2.17),

$$L^{2k} \le A_{m_k} \le LA_{m_k - 1} < L^{2k+1} \text{ for all } k. \tag{2.18}$$

Let $n_0 = 0$, $n_1 = m_1$. Suppose that we have defined n_i for $i = 1, \cdots, k$ such that

$$n_i = m_{\nu(i)} \text{ for some } i \le \nu(i) \le i(p-1)L, \text{ and} \tag{2.19}$$

$$\sum_{1 \le t \le n_{i-1}} a_{n_i,t}^2 \le \gamma A_{n_i}. \tag{2.20}$$

We now define n_{k+1} so that (2.19) and (2.20) hold for $i = k + 1$. Let $\nu = \nu(k)$. By Lemma 2(ii), there exists j such that $\nu < j \le \nu + (p-1)L$ and

$$\operatorname{tr}\{\mathbf{H}_{m_\nu}(\mathbf{H}_{m_{j-1}}^{-1} - \mathbf{H}_{m_j}^{-1})\} \le 1/L. \tag{2.21}$$

Since $\nu \le j - 1$, it follows from (2.16) and (2.21) that

$$\begin{aligned} \sum_{1 \le t \le m_\nu} a_{m_j,t}^2 &\le 2A_{m_{j-1}} + 2A_{m_j}\operatorname{tr}\{\mathbf{H}_{m_\nu}(\mathbf{H}_{m_{j-1}}^{-1} - \mathbf{H}_{m_j}^{-1})\} \\ &\le 2A_{m_{j-1}} + 2A_{m_j}/L < 4A_{m_j}/L. \end{aligned} \tag{2.22}$$

The last inequality above follows from (2.18) since $A_{m_{j-1}} < L^{2(j-1)+1}$ and $A_{m_j} \ge L^{2j}$. Defining $n_{k+1} = m_j$, it is clear that (2.19) holds for $i = k + 1$. Moreover, since $4/L < \gamma$, (2.22) implies that (2.20) also holds for $i = k + 1$. Thus by induction we have defined positive integers $n_1 < n_2 < \cdots$ such that (2.19) and (2.20) hold. Let $I_k = \{n : n_{k-1} < n \le n_k\}$. The condition (2.20) implies that (1.13) is satisfied, recalling that $a_{ni} = 0$ for $i > n$. Moreover, by (2.18) and (2.19),

$$L^{2k} \le A_{m_k} \le A_{n_k} \le A_{m_{(p-1)Lk}} < L^{2(p-1)Lk+1},$$

so $\log\log A_{n_k} \sim \log k$, and therefore (1.14) and (1.15) are satisfied. Hence Theorem 1(iii) is applicable and gives the lower half of the log log law (2.11). □

PROOF OF LEMMA 2. (i) follows from the positive definiteness of $\mathbf{H}_n^{-1}$ and the equality

$$\operatorname{tr}(\mathbf{H}_m\mathbf{H}_n^{-1}) = \sum_{i=1}^{m} \operatorname{tr}(\mathbf{T}_i'\mathbf{T}_i\mathbf{H}_n^{-1}) = \sum_{i=1}^{m} \mathbf{T}_i\mathbf{H}_n^{-1}\mathbf{T}_i',$$

noting that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$ for the last relation above. To prove (ii), suppose that for all $\nu < j \le \nu + kL$,

$$\operatorname{tr}\{\mathbf{H}_{m_\nu}(\mathbf{H}_{m_{j-1}}^{-1} - \mathbf{H}_{m_j}^{-1})\} > 1/L.$$

Then

$$k < \sum_{j=\nu+1}^{\nu+kL} \operatorname{tr}\{\mathbf{H}_{m_\nu}(\mathbf{H}_{m_{j-1}}^{-1} - \mathbf{H}_{m_j}^{-1})\} = \operatorname{tr}(\mathbf{H}_{m_\nu}\mathbf{H}_{m_\nu}^{-1} - \mathbf{H}_{m_\nu}\mathbf{H}_{m_{\nu+kL}}^{-1}) = k - \operatorname{tr}(\mathbf{H}_{m_\nu}\mathbf{H}_{m_{\nu+kL}}^{-1}),$$

contradicting (i). □

PROOF OF LEMMA 1. The relation (2.14) has been established by Anderson and Taylor [1], and simple algebra shows that both sides of (2.15) are equal to $\sum_{i=1}^{m} x_{i1}^2 - \mathbf{K}_m\mathbf{H}_m^{-1}\mathbf{K}_m'$.

To prove (2.16), we first prove the identity

$$\sum_{i=1}^{m} a_{ni}^2 = A_m + \sum_{i=1}^{m} \{(\sum_{j=m}^{n-1} d_{j+1}\mathbf{T}_{j+1}\mathbf{H}_j^{-1})\mathbf{T}_i'\}^2 \quad \text{for} \quad n \ge m > N. \tag{2.23}$$

Note that

$$\begin{aligned}\sum_{i=1}^{m} a_{ni}^2 &= \sum_{i=1}^{m} a_{mi}^2 + \sum_{i=1}^{m} (a_{ni} - a_{mi})^2, \quad \text{by (2.15)},\\ &= A_m + \sum_{i=1}^{m} \{(\mathbf{K}_m\mathbf{H}_m^{-1} - \mathbf{K}_n\mathbf{H}_n^{-1})\mathbf{T}_i'\}^2, \quad \text{by (2.12)},\\ &= A_m + \sum_{i=1}^{m} \{\sum_{j=m}^{n-1} (\mathbf{K}_j\mathbf{H}_j^{-1} - \mathbf{K}_{j+1}\mathbf{H}_{j+1}^{-1})\mathbf{T}_i'\}^2;\end{aligned} \tag{2.24}$$

$$\begin{aligned}\mathbf{K}_{j+1}\mathbf{H}_{j+1}^{-1} &= \mathbf{K}_{j+1}\mathbf{H}_{j+1}^{-1}(\mathbf{H}_{j+1} - \mathbf{T}_{j+1}'\mathbf{T}_{j+1})\mathbf{H}_j^{-1}, \text{ by (2.1)},\\ &= (\mathbf{K}_{j+1} - \mathbf{K}_{j+1}\mathbf{H}_{j+1}^{-1}\mathbf{T}_{j+1}'\mathbf{T}_{j+1})\mathbf{H}_j^{-1}.\end{aligned} \tag{2.25}$$

Since $\mathbf{K}_j - \mathbf{K}_{j+1} = -x_{j+1,1}\mathbf{T}_{j+1}$, it follows from (2.25) that

$$\mathbf{K}_j\mathbf{H}_j^{-1} - \mathbf{K}_{j+1}\mathbf{H}_{j+1}^{-1} = -(x_{j+1,1} - \mathbf{K}_{j+1}\mathbf{H}_{j+1}^{-1}\mathbf{T}_{j+1}')\mathbf{T}_{j+1}\mathbf{H}_j^{-1}. \tag{2.26}$$

From (2.13), (2.24), and (2.26), (2.23) follows.

The next step in proving (2.16) is to establish the inequality: For $n \ge k \ge m > N$,

$$\sum_{i=1}^{m} \{(\sum_{j=k}^{n-1} d_{j+1}\mathbf{T}_{j+1}\mathbf{H}_j^{-1})\mathbf{T}_i'\}^2 \le A_n \operatorname{tr}\{\mathbf{H}_m(\mathbf{H}_k^{-1} - \mathbf{H}_n^{-1})\}. \tag{2.27}$$

Letting $e_j = 1 + \mathbf{T}_j'\mathbf{H}_{j-1}^{-1}\mathbf{T}_j$, we note that

$$\mathbf{H}_{j+1}^{-1} = (\mathbf{H}_j + \mathbf{T}_{j+1}'\mathbf{T}_{j+1})^{-1} = \mathbf{H}_j^{-1} - (\mathbf{H}_j^{-1}\mathbf{T}_{j+1}'\mathbf{T}_{j+1}\mathbf{H}_j^{-1})/e_{j+1} \tag{2.28}$$

(cf. [8, page 29]). Since the case $n = k$ is trivial, we shall assume the $n > k$. By the Schwarz inequality,

$$\begin{aligned}&\sum_{i=1}^{m} \{(\sum_{j=k}^{n-1} d_{j+1}\mathbf{T}_{j+1}\mathbf{H}_j^{-1}\mathbf{T}_i')\}^2\\ &\quad\le \sum_{i=1}^{m} (\sum_{j=k}^{n-1} d_{j+1}^2 e_{j+1})\{\sum_{j=k}^{n-1} (\mathbf{T}_{j+1}\mathbf{H}_j^{-1}\mathbf{T}_i')^2/e_{j+1}\}\\ &\quad\le A_n \sum_{i=1}^{m} \sum_{j=k}^{n-1} (\mathbf{T}_{j+1}\mathbf{H}_j^{-1}\mathbf{T}_i'\mathbf{T}_i\mathbf{H}_j^{-1}\mathbf{T}_{j+1})/e_{j+1}, \quad \text{by (2.14)},\\ &\quad= A_n \sum_{j=k}^{n-1}(\mathbf{T}_{j+1}\mathbf{H}_j^{-1}\mathbf{H}_m\mathbf{H}_j^{-1}\mathbf{T}_{j+1}')/e_{j+1}, \quad \text{since } \mathbf{H}_m = \sum_{i=1}^{m} \mathbf{T}_i'\mathbf{T}_i,\\ &\quad= A_n \sum_{j=k}^{n-1} \operatorname{tr}(\mathbf{H}_m\mathbf{H}_j^{-1}\mathbf{T}_{j+1}'\mathbf{T}_{j+1}\mathbf{H}_j^{-1})/e_{j+1}, \quad \text{since } \operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA}),\\ &\quad= A_n \sum_{j=k}^{n-1} \operatorname{tr}\{\mathbf{H}_m(\mathbf{H}_j^{-1} - \mathbf{H}_{j+1}^{-1})\}, \quad \text{by (2.28)}.\end{aligned}$$

Hence (2.27) holds.

To prove (2.16), writing $\sum_{j=m}^{n-1}$ in (2.23) as $\sum_{j=m}^{k-1} + \sum_{j=k}^{n-1}$ and applying (2.23) and the inequality $(x+y)^2 \le 2(x^2 + y^2)$, we obtain that

$$\begin{aligned}\sum_{i\le 1}^{m} a_{ni}^2 &\le 2\sum_{i=1}^{m} a_{ki}^2 + 2\sum_{i=1}^{m} \{(\sum_{j=k}^{n-1} d_{j+1}\mathbf{T}_{j+1}\mathbf{H}_j^{-1})\mathbf{T}_i'\}^2\\ &\le 2A_k + 2A_n \operatorname{tr}\{\mathbf{H}_m(\mathbf{H}_k^{-1} - \mathbf{H}_n^{-1})\}, \quad \text{by (2.27)}. \quad \square\end{aligned}$$

3. A log log law for partial sums of linear processes. In this section we apply Theorem 1 to obtain the following:

THEOREM 3. *Let* $\cdots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots$ *be independent random variables such that* (1.2) *holds for some* $r > 2$. *Let* c_n *be constants such that* $\sum_{-\infty}^{\infty} c_n^2 < \infty$, *and let* $z_n = \sum_{i=-\infty}^{\infty} c_{n-i}\varepsilon_i$ *be the linear process generated by* $\{\varepsilon_n\}$. *Let* $S_n = \sum_1^n z_i$ *and let* $g(n) = ES_n^2$.

(i) *Suppose that* g *satisfies conditions* (1.11) *and* (1.12). *Then*

$$\limsup_{n\to\infty} |S_n|/\{2g(n)\log\log g(n)\}^{1/2} \le 1 \quad \text{a.s.} \tag{3.1}$$

(ii) *Suppose that* $\liminf_{n\to\infty} (\log\log g(n))/(\log\log n) > 0$, *and*

$$\sum_{|i|\le n\exp(-(\log n)^\alpha)} \left(\sum_{j=1-i}^{n-i} c_j\right)^2 = o(g(n))\ \forall\ \alpha > 0, \tag{3.2}$$

$$\sum_{|i|\ge n\exp((\log n)^\alpha)} \left(\sum_{j=1-i}^{n-i} c_j\right)^2 = o(g(n))\ \forall\ \alpha > 0. \tag{3.3}$$

Then for every $-1 \le q \le 1$,

$$\liminf_{n\to\infty} |\{2\,g(n)\log\log g(n)\}^{-1/2}S_n - q| = 0 \qquad \text{a.s.} \tag{3.4}$$

REMARKS AND EXAMPLES. (I) Condition (3.3) is satisfied if

$$\sum_{|i|\ge n\exp((\log n)^\alpha)} c_i^2 = o(n^{-2}g(n)) \quad \text{for all} \quad \alpha > 0. \tag{3.5}$$

To see this, apply the Schwarz inequality to obtain that for $m > n$

$$\sum_{|i|\ge m} \left(\sum_{j=1-i}^{n-i} c_j\right)^2 \le n \sum_{|i|\ge m} \sum_{j=1-i}^{n-i} c_j^2 \le n^2 \sum_{|j|\ge m-n} c_j^2,$$

and therefore (3.5) implies (3.3).

(II) The linear process $z_n = \sum_{i=-\infty}^{\infty} c_{n-i}\varepsilon_i$ is covariance stationary and has an absolutely continuous spectral distribution (cf. [2]), and therefore by the Riemann-Lebesgue lemma, $\rho(n) = Ez_0z_n \to 0$ as $n \to \infty$. Consequently,

$$g(n) = n\rho(0) + 2\sum_{i=1}^{n-1} (n-i)\rho(i) = o(n^2) \quad \text{as} \quad n \to \infty. \tag{3.6}$$

(III) Suppose that $z_n = \varepsilon_n$ (i.e., $c_0 = 1$ and $c_i = 0$ for $i \ne 0$) and that $\sigma^2 (= E\varepsilon_0^2) > 0$. Then $S_n = \sum_1^n \varepsilon_i$, $g(n) = \sigma^2 n$, and (3.2) and (3.3) trivially hold. Moreover, g obviously satisfies conditions (1.11) and (1.12). Therefore, in this case, Theorem 3 reduces to the classical log log law for partial sums of the independent random variables ε_n.

(IV) Lai and Stout [4, Theorem 1] obtained the upper half of the law of the iterated logarithm for partial sums $\sum_1^n y_i$ of stationary Gaussian sequences $\{y_n\}$ with $g(n) = E(\sum_1^n y_i)^2$ satisfying (1.11) and (1.12). It is known (cf. [2]) that if the spectral distribution of $\{y_n\}$ is absolutely continuous, then y_n is a linear process of the form $y_n = \sum_{i=-\infty}^{\infty} c_{n-i}\varepsilon_i$, where the ε_i are i.i.d. standard Gaussian random variables (and therefore satisfy (1.2) for all $r > 0$). Thus, under the additional assumption of an absolutely continuous spectral distribution for $\{y_n\}$, the Lai-Stout result for stationary Gaussian sequences is a special case of Theorem 3(i).

(V) As pointed out in [4], the conditions (1.11) and (1.12) cover a wide range of correlation structures for the sequences $\{z_n\}$. In particular, letting $\rho(n) = E(z_0z_n)$, if $\rho(0) > 0$ and $\rho(n) \ge 0$ for all n, then $g(n)$ is increasing and satisfies (1.12), $g(2n) \ge 2g(n)$ and therefore (1.11) holds with $K = 2$ (since $2/r < 1$). Another example considered in [4] is

$$\rho(n) \sim n^{\lambda-2}L(n) \quad \text{for some} \quad 1 < \lambda < 2, \tag{3.7}$$

where $L(n)$ is a positive slowly varying function. Then

$$g(n) = n\rho(0) + 2\sum_{i=1}^{n-1} (n-i)\rho(i) \sim 2\{\lambda(\lambda-1)\}^{-1}n^\lambda L(n), \tag{3.8}$$

and therefore g satisfies (1.11) and (1.12). Since $\rho(n) = \sigma^2\sum_{i=-\infty}^{\infty} c_ic_{n+i}$, an example of c_i for which $\rho(n)$ satisfies (3.7) is

$$c_i \sim K|i|^{-(3-\lambda)/2}L^{1/2}(|i|) \qquad \text{as } |i| \to \infty, \qquad \text{where} \tag{3.9}$$

$$(\sigma K)^{-2} = \int_{-\infty}^{\infty} |t|^{-(3-\lambda)/2}|1+t|^{-(3-\lambda)/2}\,dt.$$

As can be easily shown, the above choice of c_i also satisfies conditions (3.2) and (3.5).

(VI) Suppose that $\sum_{-\infty}^{\infty} |c_i| < \infty$, $\sum_{-\infty}^{\infty} c_i \ne 0$, *and* $\sigma^2\ (=E\epsilon_0^2) > 0$. *Then* $\sum_{-\infty}^{\infty} |E(z_0z_n)| \le \sigma^2(\sum_{-\infty}^{\infty} |c_i|)^2\ \infty$ and

$$g(n) \sim n\sigma^2(\textstyle\sum_{-\infty}^{\infty} c_i)^2 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Obviously g satisfies (1.11) and (1.12). Suppose furthermore that

(3.11) $$\sum_{|i|\ge n} c_i^2 = o(\{n^{-1}\exp((\log n)^\alpha)\}) \quad \text{for every} \quad \alpha > 0.$$

Then in view of (3.10), (3.5) is satisfied. Moreover,

$$\sum_{|i|\le n\exp(-(\log n)^\alpha)} \left(\sum_{j=1-i}^{n-i} c_j\right)^2 \le \left(\sum_{-\infty}^{\infty} |c_j|\right)^2 n\exp(-(\log n)^\alpha),$$

and therefore (3.2) holds.

PROOF OF THEOREM 3. We note that for $n \ge 1$,

(3.12) $$S_n = \sum_1^n z_k = \sum_{i=-\infty}^{\infty} \left(\sum_{k=1}^n c_{k-i}\right)\varepsilon_i = \sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i,$$

where $a_{ni} = \sum_{j=1-i}^{n-i} c_j$. Therefore

(3.13) $$g(n) = ES_n^2 = \sigma^2 \sum_{i=-\infty}^{\infty} a_{ni}^2 = \sigma^2 A_n.$$

Note that $\lim_{n\to\infty} g(n) = \infty$ under the assumptions of part (i) or (ii) of the theorem (cf. [4]). Therefore $\sigma > 0$, $\lim_{n\to\infty} A_n = \infty$ and (1.5) holds. We now show that (1.6) also holds. By the Schwarz inequality,

(3.14) $$\sum_{m\le j<m+k} |c_j| \le Ck^{1/2}, \quad \text{where} \quad C^2 = \sum_{-\infty}^{\infty} c_j^2.$$

Since $\sup_i |a_{ni}| \le A_n^{1/2}$, it then follows from (3.14) that for $h \le i \le h+k$ and $n = 1, 2, \cdots$,

(3.15) $$\begin{aligned} |a_{ni}^2 - a_{nh}^2| &\le 2A_n^{1/2}\left\{\sum_{1-i\le j<1-h} |c_j| + \sum_{n-i<j\le n-h} |c_j|\right\} \\ &\le 4CA_n^{1/2}k^{1/2}. \end{aligned}$$

To the contrary of (1.6), suppose that there exist $\rho > 0$, $B > 0$ and integers i_m such that

(3.16) $$a_{m,i_m}^2 \ge 2BA_m(\log A_m)^{-\rho}$$

along some subsequence $\{m\}$ of $\{1, 2, \cdots\}$. Let $\lambda > \rho$. Then

$$\begin{aligned} A_m &\ge \sum_{i_m\le i<i_m+(\log A_m)^\lambda} a_{mi}^2 \\ &\ge \sum_{i_m\le i<i_m+(\log A_m)^\lambda} \{a_{m,i_m}^2 - 4CA_m^{1/2}(\log A_m)^{\lambda/2}\}, \qquad \text{by (3.15)}, \\ &\ge BA_m(\log A_m)^{\lambda-\rho} \quad \text{for all large } m, \qquad \text{by (3.16)}. \end{aligned}$$

Since $\lambda > \rho$, the above string of inequalities leads to a contradiction, so (1.6) holds.

Since $g(n-m) = E(\sum_{j=m+1}^n z_j)^2 = \sigma^2\sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})^2$ by (3.12), part (i) of Theorem 3 follows from Theorem 1(ii). To prove part (ii) of Theorem 3, given $\gamma > 0$, choose $1 < \delta < 1 + \gamma$ and define for $k = 1, 2, \cdots$,

(3.17) $$m_k = [\exp(k^\delta)], \qquad n_k = m_{2k}, \qquad I_k = \{n : m_{2k-1} < |n| < m_{2k+1}\}.$$

We now show that the assumptions (1.13), (1.14) and (1.15) of Theorem 1 (iii) are satisfied by this choice of n_k and I_k. By (3.6), $A_{n_k} = \sigma^{-2}g(n_k) = o(n_k^2)$, *and therefore*

$$\limsup_{k\to\infty} (\log\log A_{n_k})/(\log k) \le \lim_{k\to\infty} (\log\log n_k^2)/(\log k) = \delta.$$

Hence (1.14) holds. Moreover, since $\liminf_{n\to\infty} (\log\log g(n))/(\log\log n) > 0$, (1.15) is satisfied. To show that (1.13) also holds, we note that

$$\begin{aligned} m_{2k+1} &= m_{2k}\exp\{(\delta + o(1))(2k)^{\delta-1}\} = n_k\exp\{(\delta + o(1))(\log n_k)^{(\delta-1)/\delta}\}, \\ m_{2k-1} &= n_k\exp\{-(\delta + o(1))(\log n_k)^{(\delta-1)/\delta}\}. \end{aligned}$$

Therefore by (3.2) and (3.3),

$$\sum_{i\notin I_k} a_{n_k,i}^2 = \sum_{|i|\le m_{2k-1}} \left(\sum_{j=1-i}^{n_k-i} c_j\right)^2 + \sum_{|i|\ge m_{2k+1}} \left(\sum_{j=1-i}^{n_k-i} c_j\right)^2 = o(g(n_k)).$$

Hence Theorem 1 (iii) is applicable and gives the desired conclusion (3.4). □

4. Upper half of the log log law and proof of Theorem 1(i)-(ii). In this section we obtain the upper half of the log log law in parts (i) and (ii) of Theorem 1 from the following more general theorem.

THEOREM 4. *Let $\{Y_n\}$ be a sequence of random variables and $\{B_n\}$ be a sequence of positive constants such that $\lim_{n\to\infty} B_n = \infty$. Assume that there exist $\theta > 1$ and $\tau(\theta) > 0$ such that as $n \to \infty$*

(4.1) $$P[|Y_n| \geq \tau(\theta)(B_n \log\log B_n)^{1/2}] = O(\exp(-\theta \log\log B_n)).$$

(i) *Suppose there exist constants $c_i \geq 0$, $q > 0$ and $\lambda > \theta/(\theta - 1)$ such that*

(4.2) $$E|Y_n - Y_m|^q \leq \left(\textstyle\sum_{i=m+1}^n c_i\right)^\lambda \quad \text{for} \quad n > m \geq m_0, \quad \text{and}$$

(4.3) $$\left(\textstyle\sum_{i=m_0}^n c_i\right)^{2\lambda/q} = O(B_n) \quad \text{as} \quad n \to \infty.$$

Then

(4.4) $$\limsup_{n\to\infty} |Y_n|/(B_n \log\log B_n)^{1/2} \leq \tau(\theta) \qquad \text{a.s.}$$

(ii) *Suppose there exist $q > 0$ and $f: \{1, 2, \cdots\} \to (0, \infty)$ such that*

(4.5) $$E|Y_n - Y_m|^q \leq f(n - m) \quad \text{for} \quad n > m \geq m_0,$$

(4.6) $$f(n) = O(B_n^{q/2}) \quad \text{as} \quad n \to \infty,$$

(4.7) $$\liminf_{n\to\infty} f(Kn)/f(n) \geq K^\lambda \quad \text{for some} \quad \lambda > \theta/(\theta - 1) \quad \text{and integer } K \geq 2,$$

(4.8) $$\forall \gamma > 0, \quad \exists \delta < 1 \text{ such that} \quad \limsup_{n\to\infty} \{\max_{\delta n \leq i \leq n} f(i)/f(n)\} < 1 + \gamma.$$

Then (4.4) *still holds.*

PROOF. To prove (i), since $\lambda > \theta/(\theta - 1)$, we can choose $0 < \delta < 1$ such that

(4.9) $$(1 - \delta)\lambda > 1 \quad \text{and} \quad \theta\delta > 1.$$

Without loss of generality, we shall assume that $\sum_{i=m_0}^\infty c_i = \infty$. Let

(4.10) $$n_k = \inf\{n \geq m_0 : \textstyle\sum_{i=m_0}^n c_i \geq \exp(k^\delta)\}.$$

If $n_{k+1} = n_k$, set $m_k = n_k$; otherwise take m_k such that $n_k \leq m_k < n_{k+1}$ and

(4.11) $$B_{m_k} = \min\{B_j : n_k \leq j < n_{k+1}\}.$$

By (4.3) and (4.10),

(4.12) $$\exp(k^\delta) = O(B_{m_k}^{q/(2\lambda)}), \quad \text{so} \quad \theta \log\log B_{m_k} \geq (\theta\delta + o(1))\log k.$$

Since $\theta\delta > 1$, it follows from (4.1), (4.12), and the Borel-Cantelli Lemma that

(4.13) $$P[|Y_{m_k}| \geq \tau(\theta)(B_{m_k} \log\log B_{m_k})^{1/2} \text{ i.o.}] = 0.$$

By a result of Longnecker and Serfling [6, Lemma 2], since $\lambda > 1$, condition (4.2) implies that there exists an absolute constant $A_{q,\lambda}$ depending only on q and λ such that

$$E(\max_{m\leq j\leq n} |Y_j - Y_m'|^q) \leq A_{q,\lambda}\left(\textstyle\sum_{j=m+1}^n c_j\right)^\lambda \quad \text{for} \quad n > m \geq m_0.$$

Since $|Y_i - Y_j| \leq |Y_i - Y_m| + |Y_j - Y_m|$, the above inequality in turn implies that

(4.14) $$E(\max_{m\leq i,j\leq n} |Y_i - Y_j|^q) \leq 2^q A_{q,\lambda}\left(\textstyle\sum_{j=m+1}^n c_j\right)^\lambda \quad \text{for} \quad n > m \geq m_0.$$

By (4.14), for every $\alpha > 0$,

$$P[\max_{n_k \le j < n_{k+1}} |Y_j - Y_{m_k}| \ge \alpha(B_{m_k} \log\log B_{m_k})^{1/2}]$$
$$\le (2/\alpha)^q A_{q,\lambda}(\textstyle\sum_{n_k+1 \le j \le n_{k+1}-1} c_j)^\lambda (B_{m_k}\log\log B_{m_k})^{-q/2}$$
$$\le (2/\alpha)^q A_{q,\lambda}\{\exp((k+1)^\delta) - \exp(k^\delta)\}^\lambda (B_{m_k}\log\log B_{m_k})^{-q/2}$$
$$= O(k^{-\lambda(1-\delta)}), \quad \text{by (4.12).}$$

Therefore by the Borel-Cantelli Lemma,

(4.15) $$P[\max_{n_k \le j < n_{k+1}} |Y_j - Y_{m_k}| \ge \alpha(B_{m_k}\log\log B_{m_k})^{1/2} \text{ i.o.}] = 0.$$

Since α is arbitrary, the desired conclusion (4.4) follows from (4.11), (4.13), and (4.15).

To prove (ii), as shown in Lemma 2 of [4], the assumptions (4.7) and (4.8) imply that given $0 < \gamma < \lambda$, there exists $N \ge m_0$ such that

(4.16) $$f([an])/f(n) > a^\gamma \quad \text{for all} \quad a \ge N \quad \text{and} \quad n \ge N,$$

and therefore

(4.17) $$\liminf_{m\to\infty} m^{-\gamma} f(m) > 0.$$

As before, choose δ such that (4.9) holds. In place of (4.10), we now define

(4.18) $$n_k = [\exp(k^\delta)],$$

and choose m_k such that $n_k \le m_k < n_{k+1}$ and (4.11) holds for all large k. By (4.6) and (4.17),

$$\theta \log\log B_{m_k} \ge (\theta + o(1))\log\log f(m_k) \ge (\theta\delta + o(1))\log k.$$

Hence (4.13) still holds.

By a result of Lai and Stout [5, Theorem 5], since $\lambda > 1$, the assumptions (4.7) and (4.8) imply that there exists $C > 0$ such that

(4.19) $$E(\max_{m \le j \le n} |Y_j - Y_m|^q) \le Cf(n-m) \quad \text{for} \quad n > m \ge m_0.$$

Choose γ such that $\gamma < \lambda$ and $(1-\delta)\gamma > 1$. Since

$$n_{k+1} - n_k \le (\delta + o(1))k^{-(1-\delta)} m_k,$$

it follows from (4.16) that

(4.20) $$f(n_{k+1} - n_k) = O(k^{-(1-\delta)\gamma} f(m_k)).$$

Hence by a similar argument as before, for every $\alpha > 0$,

$$P[\max_{n_k \le j < n_{k+1}} |Y_j - Y_{m_k}| \ge \alpha(B_{m_k}\log\log B_{m_k})^{1/2}]$$
$$\le (2/\alpha)^q C f(n_{k+1} - n_k)(B_{m_k}\log\log B_{m_k})^{-q/2}$$
$$= O(k^{-(1-\delta)\gamma}), \quad \text{by (4.6) and (4.20).}$$

Since $(1-\delta)\gamma > 1$, it then follows that (4.15) still holds. □

The following lemma will be used for the proof of Theorem 1 in this and the next section.

LEMMA 3. (i) *Let* $\{X_i, -\infty < i < \infty\}$ *be a sequence of independent random variables such that* $\sum_{-\infty}^{\infty} EX_i^2 \le A$, $EX_i = 0$ *and* $|X_i| \le A^{1/2}c$ a.s. *for all* i, *where* A *and* c *are positive constants. Then for every* $\zeta > 0$ *such that* $c\zeta \le 1$,

(4.21) $$P[|\textstyle\sum_{i=-\infty}^{\infty} X_i| \ge A^{1/2}\zeta] \le 2\exp\{-\tfrac12\zeta^2(1 - \tfrac12 c\zeta)\}.$$

(ii) *Let* $\cdots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots$ *be independent random variables such that* $E\varepsilon_n = 0$ *and* $E\varepsilon_n^2 \le \sigma^2$ *for all* n, *and* $\sup_n E|\varepsilon_n|^r < \infty$ *for some* $r > 2$. *Let* $\{a_{ni}: n \ge 1, -\infty < i < \infty\}$ *be a* *double array of constants satisfying* (1.1), (1.5), *and* (1.6). *Define* S_n *as in* (1.3). *Then for all* $\zeta > 1$ *and* $\theta > 0$,

$$P[|S_n| > \zeta\sigma(2\theta A_n \log\log A_n)^{1/2}] = O(\exp(-\theta \log\log A_n)). \tag{4.22}$$

PROOF. (i) can be proved by the standard argument used to establish Kolmogorov's exponential inequalities (cf. [9], page 263). To prove (ii), we first note that

$$\begin{aligned} &P[|a_{ni}\varepsilon_i| \geq A_n^{1/2}(\log\log A_n)^{-1} \quad \text{for some} \quad i] \\ &\quad \leq \textstyle\sum_{i=-\infty}^{\infty} A_n^{-r/2}(\log\log A_n)^r |a_{ni}|^r E|\varepsilon_i|^r \\ &\quad \leq (\sup_i E|\varepsilon_i|^r)\{(\sup_i a_{ni}^2)/A_n\}^{(r-2)/2}(\log\log A_n)^r \\ &\quad = o(\exp(-\theta \log\log A_n)), \quad \text{by (1.6)}. \end{aligned} \tag{4.23}$$

Define

$$\tilde{\varepsilon}_i = \varepsilon_i I_{[|a_{ni}\varepsilon_i| < A_n^{1/2}(\log\log A_n)^{-1}]}, \qquad \tilde{S}_n = \textstyle\sum_{i=-\infty}^{\infty} a_{ni}(\tilde{\varepsilon}_i - E\tilde{\varepsilon}_i). \tag{4.24}$$

Then, since $E\tilde{\varepsilon}_i^2 \leq E\varepsilon_i^2 \leq \sigma^2$,

$$E(\tilde{S}_n)^2 = \textstyle\sum_{i=-\infty}^{\infty} a_{ni}^2 \operatorname{Var} \tilde{\varepsilon}_i \leq \sigma^2 A_n, \tag{4.25}$$

and

$$\sup_i(|a_{ni}||\tilde{\varepsilon}_i - E\tilde{\varepsilon}_i|) \leq 2A_n^{1/2}(\log\log A_n)^{-1}. \tag{4.26}$$

Let $\zeta > \zeta' > 1$. It then follows from (4.25), (4.26), and (i) that for all large n

$$P[|\tilde{S}_n| > \zeta'\sigma A_n^{1/2}(2\theta \log\log A_n)^{1/2}] \leq 2\exp(-\theta \log\log A_n). \tag{4.27}$$

Since $E\varepsilon_i = 0$, we obtain that

$$\begin{aligned} &|\textstyle\sum_{i=-\infty}^{\infty} a_{ni}E\tilde{\varepsilon}_i| \leq \sum_{i=-\infty}^{\infty} E|a_{ni}\varepsilon_i| I_{[|a_{ni}\varepsilon_i| \geq A_n^{1/2}(\log\log A_n)^{-1}]} \\ &\quad \leq A_n^{-(r-1)/2}(\log\log A_n)^{r-1} \textstyle\sum_{i=-\infty}^{\infty} E|a_{ni}\varepsilon_i|^r \\ &\quad \leq (\sup_i E|\varepsilon_i|^r)\{(\sup_i a_{ni}^2)/A_n\}^{(r-2)/2} A_n^{1/2}(\log\log A_n)^{r-1} \\ &\quad = o(A_n^{1/2}), \qquad \text{by (1.6)}. \end{aligned} \tag{4.28}$$

From (4.23), (4.27), and (4.28), (4.22) follows. □

PROOF OF THEOREM 1(i). Let $0 < \delta < 1$. In view of (1.2), we can choose $B > 0$ such that

$$E\varepsilon_i^2 I_{[|\varepsilon_i| > B]} \leq \delta^2\sigma^2 \quad \text{for all} \quad i. \tag{4.29}$$

Let $\varepsilon_i' = \varepsilon_i I_{[|\varepsilon_i| \leq B]} - E(\varepsilon_i I_{[|\varepsilon_i| \leq B]})$. Then $E(\varepsilon_i')^2 \leq E\varepsilon_i^2 = \sigma^2$ and therefore by Lemma 3(ii),

$$P[|\textstyle\sum_{i=-\infty}^{\infty} (a_{ni}\varepsilon_i'| > (1 + 2\delta)\sigma(2A_n \log\log A_n)^{1/2} = O(\exp\{-(1+\delta)\log\log A_n\}). \tag{4.30}$$

Moreover, by the Marcinkiewicz-Zygmund inequality [7], since $\sup_n |\varepsilon_n'| \leq 2B$, there exists for every $p > 1$ and absolute constant C_p depending only on p such that for $n \geq m$

$$E|\textstyle\sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})\varepsilon_i'|^p \leq C_p(2B)^p\{\sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})^2\}^{p/2}. \tag{4.31}$$

From (1.7) and (4.31), it then follows that for $n > m \geq n_0$

$$E|\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i' - \sum_{i=-\infty}^{\infty} a_{mi}\varepsilon_i'|^p \leq C_p(2B)^p(\sum_{i=m+1}^{n} c_i)^{pd/2}. \tag{4.32}$$

Choose p large enough such that $pd/2 > (1 + \delta)/\delta$. Then in view of (4.30), (4.32), and (1.8), we can apply Theorem 4(i) and obtain that

$$\limsup_{n\to\infty} |\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i'|/(A_n \log\log A_n)^{1/2} \le (1 + 2\delta)2^{1/2}\sigma \quad \text{a.s.} \tag{4.33}$$

Let $\varepsilon_i'' = \varepsilon_i - \varepsilon_i'$. Since $d > 2/r$, we can choose θ sufficiently large so that

$$\theta > 1, \qquad \theta/(\theta - 1) < rd/2. \tag{4.34}$$

By (4.29), $E(\varepsilon_i'')^2 \le E\varepsilon_i^2 I_{[|\varepsilon_i|>B]} \le \delta^2\sigma^2$ for all i, and therefore it follows from Lemma 3(ii) that

$$P[|\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i''| > (1 + \delta)\delta\sigma(2\theta A_n \log\log A_n)^{1/2}] = O(\exp(-\theta \log\log A_n)). \tag{4.35}$$

Moreover, by the Marcinkiewicz-Zygmund inequality, we obtain in place of (4.31) that for $n > m \ge n_0$

$$\begin{aligned} E|\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i'' - \sum_{i=-\infty}^{\infty} a_{mi}\varepsilon_i''|^r &\le C_r(\sup_i E|\varepsilon_i''|^r)\, \{\textstyle\sum_{i=-\infty}^{\infty} (a_{ni} - a_{mi})^2\}^{r/2} \\ &\le 2^r C_r(\sup_i E|\varepsilon_i|^r)(\textstyle\sum_{i=m+1}^{n} c_i)^{rd/2}, \text{ by (1.7).} \end{aligned} \tag{4.36}$$

In view of (4.34), (4.35) and (4.36), we can again apply Theorem 4(i) and obtain that

$$\limsup_{n\to\infty} |\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i''|/(A_n \log\log A_n)^{1/2} \le (2\theta)^{1/2}(1 + \delta)\delta\sigma \quad \text{a.s.} \tag{4.37}$$

Since δ is arbitrary, the desired conclusion (1.9) follows from (4.33) and (4.37). □

PROOF OF THEOREM 1(ii). We proceed as in the proof of Theorem 1(i) and replace (4.32) by

$$E|\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i' - \sum_{i=-\infty}^{\infty} a_{mi}\varepsilon_i'|^p \le C_p(2B)^p g^{p/2}(n - m) \tag{4.38}$$

for $n > m \ge m_0$. Choose p large enough such that $p/r > (1 + \delta)/\delta$. Then in view of (1.10), (1.11), and (1.12), we can apply Theorem 4(ii) (with $q = p$ and $f = C_p(2B)^p g^{p/2}$) to obtain (4.33). Likewise, letting $\liminf_{n\to\infty} g(Kn)/g(n) = K^d$ with $d > 2/r$, where K is as given in (1.11), and choosing θ as in (4.34), we replace (4.36) by

$$E|\textstyle\sum_{i=-\infty}^{\infty} a_{ni}\varepsilon_i'' - \sum_{i=-\infty}^{\infty} a_{mi}\varepsilon_i''|^r \le 2^r C_r(\sup_n E|\varepsilon_n|^r) g^{r/2}(n - m), \tag{4.39}$$

and apply Theorem 4(ii) (with $q = r$) to obtain (4.37). □

5. Lower half of the log log law and proof of Theorem 1(iii). We preface the proof of Theorem 1(iii) by the following lemma.

LEMMA 4. *Let* $\cdots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \cdots$ *be independent random variables such that* (1.2) *holds for some* $r > 2$. *Let* $\{a_{ni}: n \ge 1, -\infty < i < \infty\}$ *be a double array of constants satisfying* (1.1), (1.5), *and* (1.6). *Define* S_n *as in* (1.3). *Then for all* $\zeta > 0$, $\xi > 0$ *and* $\theta \ne 0$,

$$P[(\theta - \zeta)\sigma \le (2A_n \log\log A_n)^{-1/2} S_n \le (\theta + \xi)\sigma] \ge \exp(-\theta^2 \log\log A_n) \tag{5.1}$$

for all large n. Moreover, for all $\xi > \zeta > 0$,

$$P[0 \le (2A_n \log\log A_n)^{-1/2} S_n \le \xi\sigma] \ge \exp(-\zeta^2 \log\log A_n) \tag{5.2}$$

for all large n.

PROOF. To prove (5.1), we only consider the case $\theta > 0$, as a similar argument can be applied to the case $\theta < 0$. Take $0 < \zeta' < \zeta$ and $0 < \xi' < \xi$ such that $\theta - \zeta' > 0$. By Lemma 3(ii),

$$P[S_n > (\theta + \xi)\sigma(2A_n \log\log A_n)^{1/2}] = O(\exp\{-(\theta + \xi')^2 \log\log A_n\}). \tag{5.3}$$

Define $\tilde{\varepsilon}_i$, $\tilde{S}_n$ as in (4.24). Then $E(\tilde{S}_n)^2 = \sum_{i=-\infty}^{\infty} a_{ni}^2 \operatorname{Var} \tilde{\varepsilon}_i = (\sigma^2 + o(1))A_n$ and (4.26) holds, so we can apply Kolmogorov's lower exponential bounds (cf. [9], Theorem 5.22(iii)) and obtain that for all large n

(5.4) $$P[\tilde{S}_n \geq (\theta - \zeta')\sigma(2A_n \log\log A_n)^{1/2}] \geq 3\exp(-\theta^2 \log\log A_n).$$

By (4.23), (4.28), and (5.4),

(5.5) $$P[S_n \geq (\theta - \zeta)\sigma(2A_n \log\log A_n)^{1/2}] \geq 2\exp(-\theta^2 \log\log A_n)$$

for all large n. From (5.3) and (5.5), (5.1) follows.

To prove (5.2), letting $\zeta < \tilde{\zeta} < \xi$, we obtain as in (5.3) that

(5.6) $$P[S_n > \xi\sigma(2A_n \log\log A_n)^{1/2}] = O(\exp(-\tilde{\zeta}^2 \log\log A_n)).$$

Take $0 < \gamma < \zeta$. Then as in (5.5),

(5.7) $$P[S_n \geq \gamma\sigma(2A_n \log\log A_n)^{1/2}] \geq 2\exp(-\zeta^2 \log\log A_n)$$

for all large n. From (5.6) and (5.7), (5.2) follows. □

PROOF OF THEOREM 1(iii). Since the case $\sigma = 0$ is trivial, we shall asssme that $\sigma > 0$. By (1.15),

(5.8) $$\log\log A_{n_k} \geq d \log k \quad \text{for all large } k \text{ and some } d > 0.$$

Take $0 < \gamma < d^2$. We first show that

(5.9) $$P[|\textstyle\sum_{i \notin I_k} a_{n_k,i}\varepsilon_i| \leq \sigma(1 + 2\gamma)\gamma^{1/4}(2A_{n_k}\log\log A_{n_k})^{1/2} \quad \text{for all large } k] = 1.$$

Since $\log A_{n_k} \geq k^d$, it follows from (1.6) by an argument as in (4.23) that

(5.10) $$\textstyle\sum_{k=1}^{\infty} P[|a_{n_k,i}\varepsilon_i| \geq A_{n_k}^{1/2}(\log\log A_{n_k})^{-1} \quad \text{for some } i] < \infty.$$

Define $\tilde{\varepsilon}_i$ by (4.24) with $n = n_k$, and note that

(5.11) $$\begin{aligned}\textstyle\sum_{i \notin I_k} a_{n_k,i}^2 \operatorname{Var} \tilde{\varepsilon}_i &= (\sigma^2 + o(1)) \textstyle\sum_{i \notin I_k} a_{n_k,i}^2 \\ &\leq \sigma^2\gamma(1+\gamma)^2 A_{n_k} \qquad \text{for all large } k, \text{ by (1.13).}\end{aligned}$$

In view of (4.26) and (5.11), we can apply Lemma 3(i) to obtain that

(5.12) $$\begin{aligned}P[|\textstyle\sum_{i \notin I_k} a_{n_k,i}(\tilde{\varepsilon}_i - E\tilde{\varepsilon}_i)| &> \sigma\gamma^{1/4}(1+\gamma)(2A_{n_k}\log\log A_{n_k})^{1/2}] \\ &\leq 2\exp\{-(\tfrac{1}{2} + o(1))\gamma^{-1/2}(2\log\log A_{n_k})\} \\ &\leq 2\exp\{-(\gamma^{-1/2} d + o(1))\log k\}, \quad \text{by (5.8).}\end{aligned}$$

Since $d > \gamma^{1/2}$, we obtain (5.9) from (5.12) together with (4.28) and (5.10).

Let $b_{ki} = a_{n_k,i}$ if $i \notin I_k$ and $b_{ki} = 0$ if $i \notin I_k$. Let $B_k = \sum_{i=-\infty}^{\infty} b_{ki}^2$. Then by (1.13),

(5.13) $$(1 - \gamma + o(1))A_{n_k} \leq B_k \leq A_{n_k}, \qquad \log\log B_k \sim \log\log A_{n_k}.$$

Moreover, by (1.14),

(5.14) $$\log\log B_k \leq (1+\gamma)^2 \log k \qquad \text{for all large } k.$$

Let $U_k = \sum_{i \in I_k} a_{n_k,i}\varepsilon_i = \sum_{i=-\infty}^{\infty} b_{ki}\varepsilon_i$. Since the sets I_k are disjoint, the random variables U_k are independent. In view of (1.6), (5.13), and (5.14), we can apply Lemma 4 and the Borel Cantelli Lemma to obtain that for every $-1 \leq \theta \leq 1$ and $\eta > 0$,

(5.15) $$P[|(2B_k \log\log B_k)^{-1/2} U_k - (1+\gamma)^{-1}\theta\sigma| \leq \eta \quad \text{i.o.}] = 1.$$

We shall assume that $0 \le \theta \le 1$, as the case $-1 \le \theta < 0$ is similar. By (5.13), for all large k,

(5.16) $$A_{n_k}\log\log A_{n_k} \ge B_k\log\log B_k \ge (1-\gamma)^2 A_{n_k}\log\log A_{n_k}.$$

By (5.15) and (5.16),

(5.17) $$P[(1-\gamma)\{(1+\gamma)^{-1}\theta\sigma - \eta\} \le (2A_{n_k}\log\log A_{n_k})^{-1/2}U_k \le (1+\gamma)^{-1}\theta\sigma + \eta \quad \text{i.o.}] = 1.$$

By (5.9) and (5.17),

(5.18) $$\begin{aligned} P[(1-\gamma)\{(1+\gamma)^{-1}\theta\sigma - \eta\} - \sigma(1+2\gamma)\gamma^{1/4} & \\ \le (2A_n\log\log A_n)^{-1/2}S_n & \\ \le (1+\gamma)^{-1}\theta\sigma + \eta + \sigma(1+2\gamma)\gamma^{1/4} \quad \text{i.o.}] = 1. \end{aligned}$$

Since γ and η are arbitrary, (1.16) follows from (5.18). □

In [12], Tomkins studied the lower half of the log log law for weighted sums of the form $\sum_{i=1}^n a_{ni}\varepsilon_i$ with triangular arrays $\{a_{ni}: n \ge 1, 1 \le i \le n\}$ of weights. We now apply Theorem 1(iii) to improve some of his results.

COROLLARY 1. *Let $\varepsilon_1, \varepsilon_2, \cdots$ be independent random variables such that* (1.2) *holds for some $r > 2$. Let $\{a_{ni}: n \ge 1, 1 \le i \le n\}$ be a triangular array of constants such that $A_n = \sum_{i=1}^n a_{ni}^2 \to \infty$ and* (1.6) *is satisfied. Suppose that*

(5.19) $$\textstyle\sum_{i \le n\exp(-(\log n)^\alpha)} a_{ni}^2 = o(A_n) \quad \text{for all} \quad \alpha > 0, \qquad \text{and}$$

(5.20) $$1 + o(1) \ge (\log\log A_n)/(\log\log n) \ge \beta + o(1) \quad \text{for some} \quad 0 < \beta < 1.$$

Then (1.16) *holds for every $-\sigma \le q \le \sigma$, and therefore in particular,*

(5.21) $$\limsup_{n\to\infty} S_n/(2A_n\log\log A_n)^{1/2} \ge \sigma \qquad \text{a.s.}$$

REMARKS. (i) Let $\{a_{ni}\}$ be a bounded triangular array of constants such that $\liminf_{n\to\infty} n^{-1}\sum_{i=1}^n a_{ni}^2 > 0$. Then (1.6), (5.19) and (5.20) are satisfied. Tomkins [12, Corollary 1] established (5.21) in this case under the assumption that (1.2) holds for $r = 3$.

(ii) Let $\{a_{ni}\}$ be a triangular array of constants such that $A_n = \sum_{i=1}^n a_{ni}^2 \to \infty$. Setting $L_n = \max_{1\le i\le n}|a_{ni}|$, assume that

(5.22) $$\limsup_{n\to\infty}(\log L_n^2)/(\log A_n) < 1, \qquad \text{and}$$

(5.23) $$nL_n^2\exp(-(\log n)^\alpha) = o(A_n) \quad \text{for all} \quad \alpha > 0.$$

Since $A_n \le nL_n^2$, it follows from (5.22) and (5.23) that $\log\log A_n \sim \log\log n$, and therefore (5.20) holds. Moreover, (5.23) implies (5.19), while (5.22) implies (1.6). Tomkins [12, Corollary 2] replaced (5.23) by the stronger assumption

(5.24) $$nL_n^2 = O(A_n),$$

and established (5.21) under the conditions (5.22), (5.24), and the additional assumptions $r = 3$ and $a_{ni} = \alpha_{n-i+1}$.

PROOF OF COROLLARY 1. Given $0 < \gamma < 1$, choose $1 < \delta < 1 + \gamma$ and define

$$n_k = [\exp(k^\delta)], \qquad I_k = \{n: n_{k-1} < n \le n_k\}.$$

Since $n_{k-1} = n_k\exp\{-(\delta + o(1))k^{\delta-1}\} = n_k\exp\{-(\delta + o(1))(\log n_k)^{(\delta-1)/\delta}\}$, it follows from (5.19) that

$$\textstyle\sum_{i\notin I_k} a_{n_k,i}^2 = \sum_{i\le n_{k-1}} a_{n_k,i}^2 = o(A_{n_k}),$$

so (1.13) holds. Moreover, (5.20) implies that (1.14) and (1.15) both hold. Hence Theorem 1(iii) can be applied to give the desired conclusion. □

We now apply Corollary 1 together with Theorem 1(ii) to obtain a refinement of the results of Tomkins [11], [12] concerning iterated logarithm behavior for weighted sums of the form $\sum_{i=1}^n f(i/n)\varepsilon_i$.

COROLLARY 2. *Let* $\varepsilon_1, \varepsilon_2, \cdots$ *be independent random variables such that* (1.2) *holds for some* $r > 2$. *Let* $f \in L^2(0, 1)$ *such that*

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n f^2(i/n) = \int_0^1 f^2(t)\,dt > 0, \tag{5.25}$$

$$\lim_{a\to 0}\{\limsup_{n\to\infty} n^{-1} \textstyle\sum_{i\le an} f^2(i/n)\} = 0, \qquad \textit{and} \tag{5.26}$$

$$\max_{1\le i\le n} f^2(i/n) = o(n(\log n)^{-\rho}) \quad \textit{for all} \quad \rho > 0. \tag{5.27}$$

Let $S_n = \sum_{i=1}^n f(i/n)\varepsilon_i$. *Then for every q such that* $|q| \le \sigma(\int_0^1 f^2(t)\,dt)^{1/2}$,

$$\liminf_{n\to\infty} |(2n \log\log n)^{-1/2} S_n - q| = 0 \qquad \text{a.s.} \tag{5.28}$$

If furthermore $\limsup_{t\to 1} |f(t)| < \infty$ *and there exist* $K > 0$, $m_0 \ge 1$, *and* $0 < \theta_0 < 1$ *such that*

$$m^{-1} \sum_{i=1}^m (f(i/m) - f(i/n))^2 \le K(1 - m/n) \textit{ for all } m \ge m_0 \textit{ and } \theta_0 \le m/n < 1, \tag{5.29}$$

then

$$\limsup_{n\to\infty} |S_n|/(2n \log\log n)^{1/2} = \sigma\left(\int_0^1 f^2(t)\,dt\right)^{1/2} \qquad \text{a.s.} \tag{5.30}$$

PROOF. The lower half (5.28) of the log log law is an immediate consequence of Corollary 1 with $a_{ni} = f(i/n)$ and $A_n = \sum_{i=1}^n f^2(i/n) \sim n \int_0^1 f^2(t)\,dt$. To prove the upper half of (5.30), since $\limsup_{t\to 1} |f(t)| < \infty$, we can choose $B > 0$ and $1 > \theta_1 > \theta_0$ such that $f^2(t) \le B$ for $\theta_1 \le t \le 1$. Then for $n > m \ge \theta_1 n$ and $m \ge m_0$,

$$\begin{aligned} &\textstyle\sum_{i=1}^m (f(i/m) - f(i/n))^2 + \sum_{i=m+1}^n f^2(i/n) \\ &\le Km(1 - m/n) + B(n - m), \qquad \text{by (5.29)}, \\ &\le (K + B)(n - m). \end{aligned} \tag{5.31}$$

On the other hand, if $\theta_1 n > m$, then $n < (1 - \theta_1)^{-1}(n - m)$ and

$$\begin{aligned} \textstyle\sum_{i=1}^m (f(i/m) - f(i/n))^2 + \sum_{i=m+1}^n f^2(i/n) &\le 2\textstyle\sum_{i=1}^m f^2(i/m) + 2\sum_{i=1}^n f^2(i/n) \\ &\sim (2m + 2n) \int_0^1 f^2(t)\,dt, \qquad \text{by (5.25)}, \\ &\le (2\theta_1 + 2)(1 - \theta_1)^{-1}(n - m) \int_0^1 f^2(t)\,dt. \end{aligned} \tag{5.32}$$

Combining (5.31) and (5.32), we can therefore choose m_1 sufficiently large such that for all $n > m \ge m_1$,

$$\sum_{i=1}^m (f(i/m) - f(i/n))^2 + \sum_{i=m+1}^n f^2(i/n) \le g(n - m), \qquad \text{where} \tag{5.33}$$

$$g(j) = j \max\left\{K + B,\ (2\theta_1 + 3)(1 - \theta_1)^{-1} \int_0^1 f^2(t)\,dt\right\}.$$

Hence we can apply Theorem 1(ii) to obtain the desired conclusion. □

REMARKS. (i) If f^2 is bounded and Riemann-integrable on [0, 1], then (5.25), (5.26), and (5.27) obviously hold. Tomkins [11], [12] assumed f to be continuous on [0, 1] and established the lower half of (5.30) in this case.

(ii) If f is Hölder continuous on [0, 1] with exponent ½, i.e.,

(5.34) $$|f(x) - f(y)| \leq K |x - y|^{1/2} \quad \text{for some} \quad K > 0 \quad \text{and all} \quad 0 \leq x, y \leq 1,$$

then for all $0 \leq \theta \leq 1$ and $m \geq 1$,

$$m^{-1} \sum_{i=1}^{m} (f(i/m) - f(\theta i/m))^2 \leq K^2(1 - \theta)m^{-1} \sum_{i=1}^{m} (i/m) \leq K^2(1 - \theta).$$

Therefore (5.29) is also satisfied in this case. Tomkins [11] established (5.30) under the much more stringent condition that f has a power series representation on [0, 1] and some additional assumptions.

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COLUMBIA UNIVERSITY
GRADUATE SCHOOL OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICAL STATISTICS
NEW YORK, NY 10027

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MD 20742