# A least squares iterative method for singular equations

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**An iterative method for the solution of a system of simultaneous linear equations, having a singular coefficient matrix $A$, is described. The method is obtained by minimizing (in the least squares sense) the image under $A^T$ of a given residual vector.**

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## 1. Introduction

Let us consider the system

$$Ax = b, \tag{1.1}$$

where $A$ is an $m \times n$ matrix of rank $\sigma$, $x$ and $b$ are column vectors with $n$ and $m$ elements respectively. Here $\sigma \leqslant \min(m, n)$ and $b$ may or may not lie in the range $R(A)$ of $A$. It is well known (Penrose, 1956) that

$$\hat{x} = A^+b \tag{1.2}$$

is the minimum norm least squares solution of (1.1); in other words, among all vectors $x$, that minimize $||b - Ax||_2$, viz., the Euclidean length of the residual $b - Ax$, $\hat{x}$ has the least Euclidean length. $A^+$, which is called the generalised inverse of $A$, is the unique matrix satisfying each of the following relations (Penrose, 1955)

$$AA^+A = A, \tag{1.3}$$

$$A^+AA^+ = A^+, \tag{1.4}$$

$$(AA^+)^T = AA^+, \tag{1.5}$$

and

$$(A^+A)^T = A^+A, \tag{1.6}$$

where $T$ denotes the transpose of the relevant matrix.

In Section 2, we shall describe an iterative method for computing $\hat{x}$, which is an extension of the method given by Khabaza (1963) for the non-singular $A$. In Section 3, we give a technique for improving the computations when the non-singular part of $A$ is ill-conditioned. The convergence of the method is discussed in Section 4.

## 2. The method

Let $x^{(i)}$ be an approximation for $\hat{x}$ such that

$$\hat{x} = x^{(i)} + e^{(i)}, \tag{2.1}$$

where $e^{(i)}$ is the error in $x^{(i)}$. If we let $r^{(i)} = b - Ax^{(i)}$ and $r = b - A\hat{x}$, then we have

$$\begin{aligned} r = b - A\hat{x} &= b - Ax^{(i)} - Ae^{(i)} \\ &= r^{(i)} - Ae^{(i)} \Rightarrow Ae^{(i)} = r^{(i)} - r, \end{aligned}$$

from which it follows that (Penrose, 1956)

$$e^{(i)} = - A^+r^{(i)} - A^+r + (I_n - A^+A)\gamma, \tag{2.2}$$

where $\gamma$ is an arbitrary column vector with $n$ elements and $I_n$ is the identity matrix of order $n$. It is well known that $I - A^+A$ is the projector on the null space $\eta(A)$ of $A$ (Ben-Israel, 1963). Now, in view of (1.2) and (1.4),

$$(I_n - A^+A)\hat{x} = (I_n - A^+A)A^+b = (A^+ - A^+AA^+)b = 0,$$

which implies that $\hat{x}$ has no component in $\eta(A)$ viz., $\hat{x} \in \eta(A)^\perp$, the orthogonal complement of $\eta(A)$, and therefore from (2.1) and (2.2), it follows that $\gamma = 0$. Furthermore, using (1.2) and (1.4), we get

$$\begin{aligned} A^+r = A^+(b - A\hat{x}) &= A^+b - A^+AA^+b \\ &= A^+b - A^+b = 0, \end{aligned}$$

which, in view of (2.2) and (2.1) implies that

$$\hat{x} = x^{(i)} + A^+r^{(i)}. \tag{2.3}$$

In order to get an approximation for $A^+$, we will make use of the following result.

Theorem 2.1 (Decell, 1965). Let

$$\begin{aligned} f(\lambda) = (-1)^n(a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_k\lambda^{n-k} \\ + \ldots + a_{n-1}\lambda + a_n), \end{aligned}$$

with $a_0 = 1$, be the characteristic polynomial of $A^TA$. If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalised inverse of $A$ is given by

$$\begin{aligned} A^+ = - a_k^{-1}[(A^TA)^{k-1} + a_1(A^TA)^{k-2} \\ + \ldots + a_{k-1}I_n]A^T. \end{aligned} \tag{2.4}$$

Note that the above theorem is obtained as a result of restating the theorem, given by Decell (1965) for $A^T$ instead of $A$, and using the fact that $(A^{T+})^T = A^+$ (Penrose, 1955). Equation (2.4) can be rewritten as

$$A^+ = [c_1I_n + c_2A^TA + \ldots + c_k(A^TA)^{k-1}]A^T, \tag{2.5}$$

where $c_1 = - a_{k-1}/a_k$, $c_2 = - a_{k-2}/a_k, \ldots, c_{k-1} = - a_1/a_k$ and $c_k = - 1/a_k$.

If we let

$$A^+ \approx [c_1I_n + c_2A^TA + \ldots + c_p(A^TA)^{p-1}]A^T = f(A), \tag{2.6}$$

where $p \leqslant k$, then from (2.3), it follows that

$$\hat{x} \approx x^{(i)} + f(A)r^{(i)} = x^{(i+1)}, \text{ (say)}. \tag{2.7}$$

In order to compute $c_1, c_2, \ldots, c_p$ we will need the following result

$$A^Tr = 0, \tag{2.8}$$

which can be proved by using (1.2), (1.5) and (1.3) as follows:

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$$\begin{aligned}A^Tr &= A^T(b - A\hat{x}) = A^Tb - A^TAA^+b \\ &= A^Tb - A^T(AA^+)^Tb = A^Tb - (AA^+A)^Tb \\ &= A^Tb - A^Tb = 0.\end{aligned}$$

Let

$$\begin{aligned}r^{(i+1)} &= b - Ax^{(i+1)} \\ &= b - A[x^{(i)} + f(A)r^{(i)}], \text{ using (2.7)} \\ r^{(i+1)} &= r^{(i)} - Af(A)r^{(i)}. \qquad (2.9)\end{aligned}$$

In view of (2.8), $f(A)$ should be chosen such that we minimize (in the least squares sense) the vector

$$\begin{aligned}A^Tr^{(i+1)} &= A^T[r^{(i)} - Af(A)r^{(i)}], \text{ using (2.9)} \\ &= [A^T - A^TAf(A)]r^{(i)}. \qquad (2.10)\end{aligned}$$

Substituting the value of $f(A)$ from (2.6) in (2.10), we now have the problem of minimizing (in the least squares sense) the column vector

$$\begin{aligned}&[A^T - A^TA\{c_1I_n + c_2A^TA + \ldots + c_p(A^TA)^{p-1}\}A^T]r^{(i)} \\ &= [I_n - c_1A^TA - \ldots - c_p(A^TA)^p]A^Tr^{(i)}. \qquad (2.11)\end{aligned}$$

Let

$$A^Tr^{(i)} = r_0,\ A^TAr_0 = r_1, \ldots, A^TAr_{p-1} = r_p, \qquad (2.12)$$

then from (2.11) it follows that we have to find the least squares solution of

$$r_0 - (c_1r_1 + c_2r_2 + \ldots + c_pr_p) = 0,$$

or

$$Rc = r_0, \qquad (2.13)$$

where $c = (c_1, c_2, \ldots, c_p)^T$ and $R = (r_1, r_2, \ldots, r_p)$. It therefore follows that (Penrose, 1956)

$$c = R^+r_0 + (I_p - R^+R)\theta, \qquad (2.14)$$

where $\theta$ is an arbitrary column vector with $p$ elements. If $r_1, r_2, \ldots, r_p$, are linearly independent, viz., rank of $R$ is $p$ then

$$R^+ = (R^TR)^{-1}R^T, \qquad (2.15)$$

which can be verified by direct substitution in the four defining relations (1.3) — (1.6). Furthermore, $I_p - R^+R = I_p - (R^TR)^{-1}R^TR = I_p - I_p = 0$ and (2.14) gives

$$c = (R^TR)^{-1}R^Tr_0. \qquad (2.16)$$

Now (2.7), in view of (2.6) and (2.12), gives

$$\begin{aligned}x^{(i+1)} &= x^{(i)} + f(A)r^{(i)} \\ &= x^{(i)} + [c_1I_n + c_2A^TA + \ldots + c_p(A^TA)^{p-1}]A^Tr^{(i)}\end{aligned}$$

or

$$x^{(i+1)} = x^{(i)} + c_1r_0 + c_2r_1 + \ldots + c_pr_{p-1}. \qquad (2.17)$$

Notice that in the above scheme $r_0, r_1, \ldots, r_{p-1}$ have already been calculated earlier when $c_1, c_2, \ldots, c_p$ were computed.

## 3. Computational considerations

There exist orthogonal matrices $Q$ and $S$ such that

$$QAS = \begin{pmatrix}\hat{D} & 0 \\ 0 & 0\end{pmatrix} = D, \qquad (3.1)$$

where $\hat{D}$ is a diagonal matrix with elements $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_\sigma > 0$, which are called the singular values of $A$ (Forsythe and Moler, 1967). As in Tewarson (1968), a condition number of $A$ can be defined as

$$cond\,(A) = \frac{\mu_1}{\mu_\sigma} \geqslant 1. \qquad (3.2)$$

We will make use of the following results.

Theorem 3.1 (Tewarson and Ramnath, 1968). For any $\in > 0$,

$$cond\,(A) \geqslant cond\,(A + \in A^{T+}). \qquad (3.3)$$

Corollary 3.1. (Tewarson and Ramnath, 1968.) The minimum non-zero singular value of $[A + \in A^{T+}] > \mu_\sigma$.

In view of the above results, if we want to improve the accuracy of the computations then instead of solving (1.1), we can find the minimum norm least squares solution of

$$(A + \in A^{T+})x = b, \qquad (3.4)$$

which we will show is the same as solving

$$\begin{pmatrix}A \\ \in^{1/2}I_n\end{pmatrix} x = \begin{pmatrix}b \\ 0\end{pmatrix}, \qquad (3.5)$$

in the least squares sense. To this end we will need the following.

Theorem 3.2. For any $\in > 0$

$$(A + \in A^{T+})^+ = (A^TA + \in I_n)^{-1}A^T. \qquad (3.6)$$

Proof: From (3.1) and the fact that $Q$ and $S$ are orthogonal we have

$$\begin{aligned}D^+ = \begin{pmatrix}\hat{D}^{-1} & 0 \\ 0 & 0\end{pmatrix},\ A &= Q^TDS^T, \\ A^T = SDQ,\ A^TA &= SD^2S^T. \qquad (3.7)\end{aligned}$$

Also, by direct substitution in (1.3) — (1.6), it can be verified that

$$A^{T+} = Q^TD^+S^T \text{ and } A^+ = SD^+Q. \qquad (3.8)$$

Therefore

$$\begin{aligned}(A + \in A^{T+})^+ &= (Q^TDS^T + \in Q^TD^+S^T)^+ \\ &= S(D + \in D^+)^+Q \qquad (3.9)\end{aligned}$$

and

$$\begin{aligned}(A^TA + \in I_n)^{-1}A^T &= (SD^2S^T + \in SS^T)^{-1}SDQ \\ &= S(D^2 + \in I_n)^{-1}DQ. \qquad (3.10)\end{aligned}$$

But, in view of the definition of $D$, the $i$th element of $(D + \in D^+)^+$ is $\left(\mu_i + \frac{\in}{\mu_i}\right)^{-1} = (\mu_i^2 + \in)^{-1}\mu_i$, $1 \leqslant i \leqslant \sigma$ and zero otherwise, which is the same as the $i$th element of $(D^2 + \in I)^{-1}D$ and the result (3.6) follows from (3.9) and (3.10).

Now, in view of the fact that the coefficient matrix in (3.5) has full column rank, as in (2.16), the solution of (3.5) is

$$\begin{aligned}x &= \left[(A^T, \in^{1/2}I_n)\begin{pmatrix}A \\ \in^{1/2}I_n\end{pmatrix}\right]^{-1}(A^T, \in^{1/2}I_n)\begin{pmatrix}b \\ 0\end{pmatrix} \\ &= (A^TA + \in I_n)A^Tb. \qquad (3.11)\end{aligned}$$

On the other hand, the minimum norm least squares solution of (3.4) is

$$x = (A + \in A^{T+})^+b \qquad (3.12)$$

which in view of Theorem 3.2 is the same as (3.11). This shows that the minimum norm least squares solution of (3.4) is the same as the least squares solution of (3.5).

Another way of looking at the advantage of solving (3.5) instead of (1.1) is as follows. The least squares solution of (3.5) minimizes

$$\left\|\begin{pmatrix} b \\ 0 \end{pmatrix} - \begin{pmatrix} A \\ \in^{1/2} I \end{pmatrix} x\right\|_2^2 = ||b - Ax||_2^2 + \in ||x||^2,$$

instead of first finding all the $x$'s which minimize $||b - Ax||_2$ and out of all such $x$'s finding $\hat{x}$ of minimum norm, which is the case when (1.1) is directly solved (Penrose, 1956). In the later case, if $A$ is ill-conditioned then it may very well happen that the norm of each $x$ that minimizes $||b - Ax||_2$ is arbitrarily large. The simultaneous minimization of $||b - Ax||_2^2$ and $||x||_2^2$ with weights unity and $\in$ respectively, which results when (3.5) is solved, avoids this trouble by keeping $||x||_2$ bounded. Equation (3.5) is a generalisation of the so-called method of damped least squares (Levenberg, 1944) to the case when $A$ does not have full column rank.

We will now modify the scheme (2.17) for the system (3.5). To this end, let

$$\begin{pmatrix} A \\ \in^{1/2} I_n \end{pmatrix} = \hat{A},\ \begin{pmatrix} b \\ 0 \end{pmatrix} = \hat{b},\ \hat{r}^{(i)} = \hat{b} - \hat{A}x^{(i)} = \begin{pmatrix} r^{(i)} \\ -\in^{1/2} x^{(i)} \end{pmatrix}.$$

Therefore, we have

$$\hat{A}^T\hat{A} = A^TA + \in I = H \text{ (say)}, \tag{3.13}$$

and (2.12) gives

$$\hat{r}_0 = \hat{A}^T\hat{r}^{(i)} = (A^T, \in^{1/2} I_n)\begin{pmatrix} r^{(i)} \\ -\in^{1/2} x^{(i)} \end{pmatrix} = A^Tr^{(i)} - \in x^{(i)} \tag{3.14}$$

and

$$\hat{r}_1 = H\hat{r}_0,\ \hat{r}_2 = H\hat{r}_1, \ldots, \hat{r}_p = H\hat{r}_{p-1}. \tag{3.15}$$

and from this point on the computations are identical to those given earlier for (2.13) to (2.17).

The method given above for improving the condition of $A$ can also be applied to (2.13) to keep $||c||_2$ from getting arbitrarily large in case $R$ is ill-conditioned. In fact the direct application of Householder transformations (Golub, 1965) gives the solution $c$ in a few steps, as shown below. Let $Q$ be the Householder orthogonal matrix such that

$$Q\begin{pmatrix} R \\ \in^{1/2} I_p \end{pmatrix} c = Q\begin{pmatrix} r_0 \\ 0 \end{pmatrix}, \tag{3.16}$$

or

$$\begin{bmatrix} U \\ 0 \end{bmatrix} c = \begin{bmatrix} e \\ f \end{bmatrix}, \tag{3.17}$$

where $U$ is an upper triangular non-singular matrix of rank $p$, the column vectors $e$ and $f$ have $p$ and $n$ elements respectively. Since the Euclidean norm is invariant under an orthogonal transformation the least squares solutions of (3.16) and (3.17) are identical, and therefore from (3.17) we have (Penrose, 1956)

$$c = \begin{pmatrix} U \\ 0 \end{pmatrix}^+ \begin{pmatrix} e \\ f \end{pmatrix} = \left[(U^T, 0)\begin{pmatrix} U \\ 0 \end{pmatrix}\right]^{-1} (U^T, 0)\begin{pmatrix} e \\ f \end{pmatrix}$$
$$= (U^TU)^{-1}U^Te = U^{-1}e. \tag{3.18}$$

The evaluation of $U^{-1}$ is easy since $U$ is an upper triangular matrix. The principal advantages of using the Householder transformation are (i) stability is unconditionally guaranteed (Wilkinson, 1965), (ii) during the reduction no dangerous growth of elements can take place (Wilkinson, 1965), especially where $p$ itself is large.

## 4. Convergence of the method

Equation (2.7) can be rewritten as

$$x^{(i+1)} = x^{(i)} + f(A)(b - Ax^{(i)}) = [I_n - f(A)A]x^{(i)} + f(A)b$$

or

$$x^{(i+1)} = Bx^{(i)} + d, \text{ where } B = I_n - f(A)A \text{ and } f(A)b = d. \tag{4.1}$$

If we start with some approximation $x_0$, then the repeated application of (4.1) gives

$$x^{(k)} = B^kx_0 + (B^{k-1} + \ldots + B + I_n)d. \tag{4.2}$$

In other words, (4.2) is the equation resulting from using the $f(A)$ that was determined for the initial approximation, in the succeeding steps as well. We will now discuss the conditions under which (4.2) converges. In view of the fact that $Q$ and $S$ are orthogonal, from (3.7) we get

$$A^TA = SD^2S^T, (A^TA)^2 = SD^4S^T, \ldots, (A^TA)^p = SD^{2p}S^T. \tag{4.3}$$

From (2.6), (3.7) and the above equation, we have

$$f(A) = S(c_1D + c_2D^3 + \ldots + c_pD^{2p-1})Q = SNQ, \text{ say.} \tag{4.4}$$

Also, from (4.1), (3.7) and (4.4) it follows that

$$B = S(I_n - c_1D^2 - c_2D^4 - \ldots - c_pD^{2p})S^T = SMS^T, \text{ say} \tag{4.5}$$

and

$$B^2 = SM^2S^T, \ldots, B^{k-1} = SM^{k-1}S^T. \tag{4.6}$$

From (4.4) and (3.1) we get

$$N = \begin{pmatrix} \hat{N} & 0 \\ 0 & 0 \end{pmatrix}, \text{ where } \hat{N} = c_1\hat{D} + c_2\hat{D}^3 + \ldots + c_p\hat{D}^{2p-1}. \tag{4.7}$$

Similarly (4.5) and (3.1) yield

$$M = \begin{pmatrix} \hat{M} & 0 \\ 0 & I_{n-\sigma} \end{pmatrix}, \text{ where } \hat{M} = I_\sigma - c_1\hat{D}^2 - c_2\hat{D}^4 - \ldots - c_p\hat{D}^{2p}, \tag{4.8}$$

and

$$MN = \begin{pmatrix} \hat{M}\hat{N} & 0 \\ 0 & 0 \end{pmatrix}, M^2N = \begin{pmatrix} \hat{M}^2\hat{N} & 0 \\ 0 & 0 \end{pmatrix}, \ldots,$$
$$M^{k-1}N = \begin{pmatrix} \hat{M}^{k-1}\hat{N} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.9}$$

Now, making use of (4.6), (4.1) and (4.4) we get

$$(B^{k-1} + \ldots + B + I)d = S(M^{k-1} + \ldots + M + I)NQb,$$
$$= S\begin{bmatrix} (\hat{M}^{k-1} + \ldots + \hat{M} + I_\sigma)\hat{N} & 0 \\ 0 & 0 \end{bmatrix} . Qb, \text{ using } (4.9),$$

$$= S_\sigma(\hat{M}^{k-1} + \ldots + \hat{M} + I_\sigma)\hat{N}Q_\sigma b, \quad (4.10)$$

where the first $\sigma$ columns of $S$ are denoted by $S_\sigma$ and first $\sigma$ rows of $Q$ by $Q_\sigma$. For convergence in (4.10), in view of (4.8), we must have

$$\rho(\hat{M}) < 1 \Rightarrow \rho(I_\sigma - c_1\hat{D}^2 - c_2\hat{D}^4 - \ldots - c_p\hat{D}^{2p}) < 1, \quad (4.11)$$

where $\rho(G)$ denotes the spectral radius of $G$ (Wilkinson, 1965, p. 59). Condition (4.11) implies that for $i = 1, 2, \ldots, \sigma$

$$0 < \mu_i^2(c_1 + c_2\mu_i^2 + \ldots + c_p\mu_i^{2p-2}) < 2. \quad (4.12)$$

It is well known (e.g. Wilkinson, 1965, p. 56) that

$$\mu_i^2 \leqslant \mu_1^2 = \rho(A^TA) \leqslant ||A^TA||_\infty \leqslant ||A^T||_\infty||A||_\infty$$
$$= ||A||_1||A||_\infty$$
$$= \left[\max_j \sum_i |a_{ij}|\right]\left[\max_i \sum_j |a_{ij}|\right], \quad (4.13)$$

where $a_{ij}$ is the $i$th row $j$th column element of $A$. Therefore, in view of (4.13) and (4.12), we can see that dividing each row of $A$ by the sum of the absolute values of its elements and then dividing each column of the resulting matrix by the sum of the absolute values of its elements, should in general, improve the convergence. Of course, the above mentioned row scaling should also be applied to $b$ and the column scale factors should be saved in order to update the final value of $\hat{x}$. In spite of the above, discussion (4.12) is a rather severe restriction on $A$. Therefore, in general, it is recommended that the $f(A)$ computed at a particular step should be used for only a few succeeding steps and then recomputed again. This was suggested in the non-singular case by Khabaza (1963).

We still have to investigate $B^kx_0$, before the question of convergence of (4.2) can be completely settled. To this end, let

$$x_0 = A^+Ax_0 + (I - A^+A)x_0 = s + t, \text{ (say)}. \quad (4.14)$$

In other words, $t$ and $s$ are respectively the components of $x_0$ in the null space $\eta(A)$ of $A$ and its orthogonal complement $\eta(A)$. This fact can be verified by using (1.3)–(1.6), (Tewarson, 1968). Now, from (4.6), (4.14), (3.8) and (3.7), we get

$$B^ks = SM^kS^TA^+Ax_0 = SM^kS^TSD^+QQ^TDS^Tx_0$$
$$= SM^kD^+DS^Tx_0$$

or

$$B^ks = S\begin{pmatrix}\hat{M}^k & 0\\ 0 & I_{n-\sigma}\end{pmatrix}\begin{pmatrix}I_\sigma & 0\\ 0 & 0\end{pmatrix}S^Tx_0, \text{ using (4.8) and (3.1)},$$
$$= S\begin{pmatrix}\hat{M}^k & 0\\ 0 & 0\end{pmatrix}S^Tx_0$$

Thus

$$\lim_{k\to\infty} B^ks = S\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}S^Tx_0 = 0, \quad (4.15)$$

since from (4.11) we already have $\rho(\hat{M}) < 1 \Rightarrow \lim_{k\to\infty} \hat{M}^k = 0$ (Wilkinson, 1965, p. 58). Similarly, (4.6), (4.14), (4.8), (3.8) and (3.7) yield

$$B^kt = SM^kS^T(I_n - A^+A)x_0$$
$$= S\begin{pmatrix}\hat{M}^k & 0\\ 0 & I_{n-\sigma}\end{pmatrix}S^TS\begin{pmatrix}0 & 0\\ 0 & I_{n-\sigma}\end{pmatrix}S^Tx_0$$
$$= S\begin{pmatrix}0 & 0\\ 0 & I_{n-\sigma}\end{pmatrix}S^Tx_0 = t \quad (4.16)$$

Finally, from (4.2), (4.14), (4.16), (4.15), (4.10) and the fact that $(I_\sigma + \hat{M} + \ldots + \hat{M}^{k-1} + \ldots) = (I_\sigma - \hat{M})^{-1}$ (Wilkinson, 1965, p. 59), we have

$$\lim_{k\to\infty} x^{(k)} = t + 0 + S_\sigma(I_\sigma - \hat{M})^{-1}\hat{N}Q_\sigma b$$
$$= t + S_\sigma(\hat{N}\hat{D})^{-1}\hat{N}Q_\sigma b, \text{ using (4.7) and (4.8).}$$
$$= t + S_\sigma\hat{D}^{-1}Q_\sigma b. \quad (4.17)$$

But from (4.4) and (3.7), we get

$$(f(A)A)^+f(A)b = (SNQQ^TDS^T)SNQb$$
$$= (SNDS^T)^+SNQb = (S_\sigma\hat{N}\hat{D}S^{T\sigma})^+S_\sigma\hat{N}Q_\sigma b, \text{ using (4.7) and (3.1),}$$
$$= S_\sigma\hat{D}^{-1}\hat{N}^{-1}S_\sigma^TS_\sigma\hat{N}Q_\sigma b = S_\sigma\hat{D}^{-1}Q_\sigma b, \quad (4.18)$$

since $S^TS = I_n \Rightarrow S_\sigma^TS_\sigma = I_\sigma$.
Therefore, (4.17) and (4.18) imply that

$$\lim_{k\to\infty} x^{(k)} = t + [f(A)A]^+f(A)b$$
$$\approx t + (A^+A)^+A^+b, \text{ if } f(A) \approx A^+,$$
$$= t + A^+b, \text{ using (1.3)–(1.6)},$$

or

$$\lim_{k\to\infty} x^{(k)} = t + \hat{x}, \text{ using (1.2).} \quad (4.19)$$

Thus we have seen that $x^{(k)}$ converges to the solution $\hat{x} = A^+b$ plus the component of the initial approximation $x_0$ in $\eta(A)$—the null space of $A$. It is easy to choose $x_0$ such that $t = 0$, e.g. by taking $x_0 = A^Tb$, which implies that $\lim_{k\to\infty} x^{(k)} \to A^+b$.

## 5. Concluding remarks

The method given in this paper has essentially the same advantages as those indicated by Khabaza (1963) for the systems with non-singular coefficient matrices. The choice of $\epsilon$ in (3.4) or (3.5) depends, to a large extent, on the user's practical experience with the type of problems he is most likely to handle. In general, $\epsilon$ should be chosen as large as possible, so long as it does not produce an unreasonable amount of change in the given problem. If $\alpha$ is the mean and $\beta$ the standard deviation of the absolute values of the non-zero elements of $A$, then $0 < \epsilon < \alpha - 2\beta$ is, in general, a reasonable choice. Also the value of $p$ in (2.6) should not be large, otherwise the computation of $c$ in (2.16) involves nearly as much work as the solution of (1.1) directly.

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## Alston S. Householder Award

Dr. A. S. Householder, director of the mathematics division of Oak Ridge National Laboratory retired in May. In recognition of his outstanding contributions to numerical analysis, it was decided at the Fourth Gatlinburg Symposium in April to establish a Householder prize of $400.

The prize is to be awarded to the author of the best thesis in numerical algebra, the topic of the Fourth Gatlinburg Symposium. The subject is to be interpreted as the area in which Dr. Householder has worked and its natural developments, and thus covers, for example, both theoretical and practical aspects of linear algebra, the solution of non-linear equations and the algebraic aspects of methods for solving ordinary, partial and integral equations.

The dissertations will be assessed by an international committee consisting of J. H. Wilkinson, of the National Physical Laboratory, England; F. L. Bauer, Mathematisches Institut, Munich; and Olga Taussky-Todd, California Institute of Technology.

To qualify the thesis must be for a (first research) degree awarded in the calendar years 1968, 1969, 1970. The candidate's sponsor (e.g. supervisor of his research) should submit the official abstract of the thesis together with his appraisal of it, by 1 January 1971, to **A. S. Householder Prize, SIAM, 33 S. 17th Street, Philadelphia, Pennsylvania 19103, USA**.

After a preliminary scrutiny of the abstract and the appraisal, the committee may request candidates to submit copies of the actual thesis. SIAM will announce the award not later than 1 July 1971.