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A Least-Squares Method to find a Remanence Direction from Converging Remagnetization Circles

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Summary

The direction of a secondary magnetization component is found from the intersection point of converging remagnetization circles using a method based on the least-squares fitting of great circles to points on a sphere. The technique may be applied to any problem that requires the best intersection point of convergent great circles and is thus useful in other fields besides palaeomagnetism, such as structural geology, plate tectonics and astronomy.

Introduction

In palaeomagnetic studies rocks are often encountered which possess two superimposed components of remanent magnetization. If their coercivity or blocking temperature spectra are dis-similar, one or other of the remanences will be preferentially removed at various stages of AC or thermal stepwise magnetic cleaning. Successive total magnetization vectors obtained after each cleaning step may thus have different directions but should always lie within a single plane. On a stereonet the vectors define an arc of a great circle, the so-called remagnetization circle.

It is possible to recover one or both of the component directions if demagnetization results yield stable end-points or are amenable to a vector subtraction analysis (e.g. Buchan & Dunlop 1976). The success of these procedures demands certain relations between the coercivity or blocking temperature spectra of the two components which are summarized in Fig. 1. This paper describes a least-squares method using converging remagnetization circles, which can give the component directions without the above spectral constraints.

If the spectra of two components A and B partially overlap as shown in Fig. 1(a), stepwise cleaning yields the direction of the B component from the stable end-point. During the initial cleaning stages when the B component remains undemagnetized, successive vector differences remain parallel to one another and define the direction of the A component. However, as Fig. 1(b) shows, successively parallel vector differences can be observed even if the intensity of the B component is changing. In this example successive vector differences, although parallel, do not have the same direction as the A component. Therefore if vector subtraction is used to find the A direction it is important that there is evidence for the spectral relations shown in

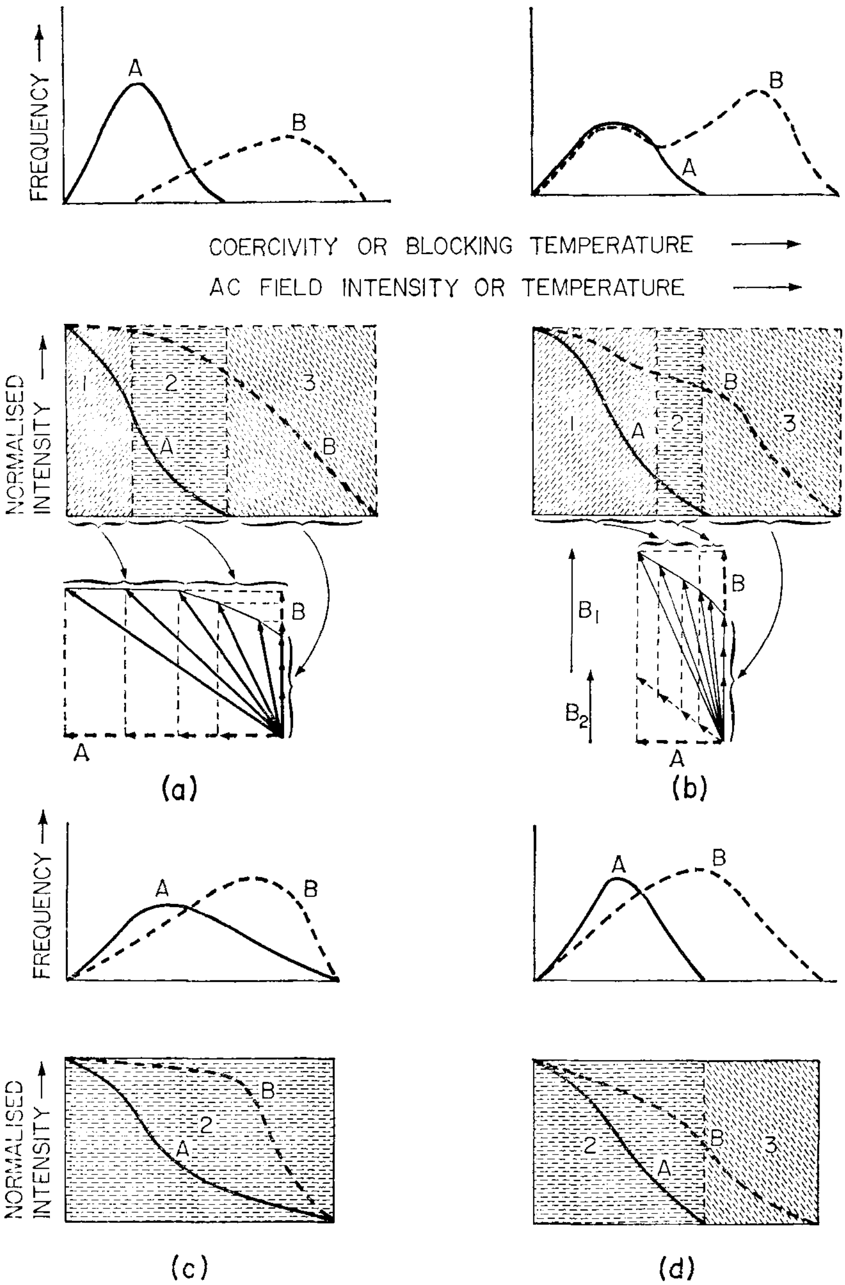

FIG. 1. Schematic diagrams to illustrate demagnetization characteristics of two superimposed remanences A and B for different relations between their coercivity or blocking temperature spectra. Regions of the demagnetization curves are specified as follows: 1, resultant vectors progressively change direction but successive vector differences remain parallel to one another; 2, resultant vectors progressively change direction but successive vector differences are not parallel to one another; and 3, resultant vectors do not change direction, implying that the A component has been completely removed. This is the stable end-point situation and gives the direction of the B component. The lower figures of 1(a) and 1(b) show the change in direction and intensity of resultant magnetization vectors on stepwise cleaning. B_1 represents that part of the B remanence, corresponding to the higher of the two spectrum peaks in Fig. 1(b), which initially remains undemagnetized. B_2 is that part which decays initially at the same rate as the A component.

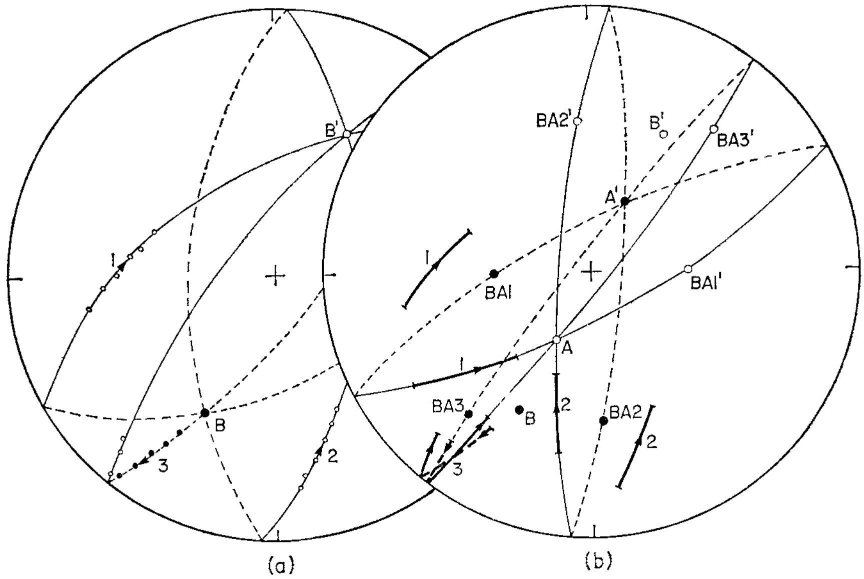

FIG. 2. Wulff net diagrams illustrate schematically how the directions and relative ages of two magnetization components can be resolved when sample cleaning yields successive magnetization vectors which define arcs of great circles, but no stable end-points. In the hypothetical example given, remagnetization circles are defined by sample data from three sites (1, 2, 3) in a structurally disturbed area. Arrows indicate sense of directional swings on cleaning; dashed/solid parts of circles are plotted on the lower/upper hemispheres; and solid/open dots are downward/upward magnetizations. In Fig. 2(a), B and B′ are the common intersection points of remagnetization circles before structural unfolding (SU). In Fig. 2(b), A and A′ are the intersection points of the same circles after SU, and BA1 and BA1′ etc. are the positions of poles B and B′ for site 1 etc. On each of the three great circles in Fig. 2(b) there are two possible directions of the primary and rotated secondary components. The ambiguity is removed because the data points defining these circles must lie between the true primary and secondary directions. Hence A and B are respectively the directions of the primary and secondary components.

Fig. 1(a). Vector subtraction cannot be used to resolve the A direction if the coercivity or blocking temperature spectra of the two components have very similar ranges (Fig. 1(c)). Also, if both spectra have similar lower limits, but the A spectrum is narrower, then it is not possible to use vector subtraction to recover the A direction, despite knowing the B direction from the stable end-point.

The above limitations in vector subtraction and end-point analyses may be removed if remagnetization circles, observed for different palaeomagnetic sites, tend to intersect at a common point. This convergence point, or its antipole, gives the direction of one of the components. The relatively large dispersion in one component over the other, necessary for converging circles to be observed, can be produced either by folding, if the period of deformation separates the times of magnetization (e.g. Irving 1964), or by factors unrelated to structure such as secular variation. If folding introduces appreciable scatter in one component relative to the other, *and* vice-versa after unfolding, then two sets of converging circles are produced, corresponding to the folded and unfolded states. In this instance, only portions of remagnetization circles need be defined to specify uniquely the relative age and direction of the two components as shown in Fig. 2. If only one of the two sets of circles shows convergence, additional data from stable end-points or vector subtraction are needed to obtain the same information. Secular variation as a possible cause of circle convergence becomes apparent if the data are obtained from rocks with a uniform structural attitude. In this case only the direction of the least dispersed component can be obtained from the converging circles, and again further data are required to find the direction and relative age of the other component.

Since the convergence of remagnetization circles will be invariably blurred due to dispersion, it is necessary to obtain a best estimate of the convergence point. This can be accomplished by fitting planes to the vectors defining each remagnetization circle by the method of least squares, and then repeating the process on vector normals to these planes. The technique, described below, is a special case of least squares fitting a plane to a set of data points (e.g. Blow 1959), and is an extension of the method outlined by Creer (1962) for finding the best-fitting great circle through a number of points on a sphere.

The least-squares method

Let $\mathbf{V}$ be the unit normal vector to a plane of remagnetization. Thus

$$\mathbf{V} = \begin{vmatrix} V_1 \\ V_2 \\ V_3 \end{vmatrix}$$

where V_1, V_2 and V_3 are the vector elements. Similarly if $\mathbf{U}_i$ are the measured magnetization vectors which define the remagnetization plane,

$$\mathbf{U}_i = \begin{vmatrix} U_{i1} \\ U_{i2} \\ U_{i3} \end{vmatrix}$$

where U_{i1} etc. are the vector elements which are known from the measured inclinations and declinations, and $i = 1, 2, \ldots, N$, N being the number of vectors. We wish to find the direction of the normal to the remagnetization plane such that the sum of squared deviations $\sum_{i=1}^{N} p_i^2$ of the vectors $\mathbf{U}_i$ from the plane is a minimum. In terms of $\mathbf{V}$ and $\mathbf{U}_i$, $p_i = \mathbf{U}_i . \mathbf{V}$. We therefore wish to minimise the quantity $G = \sum_{i=1}^{N} (\mathbf{U}_i . \mathbf{V})^2$. G cannot be immediately minimized by setting its partial derivatives with respect to V_1, V_2 and V_3 equal to zero and solving the equations because V_1, V_2 and V_3 are not independent quantities. Since the length of $\mathbf{V}$ must be held constant during the optimizing process the quantity $H = V_1^2 + V_2^2 + V_3^2$ must be a constant. The problem is solved using the Lagrange Multiplier Rule to form the three Euler equations:

$$\frac{\partial G}{\partial V_1} - \lambda \frac{\partial H}{\partial V_1} = 0, \quad \frac{\partial G}{\partial V_2} - \lambda \frac{\partial H}{\partial V_2} = 0, \quad \text{and} \quad \frac{\partial G}{\partial V_3} - \lambda \frac{\partial H}{\partial V_3} = 0$$

where λ is a constant known as the Lagrange Multiplier. These equations reduce to the following matrix equation $(\mathbf{M} - \lambda \mathbf{I})\,\mathbf{V} = 0$ where $\mathbf{I}$ is a unit matrix and $\mathbf{M}$ is a real symmetric matrix formed by the terms $m_{jk} = \sum_{i=1}^{N} (U_{ij} U_{ik})$ where j and k both take values from 1 to 3. The problem, which consists of finding the eigenvalues λ_k and their corresponding eigenvectors $\mathbf{V}_k$ for the matrix $\mathbf{M}$, is solved using a standard FORTRAN computer program (EIGEN) from the IBM Scientific Subroutine Package (SSP). The three eigenvectors define normals to three orthogonal planes, one of which is the least squares plane. It can be shown from the three Euler equations that the values of λ_k equal the sum of the squared deviations of the vectors $\mathbf{U}_i$ from each of the three planes $\left(\text{i.e. } \sum_{i=1}^{N} p_i^2\right)$. The minimum value of the λ_k together with its corresponding eigenvector thus yield respectively the least sum of squared deviations and the direction of the normal to the least squares plane.

Once the direction of the least squares normal has been found for one remagnetization plane, the process is repeated for all others obtained from different samples. A sample/site normal can then be defined by finding the direction of the resultant of all specimen/sample normals. If all the remagnetization circles tend to converge as shown in Fig. 3(a), their normals will be distributed along a great circle, i.e. they will tend to lie within a single plane. The normal to the least-squares plane fitted to these normals by the method described above will then define the line of intersection of all the remagnetization planes and hence the direction of one of the remanence components. The procedure is shown schematically in Fig. 3(b).

Applications of the method

The use of remagnetization circles to find the direction of a secondary remanence is illustrated for a case where the vector subtraction method fails. The example is taken from a study of late Precambrian Keweenawan lavas and feeder dikes from the Slate Islands in northern Lake Superior (Halls 1975). More than 60 specimens, each representing a different sample collected from 12 palaeomagnetic sites, yielded upon AC cleaning above 100 Oe, successive magnetization vectors which defined segments of remagnetization circles. Before structural unfolding all the circles converged upon a point whose pole (or antipole), given by the least-squares method, specifies the orientation of the common intersection line of all the remagnetization planes. Structural diversity is insufficient to permit use of the method shown in Fig. 2 to resolve uniquely the relative age and directions of the two components. Palaeomagnetic data in addition to those defining the orientation of the remagnetization circles are thus needed. In the present example a small number of specimens attained stable end-points on magnetic cleaning (Fig. 3(a)). After structural unfolding all these points group more tightly in the SE quadrant of the stereonet and yield a mean direction within 10° of that found for reversely magnetized Keweenawan igneous rocks elsewhere around Lake Superior (e.g. Palmer 1970; Halls 1974). The stable end-points in Fig. 3(a) thus record the direction of the primary remanence before unfolding and do not merely reflect stages in the AC cleaning process at which further differentiation between the two components was not possible. Hence the convergence point in Fig. 3(a) represents the pole of the secondary component, rather than its antipole, because as shown in Fig. 2 data points defining remagnetization circles must lie *between* the primary and secondary directions.

The convergence of circles in Fig. 3(a) is only in part due to structural diversity. The two groups of primary directions with declinations (D) approximately 180° and 90° are respectively from volcanics which dip $40 \pm 10°$ SW and vertical diabase dikes some three miles distant which appear unrotated. However circle convergence is also apparent *within* these two groups, especially the one with $D \doteqdot 180°$. Here the primary dispersion is principally due to variations in remanence directions *between* flows and hence is likely caused by secular variation. In comparison, the secondary dispersion must be very much less, otherwise convergent circles would not be observed. A possible reason for this anomalously low dispersion is that the secondary component was acquired in a time interval short compared to periods of scatter-inducing secular variation. The origin of such a rapidly-formed remanence is ascribed to the passage of a shock wave generated by a meteorite impact which occurred on the Slate Islands sometime after the Keweenawan volcanics were erupted and tilted (Halls 1975). All the rocks on the Slate Islands, which range in age from Archean to Keweenawan, are locally shatter-coned and cut by breccia dikes carrying shatter-coned fragments. Preliminary palaeomagnetic results on the breccias yield single,

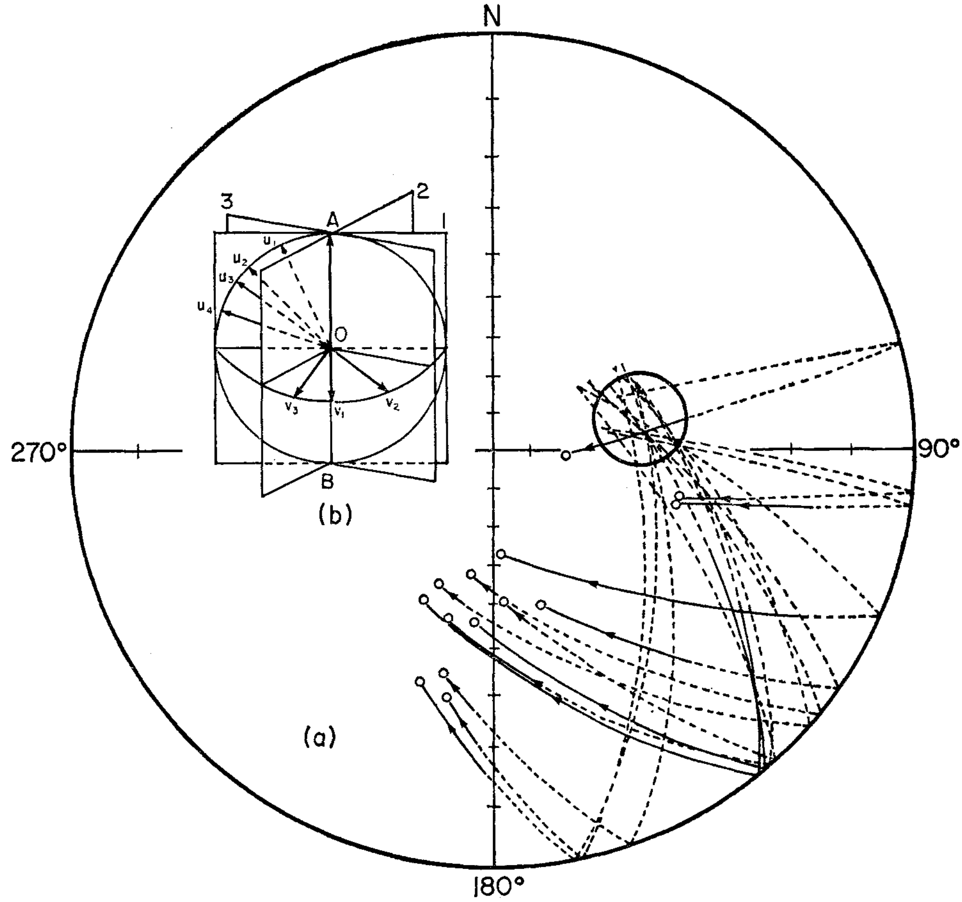

FIG. 3. Diagram 3(a) is a Wulff stereonet showing remagnetization circles obtained after AC demagnetization on samples of Late Precambrian igneous rocks from the Slate Islands, northern Lake Superior (after Halls 1975). The solid portions of circles are defined by data; the dashed parts are extrapolated. Arrows indicate direction of vector movement on stepwise cleaning. The diagram includes only data from specimens which attained stable end-points. These points are upward magnetizations and are shown as open circles. The large open circle is the approximate area of great circle convergence. The data have not been structurally unfolded. The inset diagram 3(b) illustrates schematically the principles behind the least squares method. Three remagnetization planes (1, 2, 3) intersect along the line AB. These planes are least squares fits of vector sets which define successive magnetization directions obtained on stepwise cleaning. Only one set (U_i, i = 1 to 4) for plane 1 is shown. The vector normals to the planes 1, 2 and 3 are respectively V_1, V_2 and V_3. The normal to the least-squares plane through V_1, V_2 and V_3 thus lies along AB. If U_4 is a stable end-point corresponding to the direction of one of the remanence components, then the vector OA gives the direction of the other component.

stable magnetization directions within about 10° of that given by the convergence point in Fig. 3(a).

Since the direction of both magnetization components in Fig. 3(a) is known and the magnitude and direction of the resultant is measured after successive cleaning stages, demagnetization curves for both components can be obtained. Mean curves for the 13 specimens in Fig. 3(a) are presented in Fig. 4 and indicate demagnetization of both components simultaneously until ultimate removal of the softer secondary

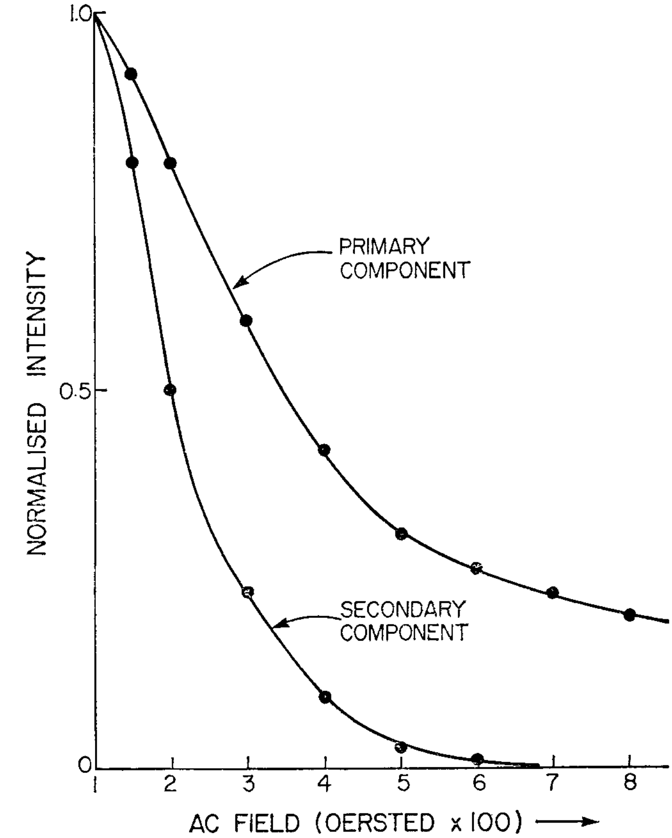

FIG. 4. Average demagnetization curves for the primary and secondary components derived from the data in Fig. 3(a). Results below AC fields of 100 Oe are not shown because in this range a third component of magnetization is often present in the specimens.

one at AC fields of about 600–700 Oe. This situation corresponds to that shown in Fig. 1(d) which explains why the direction of the secondary component could not be recovered using vector subtraction.

Convergent great circles are observed in many other fields of study besides palaeomagnetism, including for example structural geology, plate tectonics and astronomy. In structural geology the direction of a cylindrical fold axis is given by the common intersection point of great circles defined by measured tangent planes to the folded surface (Ramsay 1967). Likewise the rotation pole of lithospheric plates is found from the intersection point of great circles drawn perpendicular to transform faults (Morgan 1968). In astronomy the movement direction of a star cluster can be obtained by finding the convergence point of observed individual star trajectories projected onto the celestial sphere (e.g. Hogg 1959). A least-squares method to find the best intersection point is thus common to all these problems. While the method described above has undoubtedly been used in these types of studies it does not seem to have been fully exploited in palaeomagnetism. Furthermore, alternative but less rigorous methods have been described (e.g. Ramsay 1967) which yield solutions for least-squares planes which are not independent of the choice of reference axes, and which can also lead to large errors in the position of the best convergence point if great circle planes are used which intersect at small angles.

A further aspect of the least-squares method is that the eigenvector corresponding to the *maximum* eigenvalue defines the direction of a line from which the sum of squared deviations of all the vectors $\mathbf{U}_i$ is a minimum. This line would thus represent the best direction for a magnetization which was measured from a site population which included both normal and reversed polarities. Another use of the maximum eigenvalue would be in the directional analysis of fabric elements in rocks, such as fold axes, lineations, and normals to foliation planes.

A statistical analysis of the least-squares method is presently under study so that confidence limits can be assigned to a magnetization direction obtained from convergent great circles.

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