

and that, since $2|uv| \leq |u|^2 + |v|^2$, the latter three conditions are equivalent to the single condition (3). Accordingly, (3) assures the validity of the limit relations (9) for the general solution (p, q) of the case (10) of (7). But (11) reduces to (4), and (11) to (5), by virtue of (9).

This completes the proof of the theorem. Its assumption (3) is independent of the choice of the two, linearly independent, solutions $x = u(t)$, $x = v(t)$ of (1) which occur in (3). For, on the one hand, $u(t)$ and $v(t)$ cannot vanish at the same t and, on the other hand, the ratio of $|u^*(t)|^2 + |v^*(t)|^2$ to $|u(t)|^2 + |v(t)|^2$ stays between two positive constant bounds as $t \rightarrow \infty$. This is clear from the fact that $u^*(t) = au(t) + bv(t)$ and $v^*(t) = cu(t) + dv(t)$, where a, b, c, d are constants of non-vanishing determinant $ad - bc$.

A particular, but interesting, case of the theorem results if condition (3) is strengthened so as to make possible the elimination of the solution pair (u, v) occurring in (3). Such an elimination is made possible by using a well-known estimate, which was applied in more general forms by Liapounoff⁵ and others (L. Schlesinger, G. D. Birkhoff, O. Perron)⁶ and which, when particularized to the case of (1), states that

$$|x(t)| < \text{const.} \exp \left\{ \frac{1}{2} \int^t |f(s) - 1| ds \right\} \quad (13)$$

holds for every solution $x(t)$ of (1). What then results avoids the implicit hypothesis of the theorem, namely, that (1) has already been solved. In fact, the resulting corollary of the theorem can be formulated as follows.

If the coefficient function $g(t)$ of (2) is so "close" (for large t) to the coefficient function $f(t)$ of (1) that

$$\int_0^\infty |g(t) - f(t)| \exp \left\{ \int^t |f(s) - 1| ds \right\} dt < \infty, \quad (14)$$

then the asymptotic behavior of all solutions $y(t)$ of (2) and of their derivatives $y'(t)$ is given by (4) and (5), where c_1, c_2 is an arbitrary pair of integration constants and $u(t), v(t)$ is a pair of linearly independent solutions $x(t)$ of (1).

In fact, if (13) is applied to $x = u$ and $x = v$, then (3) reduces to (14).

A LEBEDEV TRANSFORM AND THE "BAFFLE" PROBLEM*

By C. P. WELLS AND A. LEITNER (*Michigan State University*)

1. Introduction. This note is concerned with the application of the Lebedev transform to what we term the "baffle problem," i.e. the problem of sound radiated by a vibrating circular disk in an infinite rigid baffle. The solution of this problem is not new. It has been solved by Sommerfeld [1] in terms of cylindrical waves, using the Hankel transform, and by Bouwkamp [2] and others in terms of spheroidal waves, using series representation. However, the Lebedev transform offers an equally straightforward method, representing the radiation in terms of spherical waves, and moreover is in

⁵E. Picard, *Traité d'Analyse*, 3rd ed., vol. 3, 1928, p. 385.

⁶For the particular case (13) of the general theorem, see N. Levinson, *Duke Math. J.* **8**, 2-3 (1941).

*Received November 15, 1956. Sponsored by Office of Ordnance Research, U. S. Army.

itself of considerable inherent interest. It has been applied by Kontorovich and Lebedev [3] and by Oberhettinger [4] to problems of diffraction by a wedge, and by Leitner and Wells [5] to the problem of a freely vibrating disk.

We use spherical coordinates r, θ, φ with the baffle in the plane $\theta = \pi/2$ and seek solutions, independent of φ , of

$$\nabla^2 u + k^2 u = 0, \quad (1)$$

where $k = 2\pi/\lambda$, $\lambda =$ wave length, and $u = u(r, \theta)$, the velocity potential. The boundary conditions on u are:

$$\frac{\partial u}{\partial n} = \begin{cases} v, & a \text{ constant, when } r < a, \theta = \pi/2, \\ 0, & r > a, \theta = \pi/2 \end{cases}$$

where $\partial/\partial n$ is the normal derivative, together with the radiation condition $\tau(iku + \partial u/\partial \tau) \rightarrow 0$ as $\tau \rightarrow \infty$. We now try to represent $u(r, \theta)$ as:

$$r^{1/2} u(r, \theta) = \int_L \mu g(\mu) P_{-\frac{1}{2}+\mu}(\cos \theta) J_\mu(kr) d\mu, \quad (2)$$

where $P_{1/2+\mu}(\cos \theta)$ is the Legendre function, $J_\mu(kr)$ the Bessel function, and $g(\mu)$ an unknown function of the complex variable $\mu = \sigma + i\tau$. L is a contour in a strip of finite width surrounding the imaginary μ axis from $\sigma - i\infty$ to $\sigma + i\infty$. The function $g(\mu)$ is to be determined by applying the boundary conditions to (2) and using the theorem of Kontorovich and Lebedev [3] which states that if

$$\varphi(kr) = \int_L \mu \Lambda(\mu) e^{i\pi\mu/2} J_\mu(kr) d\mu, \quad (3a)$$

then

$$\pi i \Lambda(\mu) = \int_0^\infty \varphi(kr) e^{-i\pi\mu/2} H_\mu^{(2)}(kr) dr/r. \quad (3b)$$

The conditions for validity of this theorem and other details can be found in the reference given.

One now applies the boundary conditions to (2) only to find that the resulting $\Lambda(\mu)$ is such that the conditions of the theorem are not satisfied and the integrals diverge. However, there is an analogous theorem [6] in terms of modified Bessel functions of real argument which imposes considerably milder restrictions on $\Lambda(\mu)$. This suggests making the transition $k = -i\gamma$, γ real and positive, constructing a solution using the modified functions $I_\mu(\gamma r)$ and $K_\mu(\gamma r)$ and then returning to real k and the original Bessel functions. Obviously the integrals would still diverge if the contour L remains unchanged, but they will converge if L is first deformed to surround the positive real μ axis. This was recently demonstrated by Oberhettinger [4] and verified again by the result of the present problem.

The representation of $\mu(r, \theta)$ corresponding to real γ is

$$r^{1/2} u(r, \theta) = \int_L \mu g(\mu) P_{-\frac{1}{2}+\mu}(\cos \theta) I_\mu(\gamma r) d\mu. \quad (4)$$

The "baffle problem" can now be solved by means of the following Lebedev [6] theorem: If

$$\varphi(\gamma r) = \int_L \mu \Lambda(\mu) I_\mu(\gamma r) d\mu, \tag{5a}$$

then

$$\pi i \Lambda(\mu) = \int_0^\infty \varphi(\gamma r) K_\mu(\gamma r) dr/r. \tag{5b}$$

The conditions for validity are that $\Lambda(\mu)$ be an even function of μ , analytic in a strip of finite width including the imaginary axis and of decay at least as fast as

$$|\tau|^{-1-\epsilon} \exp(-\pi |\tau|/2), \text{ where } \epsilon > 0.$$

2. Some properties of $I_\mu(\gamma r)$ and $K_\mu(\gamma r)$ as functions of μ . The functions $I_\mu(\gamma r)$ and $K_\mu(\gamma r)$ with real argument γr are entire functions with an infinite number of simple zeros [7]. For $|\mu/\gamma r| \gg 1$,

$$I_\mu(\gamma r) = [(\gamma r/2)^\mu / \Gamma(1 + \mu)] [1 + O(\mu^{-1})]. \tag{6}$$

Hence $I_\mu(\gamma r)$ decays rapidly as $\text{Re } \mu \rightarrow +\infty$. On the left half plane, as $\text{Re } \mu \rightarrow -\infty$, it has Γ function-like growth since $1/\Gamma(1 + \mu) = -\sin \pi \mu \Gamma(-\mu)/\pi$. Along the imaginary axis $\mu = i\tau$, $I_\mu(\gamma r)$ has exponential growth as $\tau \rightarrow \pm \infty$.

The function $K_\mu(\gamma r)$ is defined by

$$K_\mu(\gamma r) = (\pi/2) [I_{-\mu}(\gamma r) - I_\mu(\gamma r)] / \sin \pi \mu. \tag{7}$$

As $\text{Re } \mu \rightarrow \infty$, $I_{-\mu}(\gamma r)$ is dominant and as $\text{Re } \mu \rightarrow -\infty$, $I_\mu(\gamma r)$ is dominant. Hence $K_\mu(\gamma r)$ has Γ function-like growth on both right and left half planes. Along the imaginary axis $K_\mu(\gamma r)$ decays exponentially as $\tau \rightarrow \pm \infty$. Asymptotic forms for $I_\mu(\gamma r)$ and hence for $K_\mu(\gamma r)$ can, of course, be found from (6) using Stirling's formula for $\Gamma(1 + \mu)$.

3. Solution of the "baffle" problem. We represent $u(r, \theta)$ by means of (4) and enforce the boundary conditions as modified by transition to real γ :

$$\left. \begin{aligned} r \leq a, & \quad \gamma^{3/2} v \\ r \geq a, & \quad 0 \end{aligned} \right\} = \int_L \mu \Lambda(\mu) I_\mu(\gamma r) d\mu. \tag{8}$$

Here $\Lambda(\mu) = g(\mu) P'_{-\frac{1}{2}+\mu}(0)$, where the prime indicates differentiation with respect to the argument $\cos \theta$. The formal solution is given by the inversion integral (5b):

$$\pi i \Lambda(\mu) = v \int_0^a r^{\frac{1}{2}} K_\mu(\gamma r) dr. \tag{9}$$

The integral defines an even function of μ and converges if μ is restricted to a strip of half-width $3/2$ about the imaginary axis. Within this strip $\Lambda(\mu)$ is analytic and $|\Lambda(\mu)| \approx e^{-\pi |\tau|^{1/2}/\tau^{3/2}}$ as $|\tau| \rightarrow \infty$. From this it is seen that $\Lambda(\mu)$ satisfies the conditions of the Levedev theorem.

The integral (9) can be expressed in terms of the Lommel functions and they in terms of their series representation [8]. Thus one can continue $\Lambda(\mu)$ into the remainder of the μ plane where it is analytic except at the points $\mu = \pm (2n + 3/2)$, $n = 0, 1, 2, \dots$, where it has simple poles with residues $[2^{1/2} \nu(-)^n i / \pi \gamma^{3/2}] [\Gamma(n + 3/2)/n!]$. For $\text{Re } \mu \rightarrow \pm \infty$, we find also that $\Lambda(\mu) \approx I_{\mp \mu}(\gamma a) / (3/2 \pm \mu) \sin \pi \mu$. We now have $\Lambda(\mu)$ defined for the entire μ plane and are ready to convert the integral (4) into an eigenfunction expansion.

Substituting for $g(\mu)$ in (4) gives

$$r^{\frac{1}{2}}u(r, \theta) = \int_L \mu \Lambda(\mu) \frac{P_{-\frac{1}{2}+\mu}(\cos \theta)}{P_{-\frac{1}{2}+\mu}(0)} I_\mu(\gamma r) d\mu, \tag{10}$$

in which $\mu \Lambda(\mu) I_\mu(\gamma r) \approx (r/a)^\mu / \mu$ for large μ on the right half plane. Further the ratio of the Legendre functions behaves like $e^{(\theta-\pi/2)\mu}$ when $\theta \leq \pi/2$ the only values of θ in our problem. Hence for $r \leq a$ the contour Γ may be closed on the right and the integral is unchanged in value. The poles of the integrand are at $2n + 1/2$ and $2n + 3/2$ and the series of residues is the eigenfunction representation for the region $r \leq a$.

For $r \geq a$ the contour of (10) in its present form can not be closed. However the decomposition

$$\Lambda(\mu) = (\pi/2)[\lambda(-\mu) - \lambda(\mu)]/\sin \pi\mu$$

allows the contour to be closed. Then

$$\pi i \lambda(\mu) = v \int_0^a r^{\frac{1}{2}} I_\mu(\gamma r) dr, \tag{11}$$

and, by arguments similar to those used above for $\Lambda(\mu)$ we find $\lambda(\mu)$ to have poles at $-(2n + 3/2)$, analytic in the strip with growth like $I_\mu(\gamma a)/(3/2 - \mu)$, for large μ .

We now have

$$r^{\frac{1}{2}}u(r, \theta) = \int_L \mu \lambda(\mu) \frac{P_{-\frac{1}{2}+\mu}(\cos \theta)}{P_{-\frac{1}{2}+\mu}(0)} K_\mu(\gamma r) d\mu, \tag{12}$$

where $\mu \lambda(\mu) K_\mu(\gamma r) \approx \mu^{-1} (v/a)^{-\mu}$, $|\mu| \rightarrow \infty$. Hence the contour can be closed on the right for $r \geq a$. The poles of the integrand lie at $2n + 1/2$, whose residues lead to the appropriate series of eigenfunctions in the space $r \geq a$. One sees now how the transition from integral representation to series of appropriate eigenfunctions resolves itself. When the transition from real γ to real k is made we see that the expansion will be in terms of Bessel functions for $r \leq a$, in terms of Hankel functions for $r \geq a$.

For completeness we record the eigenfunction expansion:

$$r \leq a: u(r, \theta) = \sum_{n=0}^{\infty} a_n P_{2n+1}(\cos \theta) j_{2n+1}(kr) + \sum_{n=0}^{\infty} b_n P_{2n}(\cos \theta) j_{2n}(kr),$$

$$r \geq a: u(r, \theta) = \sum_{n=0}^{\infty} c_n P_{2n}(\cos \theta) h_{2n}^{(2)}(kr),$$

where j and h stand for the spherical Bessel and Hankel functions and

$$a_n = (2v/k)(2n + 3/2)(-1)^n,$$

$$b_n = (2)^{\frac{1}{2}} v a i (2n + \frac{1}{2})(-1)^{n+1} [\Gamma(n + \frac{1}{2})/n!] W \{ H_{2n+\frac{1}{2}}^{(2)}(ka), s_{\frac{1}{2}, 2n+\frac{1}{2}}(ka) \},$$

$$c_n = ((2)^{\frac{1}{2}} i v/k)(2n + \frac{1}{2})(-1)^n [\Gamma(n + \frac{1}{2})/n!] \int_0^{ka} x^{\frac{1}{2}} J_{2n+\frac{1}{2}}(x) dx,$$

where W stands for Wronskian and $s_{\frac{1}{2}, 2n+\frac{1}{2}}$ is a Lommel function [8]. Note that for $r \geq a$ only the Legendre polynomials even in $\cos \theta$ appear, as required by the boundary condition.

REFERENCES

1. A. Sommerfeld, *Ann. Physik* **42**, 389 (1942)
2. C. J. Bouwkamp, Dissertation, Groningen, 1941
3. M. J. Kontorovich and N. N. Lebedev, *J. Phys. Moscow* **1**, 229 (1939)
4. F. Oberhettinger, *Communs. Pure and Appl. Math.* **7**, 553 (1954)
5. A. Leitner and C. P. Wells, *I.R.E. Trans. P.G.A.P.* **4**, October, 1956
6. N. N. Lebedev, *Doklady Akad. Nauk S.S.S.R.* **58**, 1007 (1947)
7. M. J. Coulomb, *Bull. sci. mathematiques* **60**, 297 (1936)
8. A. Erdelyi et al., *Higher transcendental functions*, vol. 2, McGraw-Hill, New York, 1953, p. 40

GEOMETRIC INTERPRETATION FOR THE RECIPROCAL DEFORMATION TENSORS*

BY C. TRUESDELL (*Indiana University and Università di Bologna*)

In a finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$, changes of infinitesimal lengths may be measured by the tensor \mathbf{C} , where

$$ds^2 = g_{km} dx^k dx^m = C_{KM} dX^K dX^M, \quad C_{KM} = g_{km} x^k_{,K} x^m_{,M}, \quad (1)$$

or by the dual tensor \mathbf{c} satisfying formulae that follow by systematic interchange of majuscules and minuscules. Geometric interpretations of \mathbf{C} and \mathbf{c} have been given by Cauchy and others. In 1894 Finger introduced the reciprocal tensors \mathbf{C}^{-1} and \mathbf{c}^{-1} , and recent exact work on isotropic elastic bodies employs them often. While formulae such as

$$(\mathbf{C}^{-1})^{KM} = g^{km} X^k_{,K} X^m_{,M} \quad (2)$$

for their expression and use are known, geometric interpretation has been lacking.

As is known, the correspondence between elements of area is given by $da^{km} = x^k_{,K} x^m_{,M} dA^{KM}$, where dA^{KM} is connected with the usual vector element of area dA_K by $dA_K = e_{KMP} dA^{MP}$, $e_{KMP} \equiv (\det G_{QR})^{1/2} \epsilon_{KMP}$. Hence¹

$$\begin{aligned} (da)^2 &= e_{pq}^k e_{kr}^s x^p_{,P} x^q_{,Q} x^r_{,R} x^s_{,S} dA^{PQ} dA^{RS}, \\ &= \frac{\det g_{uv}}{\det G_{UV}} g^{km} \left(\frac{1}{2} \epsilon_{krs} \epsilon^{KRS} x^r_{,R} x^s_{,S} \right) \left(\frac{1}{2} \epsilon_{mpq} \epsilon^{MPQ} x^p_{,P} x^q_{,Q} \right) dA_K dA_M, \\ &= \frac{\det g_{uv}}{\det G_{UV}} \left[\frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)} \right]^2 g^{km} X^k_{,K} X^m_{,M} dA_K dA_M, \\ &= [\det (\mathbf{C}^{-1})_0^p]^{-1} (\mathbf{C}^{-1})^{KM} dA_K dA_M. \end{aligned} \quad (3)$$

Comparing this result with (1) shows that the tensor $\mathbf{C}^{-1}/\det \mathbf{C}^{-1}$ measures changes of the magnitudes of infinitesimal areas in precisely the same way as \mathbf{C} measures changes of infinitesimal lengths.

A known principle of duality, which may be called the *first* principle of duality,

*Received Dec. 4, 1956. This work was done under a National Science Foundation grant to Indiana University.

¹A formula which is essentially the next to last step in (3) was given by Tonolo, *Rend. sem. mat. Padova* **14**, 43-117 (1943), Sec. V. 4, but he did not mention any connection with \mathbf{C}^{-1} .