# A length operator for canonical quantum gravity 

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Preprint HUTMP-96/B-354


#### Abstract

We construct an operator that measures the length of a curve in fourdimensional Lorentzian vacuum quantum gravity.

We work in a representation in which a $S U(2)$ connection is diagonal and it is therefore surprising that the operator obtained after regularization is densely defined, does not suffer from factor ordering singularities and does not require any renormalization.

We show that the length operator admits self-adjoint extensions and compute part of its spectrum which like its companions, the volume and area operators already constructed in the literature, is purely discrete and roughly is quantized in units of the Planck length.

The length operator contains full and direct information about all the components of the metric tensor which faciliates the construction of a new type of weave states which approximate a given classical 3-geometry.


## 1 Introduction

If one was working in a representation of canonical quantum gravity for which the intrinsic metric $q_{a b}$ of an initial data hypersurface was the configuration variable then the operator corresponding to the length of a curve would be fairly easy to construct because it would act by multiplication.
However, in this so-called geometrodynamic representation []] one of the most important operators, the Wheeler-DeWitt constraint operator, adopts an algebraic form which is so difficult that almost half a century after its discovery it is still unknown how to rigorously define it and much less how to solve it (compare [7] for a detailed analysis).
Fortunately, there is an alternative, called the connection representation, in which the Wheeler-DeWitt operator adopts a polynomial form after multiplying it by $\sqrt{\operatorname{det}\left(\left(q_{a b}\right)\right)}$. In the Lorentzian signature the underlying connection is complex valued [3] which renders the task of incorporating the correct reality conditions on the classical phase space variables at the quantum level into a difficult one. However,

[^0]these complications can be overcome in two different ways: the first possibility is to start with Euclidean gravity, in which the Wheeler-DeWitt constraint is also a polynomial [4], and to Wick rotate the Euclidean theory into the Lorentzian regime [6]. The second possibility arises from the recent discovery [6, (7] that by employing a certain novel technique it is indeed possible to obtain a finite, densely defined (and symmetric) operator corresponding to the original Wheeler-DeWitt constraint directly in the Lorentzian regime and without multiplying by a power of $\sqrt{\operatorname{det}(q)}$. This technique is independent of the one described in [8] which was restricted to the classical case and unavoidably is doomed to produce a highly singular Hamiltonian constraint operator (compare [22] where this singularity is discussed in the restricted context of lattice quantum gravity). In contrast, the one proposed in [7], 8] is perfectly finite everywhere on the Hilbert space. Also, it is independent of the technique used in [9] which suffers from other problems which have to do with taking the square root of a non-positive and not self-adjoint operator besides being a quantization of the Euclidean constraint operator only.
In the real connection representation which we will use in the sequel the metric depends on the momentum operator only, and it depends on it in a non-polynomial fashion. It is therefore much harder to define a well-defined operator corresponding to it. In fact, that is why only volume and area operators could be constructed so far [19, [1]. It turns out that the same novel technique introduced in [6, 7] can be used to derive a completely well-defined operator corresponding to the length of a curve.
The article is organized as follows:
In section 2 we fix the notation and recall the necessary information about the Hilbert space and techniques that come with the space of generalized connections modulo gauge transformations $\overline{\mathcal{A} / \mathcal{G}}$.

In section 3 we derive the length operator and show that it is an unbounded symmetric operator on the full Hilbert space with dense domain and that it has self-adjoint extensions.

In section 4 we derive several properties of its spectrum and compute it for some simple situations.

In section 5 we comment on how one can construct "weave states" which approximate a given classical geometry. We need to consider weave states which are more general than most of the ones previously considered in the literature in the sense that they necessarily involve intersections and overlappings of the loops involved and should therefore be rather called "lattice states".

## 2 The real connection representation

Let $q_{a b}$ be the intrinsic metric of an initial data hypersurface $\Sigma$ and let $K_{a b}$ be its extrinsic curvature. Introduce a triad field $e_{a}^{i}$ which transforms like a 1-form under diffeomorphisms of $\Sigma$ and according to the adjoint representation of $S U(2)$ so that $q_{a b}=\delta_{i j} e_{a}^{i} e_{b}^{j}$. Let $e_{i}^{a}$ be its inverse and define $K_{a}^{i}:=\operatorname{sgn}\left(\operatorname{det}\left(\left(e_{b}^{j}\right)\right)\right) K_{a b} e_{i}^{b}$ and $E_{i}^{a}:=\operatorname{det}\left(\left(e_{b}^{J}\right)\right) e_{i}^{a}$. Then one can show that $\left(K_{a}^{i} / \kappa, E_{i}^{a}\right)$ is a canonical pair for Lorentzian four-dimensional canonical gravity, where $\kappa$ is Newton's constant.
Now consider the spin-connection $\Gamma_{a}^{i}$, that is, the unique connection which annihi-
lates the triad $e_{a}^{i}$. One can define a $S U(2)$ connection $A_{a}^{i}:=\Gamma_{a}^{i}+K_{a}^{i}$ which has the correct dimension of an inverse length. Then one can show that the so-called real Ashtekar variables given by $A_{a}^{i} / \kappa, E_{i}^{a}$ define a canonical pair. This observation is due to Ashtekar [3, [4]. In retrospect, since this real connection formulation does not simplify the Wheeler-DeWitt constraint too much while the complex formulation suffers from the other problems mentioned in the introduction, the virtue of using a connection dynamics formulation rather than a geometrodynamical one is the following : one can use techniques normally employed in Yang-Mills theory. In particular, one can use Wilson loop variables which serve as good coordinates on $\mathcal{A} / \mathcal{G}$, the space of smooth connections modulo gauge transformations. The use of loops to probe connections is radical : those Wilson loop variables can become well-defined operators only if the excitations of geometry are string-like rather than bubble-like. On the other hand, given that assumption, it is possible to explicitly characterize the quantum configuration space $\overline{\mathcal{A} / \mathcal{G}}$ of generalized connections modulo gauge transformations. This is the precise analogue, in the connection representation, of "Wheeler's superspace" in the metric formulation which, to the best of our knowledge, was never specified precisely.
One can show that the elements of $\overline{\mathcal{A} / \mathcal{G}}$ are in one-to-one correspondence with the set of all homomorphisms from the group of holonomically equivalent loops in $\Sigma$ into $S U(2) /$ Ad [12]. Moreover, there is a $\sigma$ - additive, faithful and diffeomorphism invariant probability measure $\mu_{0}$ on $\overline{\mathcal{A} / \mathcal{G}}$ which equips us with a Hilbert space structure $\mathcal{H}:=L_{2}\left(\overline{\mathcal{A} / \mathcal{G}}, d \mu_{0}\right)$. This measure is defined as follows :
In the sequel we will denote by $\gamma$ a closed graph, that is, a collection of analytic edges $e$ which intersect in vertices $v$ such that each vertex is at least two-valent. A function $f$ on $\overline{\mathcal{A} / \mathcal{G}}$ is said to be cylindrical with respect to $\gamma$ iff it is of the form $f(A)=\left(f_{\gamma} \circ p_{\gamma}\right)(A)=f_{\gamma}\left(h_{e_{1}}(A), . ., h_{e_{n}}(A)\right)$ where $e_{1}, . ., e_{n}$ are the edges of $\gamma, h_{e}$ is the holonomy along $e_{i}$ and $f_{\gamma}$ is a complex-valued function on $S U(2)^{n}$ which is gauge invariant. So each cylindrical function is determined through a graph $\gamma$ and such an $f_{\gamma}$ and one says that $f_{\gamma}$ and $f_{\gamma}^{\prime}$ are equivalent whenever their pullbacks agree, that is $p_{\gamma}^{*} f_{\gamma}=p_{\gamma^{\prime}}^{*} f_{\gamma^{\prime}}$. Let us denote by $\operatorname{Cyl}_{\gamma}(\overline{\mathcal{A} / \mathcal{G}})$ the collection of functions cylindrical with respect to $\gamma$ modulo cylindrical equivalence and denote by $\operatorname{Cyl}(\overline{\mathcal{A} / \mathcal{G}}):=\cup_{\gamma} \operatorname{Cyl}_{\gamma}(\overline{\mathcal{A} / \mathcal{G}})$ the set of all cylindrical functions.
The measure $\left(\mu_{0}\right)$ can now be seen as the $\sigma$-additive extension [13] of the following self-consistent family of measures $\left(\mu_{0, \gamma}\right)_{\gamma}$ [14] : let $f \in \operatorname{Cyl}_{\gamma}(\overline{\mathcal{A} / \mathcal{G}})$ then

$$
\begin{equation*}
\int_{\overline{\mathcal{A} / \mathcal{G}}} d \mu(A) f(A):=\int_{\overline{\mathcal{A} / \mathcal{G}}} d \mu_{0, \gamma}(A) f(A):=\int_{S U(2)^{n}} d \mu_{H}\left(g_{1}\right) . . d \mu_{H}\left(g_{n}\right) f_{\gamma}\left(g_{1}, . ., g_{n}\right) \tag{2.1}
\end{equation*}
$$

where $\mu_{H}$ is the Haar measure on $S U(2)$ and $g_{I}=h_{e_{I}}(A)$. In other words, everything is reduced to finite dimensional integrals over $S U(2)$. One can show that $\operatorname{Cyl}(\overline{\mathcal{A} / \mathcal{G}})$ is dense in $\mathcal{H}$.
Thus integral calculus is introduced on $\overline{\mathcal{A} / \mathcal{G}}$. One can even develop differential calculus on $\overline{\mathcal{A} / \mathcal{G}}$ [15] : a differentiable cylindrical function is simply a differentiable function on $S U(2)^{n}$. Functional derivatives can be evaluated on differentiable elements of $\operatorname{Cyl}(\overline{\mathcal{A} / \mathcal{G}})$ because on a finite graph a distributional connection can be replaced by a smooth one [12, 14]. One sees that differential and integral calculus is inherited from the one on $S U(2)^{n}$. Finally, let us define the spaces
$\operatorname{Cyl}_{\gamma}^{n}(\overline{\mathcal{A} / \mathcal{G}}), \operatorname{Cyl}^{n}(\overline{\mathcal{A} / \mathcal{G}})=\cup_{\gamma} \mathrm{Cyl}_{\gamma}^{n}(\overline{\mathcal{A} / \mathcal{G}})$ of differentiable cylindrical functions of order $n=0,1, \ldots, \infty$. Each of $\operatorname{Cyl}^{n}(\overline{\mathcal{A} / \mathcal{G}})$ is dense in $\mathcal{H}$.

## 3 The length operator

The stage is now set to derive the length operator. Before we do that we wish to comment why it was up to now impossible to define this operator rigorously on $\mathcal{H}$ and why it was possible to define volume and area operators.
Consider for instance the volume of a region $R \subset \Sigma$

$$
\begin{equation*}
V(R):=\int_{R} d^{3} x \sqrt{|\operatorname{det}(q)|}=\int_{R} d^{3} x \sqrt{\left|\frac{1}{3!} \epsilon_{a b c} \epsilon^{i j k} E_{i}^{a} E_{j}^{b} E_{k}^{c}\right|} \tag{3.1}
\end{equation*}
$$

where $\epsilon_{a b c}$ carries density weight -1 . We see that this functional involves the square root of a polynomial in $E_{i}^{a}$. This is important because, according to the canonical commutation relations, we are supposed to replace $E_{i}^{a}$ by $\hat{E}_{i}^{a}=-i \ell_{p}^{2} \frac{\delta}{\delta A_{a}^{2}}$ where $\ell_{p}:=\sqrt{\kappa \hbar}$ is the Planck length. Now one tries to regularize the polynomial and defines its square root by its spectral resolution. This has been done successfully in the literature 10, 11.
We know how to regularize a polynomial in the basic operators $A, E$ but certainly we do not know how to define a non-polynomial function. This is precisely the case for

$$
\begin{equation*}
q_{a b}=\epsilon_{a c d} \epsilon_{b e f} \epsilon^{i j k} \epsilon^{i m n} \frac{E_{j}^{c} E_{k}^{d} E_{m}^{e} E_{n}^{f}}{4 \operatorname{det}\left(\left(E_{l}^{g}\right)\right)^{2}} . \tag{3.2}
\end{equation*}
$$

Even if one could define it, it is by now well known that the operator version of the denominator of (3.2) has a huge kernel so that it could not be defined on a dense domain. This is the reason why a quantization of the length of a piecewise differentiable curve $c:[0,1] \rightarrow \Sigma ; t \rightarrow c(t)$ given classically by

$$
\begin{equation*}
L(c):=\int_{[0,1]} d t \sqrt{\dot{c}^{a}(t) \dot{c}^{b}(t) q_{a b}(c(t))} \tag{3.3}
\end{equation*}
$$

has escaped its quantization in the representation $\mathcal{H}$ so far. Note that no absolute value signs are necessary in (3.3) under the square root since $\dot{c}^{a} \dot{c}^{b} q_{a b}=$ $\left(\dot{c}^{a} e_{a}^{i}\right)\left(\dot{c}^{b} e_{b}^{j}\right) \delta_{i j} \geq 0$.

### 3.1 Regularization of the length operator

The regularization of (3.3) is based on two key observations :
Observation 1)
The triad is integrable with generating functional $V:=V(\Sigma)$, the total volume of $\Sigma$. More precisely we have $e:=\operatorname{sgn}\left(\operatorname{det}\left(\left(e_{a}^{i}\right)\right)\right)$

$$
\begin{equation*}
\frac{\delta V}{\delta E_{i}^{a}}=e \frac{e_{a}^{i}}{2}=\frac{1}{\kappa}\left\{A_{a}^{i}, V\right\} \tag{3.4}
\end{equation*}
$$

which can be verified immediately.
Observation 2)

The total volume can be quantized in a mathematically rigorous way. Its action on sufficiently differentiable cylindrical functions is given by (we follow 11])

$$
\begin{align*}
\hat{V} f & =\ell_{p}^{3} \hat{v} f=\ell_{p}^{3} \sum_{v \in V(\gamma)} \hat{V}_{v} f_{\gamma} \\
& =\ell_{p}^{3} \sum_{v \in V(\gamma)} \sqrt{\left|\frac{i}{8 \cdot 3!} \sum_{e_{I} \cup e_{J} \cup e_{K}=v} \epsilon\left(e_{I}, e_{J}, e_{K}\right) \epsilon_{i j k} X_{I}^{i} X_{J}^{j} X_{K}^{k}\right|} \tag{3.5}
\end{align*} f_{\gamma} .
$$

Here $g_{I}=h_{e_{I}}(A), X_{I}=X\left(g_{I}\right)$ and $X^{j}(g):=\operatorname{tr}\left(g\left[-i \sigma_{j} / 2\right] \partial_{g}\right)$ are the components of the right invariant vector field on $S U(2)$ ( $\sigma_{j}$ are Pauli matrices) where we have assumed that the edges are outgoing at a vertex. $V(\gamma)$ is the set of vertices of $\gamma$ and finally $\epsilon\left(e_{I}, e_{J}, e_{K}\right)=\operatorname{sgn}\left(\operatorname{det}\left(\dot{e}_{I}(0), \dot{e}_{J}(0), \dot{e}_{K}(0)\right)\right)$. Note that $\hat{v}$ is a dimensionless operator. One can show that (3.5) has dense domain $\operatorname{Cyl}^{3}(\overline{\mathcal{A} / \mathcal{G}})$ and is an essentially self-adjoint operator on $\mathcal{H}$ with a purely discrete spectrum (compare the forthcoming companion paper to [11]).
These two observations motivate the following regularization strategy.
Choose the basis in $s u(2)$ given by $\tau_{j}=-i \sigma_{j} / 2$ and write $e_{a}=e_{a}^{i} \tau_{i}, A_{a}=A_{a}^{i} \tau_{i}$ so that for smooth $A$ we have $h_{e}(A)=1+A(e)+o\left(A(e)^{2}\right)$ where $A(e)=\int_{e} A$. Then, according to (3.4)

$$
\begin{equation*}
q_{a b}=-2 \operatorname{tr}\left(e_{a} e_{b}\right)=-\frac{8}{\kappa^{2}} \operatorname{tr}\left(\left\{A_{a}, V\right\}\left\{A_{b}, V\right\}\right) \tag{3.6}
\end{equation*}
$$

Clearly we now are going to replace the integral in (3.3) by a Riemann sum and take the limit. So let $t_{0}=0<t_{1}<. .<t_{n}=1$ be a partition of $[0,1]$ such that points $t$ of non-differentiability of $c$ are among the values $t_{i}$ and let $\Delta_{i}:=t_{i}-t_{i-1}$. It is understood that in the limit $n \rightarrow \infty$ also $\delta:=\max \left(\left(\Delta_{i}\right)_{i=1}^{n}\right) \rightarrow 0$. Consider then the following quantity

$$
\begin{equation*}
L_{n}(c):=\frac{1}{\kappa} \sum_{i=1}^{n} \sqrt{2 \operatorname{tr}\left(\left\{h_{c}\left(t_{i-1}, t_{i}\right), V\right\}\left\{h_{c}\left(t_{i-1}, t_{i}\right)^{-1}, V\right\}\right)} . \tag{3.7}
\end{equation*}
$$

Here $h_{c}(s, t)$ denotes the holonomy of $A$ along $c$ from the parameter value $s \rightarrow$ $t$. It is easy to see that for a classical (that is, smooth) connection we have $\left\{h_{c}\left(t_{i-1}, t_{i}\right)^{ \pm 1}, V\right\}= \pm \Delta_{i} \dot{c}^{a}\left(\tilde{t}_{i}\right)\left\{A_{a}\left(c\left(\tilde{t}_{i}\right), V\right\}+\mathrm{o}\left(\Delta_{i}^{2}\right)\right.$ where $\tilde{t}_{i}$ is some value of $t \in$ $\left[t_{i-1}, t_{i}\right]$. Therefore (3.7) converges classically to (3.3) in the limit $n \rightarrow \infty$. Note that the argument of the square root is manifestly gauge invariant. The motivation to replace the $A^{\prime} s$ by holonomies of course comes from the fact that $\hat{V}$ has finite action on holonomies as seen from (3.5).
The final step is to replace $V$ by $V$ and Poisson brackets by commutators times $1 / i \hbar$. The result is

$$
\begin{equation*}
\hat{L}_{n}(c):=\ell_{p} \sum_{i=1}^{n} \sqrt{-8 \operatorname{tr}\left(\left[h_{c}\left(t_{i-1}, t_{i}\right), \hat{v}\right]\left[h_{c}\left(t_{i-1}, t_{i}\right)^{-1}, \hat{v}\right]\right)} \tag{3.8}
\end{equation*}
$$

To complete the definition of the length operator we now define for each thrice differentiable cylindrical function $f \in \operatorname{Cyl}^{3}(\overline{\mathcal{A} / \mathcal{G}})$

$$
\begin{equation*}
\hat{L}(c) f:=\lim _{n \rightarrow \infty}\left[\hat{L}_{n}(c) f\right] \tag{3.9}
\end{equation*}
$$

As it stands, (3.9) does not make sense yet because we have not shown that the limit exists and it may matter how to choose the partition $\left(t_{i}\right)_{i=0}^{n}$. Also, it is far from obvious that the square root is well defined because we did not show that its argument is a positive and self-adjoint operator. These issues will be settled in the next subsections.

### 3.2 Finiteness and choice of the partition adapted to a graph

Let $f$ be a function cylindrical with respect to a graph $\gamma$ and let $s$ be one of the segments of $c$ into which we have partitioned it. We wish to study the action of the operator

$$
\begin{equation*}
\hat{l}_{s}^{2}:=-8 \operatorname{tr}\left(\left[h_{s}, \hat{v}\right]\left[h_{s}^{-1}, \hat{v}\right]\right) \tag{3.10}
\end{equation*}
$$

on $f$. By choosing $n$ in (3.9) large enough we may assume without loss of generality that $s$ and $\gamma$ are either disjoint, intersect in at most one point or $s$ is contained in $\gamma$ (here we have made use of the piecewise analyticity of $\gamma$ (14]). Further we adapt the partition to the graph in the following way : a) if $s$ and $\gamma$ intersect in a point then this point is a boundary point of $s$ and b ) if $s$ is contained in $\gamma$ then it is contained in an edge of $\gamma$. This is a choice that we have to make in order to achieve independence of the partition. The resulting operator is different if one assumes that $s$ and $\gamma$ do not intersect in an endpoint of $s$.
We have in general $V(s \cup \gamma)=V(\gamma) \cup V(s) \cup V(\gamma \cap s)=V(\gamma) \cup V(\gamma \cap s)$. The volume operator applied to a graph is a sum of the $\hat{V}_{v}$ for each vertex of the graph and $\hat{V}_{v}$ only depends on those edges of the graph which are incident at $v$ as follows from (3.5). Therefore $h_{s} \hat{V}_{v}=\hat{V}_{v} h_{s}$ if $s$ is not incident at $v$. It follows that

$$
\begin{align*}
& {\left[h_{s}^{-1}, \hat{v}\right] f=\sum_{v \in V(\gamma)} h_{s}^{-1} \hat{V}_{v} f-\sum_{v \in V(\gamma \cup s)} \hat{V}_{v} h_{s}^{-1} f } \\
= & \sum_{v \in V(\gamma) \cap \partial s}\left[h_{s}^{-1}, \hat{V}_{v}\right] f-\sum_{v \in V(\gamma \cap s)-V(\gamma)} \hat{V}_{v} h_{s}^{-1} f \tag{3.11}
\end{align*}
$$

and both sums in the last line have at most one non-vanishing term corresponding to an endpoint of $s$ intersecting $\gamma$.
Case 1) $\gamma \cap s=\emptyset$
Then $\hat{l}_{s}^{2} f=0$ as is immediate from (3.11).
Case 2) $p \in \gamma \cap s$
Subcase a) $p$ is neither a vertex of $\gamma$ nor a kink of $c$ and so only the second term in (3.11) survives.
This implies that $p$ is a trivalent vertex with only two independent tangent directions of the edges of $\gamma$ and the segment $s$ incident at $p$. The properties of the volume operator now imply that the contribution $\hat{V}_{p}$ of $\hat{v}$ vanishes and therefore $\hat{l}_{p}^{2} f=0$.

Subcase b) $p$ is not a vertex of $\gamma$ but a kink of $c$.
Since $s$ is only one of the segments of $c$ incident at $p$ this case is not different from a).

Subcase c) $p$ is a vertex of $\gamma$ but not a kink of $c$.
Now the result can be non-vanishing. We have for a vertex $p=v$ of $\gamma$ that $\hat{l}_{s}^{2} f=$ $-8 \operatorname{tr}\left(\left[h_{s}, \hat{V}_{v}\right]\left[h_{s}^{-1}, \hat{V}_{v}\right]\right) f$, that is, only the first term in (3.11) survives. That the volume operator in the second commutator also reduces to $\hat{V}_{v}$ follows by a similar argument.

Subcase d) $p$ is both a vertex of $\gamma$ and a kink of $s$.
Again, this case does not differ from c) for the same reason as b) was equal to a). So we conclude that for large enough $n$ the operator (3.8) reduces to

$$
\begin{equation*}
\hat{L}_{n}(c):=\ell_{p} \sum_{v \in V(\gamma)} \sum_{v \in s_{i}} \sqrt{-8 \operatorname{tr}\left(\left[h_{s_{i}}, \hat{V}_{v}\right]\left[h_{s_{i}}^{-1}, \hat{V}_{v}\right]\right)} \tag{3.12}
\end{equation*}
$$

where we have denoted the segment $s_{i}:=c\left(\left[t_{i-1}, t_{i}\right]\right)$. This demonstrates that $\hat{L}_{n}(c)$ is a finite operator for each $n$ because there are at most $|V(\gamma)|$ terms in the sum (3.12). The next question is whether the limit $n \rightarrow \infty$ exists.

### 3.3 Existence of the limit $n \rightarrow \infty$

We now show that (3.12) is actually independent of the $s_{i}$ so that the limit $n \rightarrow \infty$ is already taken and the limit therefore exists trivially.
In fact, $s_{i}$ can be an arbitrarily "short" (we put this term in inverted commas because there is no background metric available with respect to which we could measure the length of $s_{i}$ ) segment of $c$ starting at one and only one vertex $v$ of $\gamma$.
Let us first recall the notion of a spin-network state [16, [17, 18], we use the notation of 18] : Label the edges $e$ of a graph $\gamma$ with nontrivial irreducible representations of $S U(2)$, that is, assign to each of them a spin quantum number $j_{e}>0$. If $\pi_{j}$ is an irreducible representation of $S U(2)$ with weight (or spin) $j$ then the spin-network state depends on $\pi_{j_{e}}\left(h_{e}\right)$. Further, assign to each vertex $v$ of $\gamma$ a contractor matrix $m_{v}$ which contracts the matrix elements of the tensor product (over the set of edges $e$ incident at $v$ ) of the $\pi_{j_{e}}\left(h_{e}\right)$ in such a way that the resulting state is gauge invariant. We label these spin-network states by $T_{\gamma, \vec{j}, \vec{m}}$. One can show that spin-network states form an orthonormal basis of $\mathcal{H}$.
Consider first the case that $\gamma$ and $s:=s_{i}$ intersect in only one point. Since $\hat{l}_{s}^{2}$ is gauge-invariant, the result of applying $\hat{l}_{s}^{2}$ to a gauge invariant cylindrical function must be a gauge invariant cylindrical function $f^{\prime}$ which depends on the graph $s \cup \gamma$. Let us decompose $f^{\prime}$ into a basis of spin-network states. Since $f$ did not depend on $s$, the spin assigned to $s$ for a particular term $T$ in the spin-network decomposition of $f^{\prime}$ can only be $j= \pm 1,0$. In the case $j=0$ the state $T$ does not depend on $s$ at all. In case that $j= \pm 1$ then $T$ would be a spin-network state which contains an univalent vertex, namely the endpoint of $s$ distinct from $v$, and there is only one edge, namely $s$, incident at it and coloured with $j \neq 0$. Such a state is not gauge-invariant. Therefore the case $j \neq 0$ does not appear.
Now consider the case that $s:=s_{i}$ is contained in an edge $e$ of $\gamma$ and starts at a vertex $v$ of $\gamma$. By choosing $n$ high enough we may assume that $s$ is properly contained in $e$, that is, $e^{\prime}:=e-s \neq \emptyset$. Without loss of generality we may assume that $f$ is a spin-network state and thus $e$ carries spin $j>0$. Then in the spinnetwork decomposition of $f^{\prime}$ a particular term depends on $\gamma=\gamma \cup s$ in which the spins of all the edges of $\gamma$ distinct from $e$ are unchanged while $e^{\prime}$ carries spin $j$ and $s$ carries spin $j^{\prime} \in\{j, j \pm 1\}$. Consider the divalent "vertex" $p=s \cap e^{\prime}$ of $\gamma$. In case that $j^{\prime}=j$ then $T$ depends on all of $e$ with the same spin, i.e. it does not depend on $s$ and $e^{\prime}$ with different spins. The only gauge invariant way how to contract the corresponding matrix elements at $p$ is such that the resulting state depends on $s, e^{\prime}$ only through their product $e=s \circ e^{\prime}$. In case that $j^{\prime}=j \pm 1$ then $p$ is a two-valent
vertex at which two edges $e^{\prime}, s$ with distinct spins $j, j^{\prime}=j \pm 1$ are incident. There exists no vertex contractor that makes such a state gauge invariant, therefore this case cannot appear.
This furnishes the proof that for a spin-network state $f$ the function $\hat{L}_{n}(c) f$ still depends on all of the edges of $\gamma$ if $f$ did and that there is no dependence on the segments of the partition of $c$ which is to be expected since the classical length of a curve is a functional of $E_{i}^{a}$ only. Thus, the limit $n \rightarrow \infty$ is already taken in (3.12).

### 3.4 Cylindrical consistency

Since we have adapted our partition to the graph on which a cylindrical function depends, although the operator (3.12) was derived, it is not obvious any more that the family of operators $\left(L_{\gamma}(c)\right)$ constructed actually line up and qualify as the projections of an operator on $\mathcal{H}$. So let $\gamma \subset \gamma^{\prime}$. We need to show that 1) $\left(\hat{L}_{\gamma^{\prime}}(c)\right)_{\mid \gamma}=\hat{L}_{\gamma}(c)$, that is, the restriction of a projection to a smaller graph actually coincides with the projection to the smaller graph and 2) the domain of $\hat{L}_{\gamma}(c)$ is contained in that of $L_{\gamma^{\prime}}(c)$. Issue 2$)$ follows trivially by inspection if we choose the domain of $\hat{L}_{\gamma}(c)$ to be $\operatorname{Cyl}_{\gamma}^{3}(\overline{\mathcal{A} / \mathcal{G}})$ which we can since the volume operator has that domain. Issue 1) follows immediately: Given a curve $c$, the action of $\hat{L}_{\gamma}(c)$ and $\hat{L}_{\gamma^{\prime}}(c)$ on a function $f$ cylindrical with respect to $\gamma$ differ if either a) and $\gamma^{\prime}$ has more vertices intersecting $c$ than $\gamma$ or b) there are additional edges of $\gamma^{\prime}$ incident at a common vertex of $\gamma$ and $\gamma^{\prime}$ intersecting $c$. This follows directly by inspection from (3.12). So let first $v$ be a vertex of $\gamma^{\prime}$ but not of $\gamma$ and $s$ a segment of $c$ incident of $v$. Then $\left[h_{s}^{-1}, \hat{V}_{v}\right] f=-f \hat{V}_{v} h_{s}^{-1}=0$ and so this term in the sum (3.12) does not contribute. Now if case b) occurs then the cylindrical consistency of the volume operator assures that the edges of $\gamma^{\prime}-\gamma$ incident at $v$ do not contribute.
This furnishes the proof that (3.12) is consistently defined.

### 3.5 Symmetry and positivity

We wish to show that 1) every projection $\hat{L}_{\gamma}(c)$ is a symmetric operator on $\mathcal{H}$ with domain $\operatorname{Cyl}_{\gamma}^{3}(\overline{\mathcal{A} / \mathcal{G}})$ and 2) the family of projections $\left.\left(\hat{L}_{\gamma}(c)\right)_{\gamma}\right)$ comes from a positive semi-definite symmetric operator on $\mathcal{H}$ with domain $\operatorname{Cyl}^{3}(\overline{\mathcal{A} / \mathcal{G}})$.

Lemma 3.1 Every projection $\hat{L}_{\gamma}(c)$ defines a symmetric and positive semi-definite operator on $\mathcal{H}$ with domain $\mathcal{D}_{\gamma}:=C y l_{\gamma}^{3}(\overline{\mathcal{A} / \mathcal{G}})$.

Proof :
Let us begin with $\hat{l}_{s}^{2}$. Since $\hat{V}_{v}$ is a symmetric operator on $\mathcal{H}$ we find for the adjoint of $\left[h_{s}^{-1}, \hat{V}_{v}\right.$ ] using the unitarity of $S U(2)$

$$
\begin{equation*}
\left[h_{s}, \hat{V}_{v}\right]^{\dagger}=\left[\hat{V}_{v}, \overline{h_{s}}\right]=-\left[\left(h_{s}^{-1}\right)^{T}, \hat{V}_{v}\right] \tag{3.13}
\end{equation*}
$$

where (. $)^{T}$ denotes the transpose of (.). It follows that

$$
\begin{equation*}
\hat{l}_{s}^{2}:=+8 \sum_{A, B} \hat{K}_{A B}(s)^{\dagger} \hat{K}_{A B}(s) \text { where } \hat{K}_{A B}(s)=\left[\left(h_{s}^{-1}\right)_{A B}, \hat{V}\right] \tag{3.14}
\end{equation*}
$$

which shows that $\hat{l}_{s}^{2}$ is a symmetric and positive semidefinite operator on $\mathcal{H}$ and therefore possesses a square root $\hat{l}_{s}$. The statement about the domain is a consequence of the fact that $\hat{L}_{\gamma}(c)$ is defined in terms of the spectral projections of $\hat{V}_{\gamma}$ which has domain $\mathcal{D}_{\gamma}$.

Let us derive a more convenient expression for (3.12) :
By means of the basic $S L(2, \mathbb{C})$ formula $g_{A B}^{-1}=\epsilon_{A C} \epsilon_{B D} g_{D C}$ we readily derive that $\hat{l}_{s}=\hat{l}_{s^{-1}}$. This motivates the following notation:
Let be given an oriented curve $c$ and a vertex $v$ of $\gamma$ that intersects it. If $v$ is an interior point of $c$ consider any two segments $s_{v}^{ \pm}$of $c$ which start at $v$ and are "short" enough as not to intersect $\gamma$ in any other of its vertices. If $v$ is an endpoint of $c$ we need only one such segment $s_{v}^{+}=s_{v}^{-}$starting at $v$. In (3.12) the segments $s_{i}$ of $c$ starting at vertices of $\gamma$ come with the orientation of $c$, however, as we just showed, the operator $\hat{l}_{s}$ is invariant under reversal of the orientation of $s$ and the result of applying it to a gauge invariant function is independent of the choice of $s$ as long as it is "short" enough. Therefore, let

$$
N(v, c)= \begin{cases}0 & v \notin c  \tag{3.15}\\ \frac{1}{2} & v \text { is an endpoint of } c \\ 1 & v \text { is an interiour point of } c\end{cases}
$$

and define $\hat{L}_{v}^{ \pm}(c):=\hat{l}_{s_{v}^{ \pm}}(c)$. Then the length operator can be conveniently written

$$
\begin{equation*}
\hat{L}_{\gamma}(c)=\sum_{v \in V(\gamma)} N(v, c)\left[\hat{L}_{v}^{+}(c)+\hat{L}_{v}^{-}(c)\right] . \tag{3.16}
\end{equation*}
$$

Theorem 3.1 The family $\left(\hat{L}_{\gamma}(c), \mathcal{D}_{\gamma}\right)$ defines a positive semi-definite symmetric operator on $\mathcal{D}:=\operatorname{Cyb}^{3}(\overline{\mathcal{A} / \mathcal{G}})$.

Proof:
So far we have demonstrated $\hat{L}_{\gamma}(c)$ is a symmetric and positive semi-definite operator for each $\gamma$ with domain $\mathcal{D}_{\gamma}=\operatorname{Cyl}_{\gamma}^{3}(\overline{\mathcal{A} / \mathcal{G}})$. But to qualify as a symmetric projection of a symmetric operator requires more :
Let $\hat{S}$ be a symmetric operator on $\mathcal{H}$ with symmetric projections $\hat{S}_{\gamma}$. Let $f=p_{\gamma}^{*} f_{\gamma}$ and $g=p_{\gamma^{\prime}}^{*} g_{\gamma^{\prime}}$ be cylindrical functions. Then

$$
\begin{equation*}
<g, \hat{S} f>=<g, \hat{S}_{\gamma} f>=<\hat{S}_{\gamma} g, f>\stackrel{!}{=}<\hat{S} g, f>=<\hat{S}_{\gamma^{\prime}} g, f> \tag{3.17}
\end{equation*}
$$

Condition (3.17) is necessary and sufficient for a family of self-consistent symmetric projections to come from a symmetric operator. Let us check that (3.17) is satisfied for $\hat{L}(c)$ for each curve $c$.
It is sufficient to check it for the case that $f, g$ are spin-network states. Now since $\hat{L}_{\gamma}(c)$ does not change the graph $\gamma$ on which $f$ depends non-trivially, it follows that $<g, \hat{L}_{\gamma}(c) f>$ is non-vanishing if and only if $\gamma=\gamma^{\prime}$ because spin-network states depending on different graphs are orthogonal [16, 17, 18]. The same holds for $<\hat{L}_{\gamma^{\prime}}(c) g, f>$. Symmetry now follows from the cylindrical consistency and the statement about the domain is because $\hat{V}$ has this domain.

### 3.6 Self-adjointness

Although the expression of $\hat{L}_{\gamma}(c) f$ does not depend on the $s_{v}^{ \pm}$such that there must be a way to write purely in terms of the right invariant vector fields $X_{I}$ very much like as in the case of the volume operator (3.5), it is not clear to us how to do that. We can therefore not employ the essential self-adjointness of $i X_{I}$ with core $\operatorname{Cyl}^{1}(\overline{\mathcal{A} / \mathcal{G}})$ in order to show that $\hat{L}_{\gamma}(c)$ is essentially self-adjoint as well with core $\mathrm{Cyl}^{3}(\overline{\mathcal{A} / \mathcal{G}})$.
An independent proof which demonstrates that there exist self-adjoint extensions goes as follows :

Theorem 3.2 The operator $(\hat{L}(c), \mathcal{D})$ admits self-adjoint extensions on $\mathcal{H}$.
Proof :
The expression of $\hat{L}(c)$ is purely real and symmetric. Therefore it commutes with the antiunitary operator of complex conjugation. The assertion now follows from von Neumann's theorem [19], p. 143.

A possible choice of extension is given by its Friedrichs extension.

## 4 Spectral analysis of the length operator

### 4.1 Discreteness

The important feature of $\hat{L}_{\gamma}(c)$ is that it changes neither the graph $\gamma$ on which a cylindrical function depends nor the irreducible representations of the corresponding spin-network states into which a cylindrical function $f$ can be decomposed. Since the number of spin-network states associated with a fixed graph and a fixed colouring of its edges by irreducible representations is finite-dimensional [18] it follows that the length operator on each such subspace of $\mathcal{D}$ is just a symmetric, positive semi-definite, finite dimensional matrix. Its eigenvalues are real and are given by certain non-negative functions $\lambda\left(\left\{j_{I}\right\}\right)$ of the spins $j_{I}$ associated with the edges of $\gamma$. Since spin-networks span $\mathcal{D}_{\gamma}$ and provide a countable basis for its completion $\mathcal{H}_{\gamma}$ (with respect to $\mu_{0, \gamma}$ ) it follows that there exists a countable basis of eigenvectors for $\mathcal{H}_{\gamma}$. The corresponding eigenvalues therefore form a countable set and lie in the point spectrum of $\hat{L}_{\gamma}(c)$.
Recall that a point in the spectrum is said to be in the discrete spectrum if it is an isolated point and an eigenvalue of finite multiplicity (clearly 0 has infinite multiplicity, all functions cylindrical with respect to graphs not intersecting the curve $c$ are annihilated). In order to show that the point spectrum that $\hat{L}_{\gamma}(c)$ attains on $\mathcal{D}_{\gamma}$ is discrete it would be sufficient to show that $\lambda\left(\left\{j_{I}\right\}\right)$ diverges whenever $j:=j_{1}+. .+j_{n}$, $n$ being the number of edges of $\gamma$, diverges. Namely, if there was a finite condensation point in the point spectrum then an infinite number of different $n$-tuples would give an eigenvalue which lies in a finite neighbourhood of that condensation point but necessarily the corresponding values of $j$ must diverge which would be a contradiction. The same argument shows that there would not be an eigenvalue of infinite multiplicity.

Looking at the eigenvalue formulae derived below for some simple graphs, such a behaviour of $\lambda$ is plausible because $\hat{L}_{\gamma}(c)$ is an unbounded operator, however, a strict proof is missing at this point.
Discreteness of the spectrum of $\hat{L}_{\gamma}(c)$ would follow immediately if we could prove it for the volume operator. An elegant method of proof would be to show that $\hat{V}$ is (a positive root of) an elliptic operator. Since this operator acts on the compact manifold $S U(2)^{n}$ standard results form harmonic analysis would imply that its spectrum is discrete. Unfortunately this does not work : the volume operator is easily seen to be the fourth root of the operator $\hat{q}^{\dagger} \hat{q}, \hat{q}=\sum_{I J K} \epsilon_{I J K} \hat{q}_{I J K}$ (compare [6]) and so is a 6 -th order homogenous polynomial in the derivative operators $X_{I}$. Its principal symbol can be seen to be a non-negative function with an at least two-dimensional kernel and therefore is far from being invertible. Indeed, it is by now well-known that the volume operator has a kernel which includes all divalent and trivalent graphs. Summarizing, we have shown that $\hat{L}_{\gamma}(c)$ has pure point spectrum when restricted to $\mathcal{D}_{\gamma}$. Now consider the complete spectrum of $\hat{L}_{\gamma}(c)$ on all of $\mathcal{H}_{\gamma}$ Since $\hat{L}_{\gamma}(c)$ is a self-adjoint (not only symmetric) operator, it has spectral projections which are orthogonal for disjoint subsets of its spectrum. It follows that if $I$ is a subset of the real numbers not contained in the part of the spectrum that $\hat{L}_{\gamma}(c)$ attains on $\mathcal{D}_{\gamma}$ and if $\hat{P}_{I}$ is the corresponding spectral projection then $\mathcal{H}_{\gamma}^{\prime}:=\hat{P}_{I} \mathcal{H}_{\gamma}$ and $\mathcal{D}_{\gamma}$ are orthogonal. But $\mathcal{D}_{\gamma}$ is dense in $\mathcal{H}_{\gamma}$, therefore for each $v \in \mathcal{H}_{\gamma}^{\prime}, \epsilon>0$ we find $f \in \mathcal{D}_{\gamma}$ such that $\|v-f\|<\epsilon$. On the other hand by orthogonality $\|v-f\| \geq\|v\|$ which is a contradiction unless $v=0$. Therefore the complete spectrum of $\hat{L}_{\gamma}(c)$ is already attained on $\mathcal{D}_{\gamma}$ and it has no continuous part very much like its companions, the area and volume operators [10, 11]. Thus, the spectrum on $\mathcal{H}_{\gamma}$ is discrete in the sense that it does not have a continuous part.

Similarly, it follows that the spectrum of $\hat{L}(c)$ is attained on $\mathcal{D}$ and also does not have a continuous part (this property is shared by all three operators, length, area and volume) : Although the set of piecewise analytic graphs is uncountable, the matrix elements of the length operator in a spin-network basis do not depend on the graph, they only depend on the quantum numbers $\vec{j}, \vec{m}$ and are therefore diffeomorphism invariant. Therefore, the spectrum attained on $\mathcal{D}_{\gamma}$ and $\mathcal{D}_{\varphi(\gamma)}, \varphi \in \operatorname{Diff}(\Sigma)$ an analyticity preserving diffeomorphism, is identical. Moreover, the spectrum does not depend on whether two curves touch each other in a $C^{m}$ or $C^{n}$ fashion for any $1 \leq m<n$. It follows that the spectrum only depends on the $C^{\infty}$ properties of the diffeomorphism class of a graph, that is, all that matters is whether two edges are distinct or coincide. In other words all we need to know about a graph is
a) the number $N$ of its vertices
b) the valence $n(v)$ of each vertex $v$
c) the topologically different ways of connecting edges incident at different vertices in a $C^{\infty}$ manner which is a finite number.
Thus, since this characterization of a graph depends on discrete labels and the union of countable sets is a countable set it follows that the spectrum is still discrete in the sense that it does not have a continuous part. On the other hand we see that every eigenvalue of the length, area or volume operators is of uncountably infinite multiplicity when we consider all of $\mathcal{H}$.

### 4.2 Eigenvalues

By inspection, the task of giving a closed formula for the eigenvalues of the length operator requires to have a closed formula for the spectrum of the volume operator at one's disposal which we lack, however, at the present stage (see, however, [23] for a closed formula for its matrix elements). We will therefore restrict ourselves here to compute the spectrum for some simple types of graphs and thereby obtain the quantum of length.
Specifically, it is comparatively simple to compute the spectrum of the length operator when restricted to at most trivalent graphs, thus including the classical spinnetworks which were introduced in [20] and whose vertices are all precisely trivalent. Since for trivalent graphs the vertex contractors of a spin-network are unique and since the result of applying the length operator to a spin-network state is a spin-network state on the same graph and with the same spin, it follows that all spin-network states on trivalent graphs are eigenvectors of the length operator. We will see that the corresponding eigenvalues are non-vanishing in general. This is an astonishing feature because the volume operator is known to vanish on trivalent graphs. Now, classically the volume of a region is known if one can measure the length of arbitrary curves through that region and so non-vanishing length of curves results in non-vanishing volume. This indicates that trivalent graphs are rather special and are insufficient to construct weave states that approximate a given classical geometry.
The computations are largely governed by the properties of the $6 j$-symbol of the recoupling theory of angular momentum which we recall in the appendix. The way how the recoupling theory of spin systems enters the stage is as follows : in our computations we evaluate the volume operator at a given vertex $v$ on functions $f$ which transform according to an irreducible representation $j$ of $S U(2)$ under gauge transformations at $v$. This $j$ is nothing else than the resulting total angular momentum to which the angular momenta $j_{I}$ (which colour the edges $e_{I}$ incident at $v$ ) couple. There are different ways of how to couple angular momenta $j_{I}$ to resulting spin $j$ and this freedom is determined by the recoupling quantum numbers $j_{I J}, j_{I J K}, .$. (compare the appendix) and results in different contractors of the so not necessarily gauge invariant (or extended) spin networks. Here we mean by an extended spin network just any function that depends on the edges of a graph through the matrix elements of irreducible representations of $S U(2)$ evaluated at the holonomy of the corresponding edge and transforms at each vertex according to an irreducible representation of $S U(2)$ under gauge transformations. The point is now that the function $f$ is easily seen to be in the left regular representation of $S U(2)^{n}, n$ being the valence of the vertex $v$, defined by $\hat{R}\left(g_{1}, . ., g_{n}\right) f\left(h_{1}, . ., h_{n}\right)=f\left(g_{1} h_{1}, . ., g_{n} h_{n}\right)$. On the other hand, the connection with the abstract angular momentum Hilbert space where $S U(2)$ acts by the abstract unitary representation $\hat{U}(g)$ and which is spanned by the vectors of the form $\left|\left(j_{1}, m_{1}\right), . .,\left(j_{n}, m_{m}\right) ; k_{1}, . ., k_{n}>=\left|j_{1}, m_{1} ; k_{1}>\otimes . . \otimes\right| j_{n}, m_{n} ; k_{n}>\right.$ where $k_{I}$ are certain additional quantum numbers, is made as follows: Consider the special functions $\left(\pi_{j}(h)\right)_{m, m^{\prime}}$ given by the matrix elements of the $j-t h$ irreducible representation of $S U(2), m, m^{\prime} \in\{-j,-j+1, . ., j\}$. Consider the Hilbert space $L_{2}\left(S U(2), d \mu_{H}\right)$ and let $\mid g>$ be the usual Dirac generalized eigenstates of the multiplication operator $\hat{g}$. Then $\left(\pi_{j}(h)\right)_{m, m^{\prime}}=<h \mid j, m ; m^{\prime}>$. The proof is by checking the representation prop-
erty $\hat{R}(g)\left(\pi_{j}(h)\right)_{m, m^{\prime}}=\left(\pi_{j}(g h)\right)_{m, m^{\prime}}=\left(\pi_{j}(g)\right)_{m, \tilde{m}}\left(\pi_{j}(h)\right)_{\tilde{m}, m^{\prime}}=<h|\hat{U}(g)| j, m ; m^{\prime}>$ by definition of the states $\mid j, m ; m^{\prime}>$. The reader is referred to [23] for more details. It follows from these considerations that instead of doing tedious computations within the left regular representation (spin-network states or traces of the holonomy around closed loops, [22]) it is far easier to do them in the abstract representation which allows to use the powerful Clebsh-Gordan theory of angular momentum. We will do that in the sequel.
In particular, it follows immediately that the right invariant vector field $X_{I}$ is identified with $2 i J_{I}$ where $J_{I}$ is the angular momentum operator of the spin associated with the $I-t h$ edge which is the self-adjoint generator of the unitary group $\hat{U}(g)$. We are now in the position to compute the spectrum of the length operator on trivalent graphs $\gamma$. First we cast the volume operator in the more convenient form

$$
\begin{align*}
& \hat{V}=\ell_{p}^{3} \hat{v} f=\ell_{p}^{3} \sum_{v \in V(\gamma)} \hat{V}_{v} \text { where } \hat{V}_{v}:=\sqrt{\left|\frac{i}{32} \hat{q}_{v}\right|} \\
& \hat{q}_{v}=\sum_{e_{I} \cup e_{J} \cup e_{K}=v, I<J<K} \epsilon\left(e_{I}, e_{J}, e_{K}\right) \hat{q}_{I J K} \text { where } \hat{q}_{I J K}:=\left[J_{I J}^{2}, J_{J K}^{2}\right] . \tag{4.1}
\end{align*}
$$

To arrive at this expression one only has to use elementary angular momentum algebra. Expression (4.1) captures a neat interpretation of the volume operator : it measures the difference between recoupling schemes of $n$ angular momenta based on $J_{I J}$ and $J_{J K}$ respectively. This is why the recoupling theory is important and the matrix elements of the volume operator can be given purely in terms of polynomials of $6 j$-symbols [23].
Now let $v$ be a vertex of $\gamma$ which intersects $c$. From (3.16) we find $\left(h:=h_{s_{v}^{ \pm}}, \hat{L}_{v}(c):=\right.$ $\left.\hat{L}_{v}^{ \pm}(c)\right)$

$$
\begin{equation*}
\frac{1}{8} \hat{L}_{v}(c)^{2}=-\left[\operatorname{tr}\left(h^{-1} \hat{V} h\right) \hat{V}+\hat{V} \operatorname{tr}\left(h^{-1} \hat{V} h\right)\right]+2 \hat{V}^{2}+\operatorname{tr}\left(h^{-1} \hat{V}^{2} h\right) \tag{4.2}
\end{equation*}
$$

and the simplification that occurs on trivalent graphs is that the first three terms vanish identically. We have two possibilities :
Case A: $s_{v}$ lies within an edge of $\gamma$ incident at $v$ or
Case B: $s_{v}$ is not contained in an edge of $\gamma$.
We will discuss both cases separately.

### 4.2.1 Case A

We may without loss of generality assume that $s_{v}$ coincides with one, say $e_{3}$, of the three edges of $\gamma$ incident at $v$. This is because 1) if $e=s_{v} \circ e^{\prime}$ then for any irreducible representation $\pi_{j}$ of $S U(2)$ we have $\pi_{j}\left(h_{e}\right)=\pi_{j}\left(h_{s_{v}}\right) \pi_{j}\left(h_{e^{\prime}}\right)$ so a spin network state also depends on $s_{v}$ through $\pi_{j}$ and 2) for a right invariant vector field $X\left(h_{s_{v}}\right)=X\left(h_{s_{v}} h_{e^{\prime}}\right)=X\left(h_{e}\right)$ by definition of right invariance. Therefore the volume operator (4.1) contains only one term $\hat{q}_{123}$.
Consider a trivalent spin-network function $f=T_{\gamma, \vec{j}}$ with spins $j_{1}, j_{2}, j_{3}$ assigned to $e_{1}, e_{2}, e_{3}$ (we have suppressed the contractor matrices $\vec{m}$ since they are unique for trivalent spin-networks). Then $h_{e_{3}} f$ can be decomposed into extended spin network functions with total angular momentum $j=1 / 2$ and edge spins $j_{1}^{\prime}=j_{1}, j_{2}^{\prime}=j_{2}, j_{3}^{\prime}=$
$j_{3} \pm 1 / 2$ by usual Clebsh-Gordan representation theory. Let us determine the precise coefficients of that decomposition.

Lemma 4.1 Let $n=2 j$ so that $\pi_{j}(g)_{A_{1}, \ldots, A_{n} ; B_{1}, \ldots, B_{n}}=g_{\left(A_{1}, B_{1} . .\right.} g_{\left.A_{n}\right), B_{n}}$ where the round bracket means total symmetrization in the $A$ indices. Then

$$
\begin{align*}
g_{A_{0}, B_{0}} \pi_{j}(g)_{A_{1}, \ldots, A_{n} ; B_{1}, \ldots, B_{n}}= & \pi_{j+1 / 2}(g)_{A_{0}, \ldots, A_{n} ; B_{0}, . ., B_{n}} \\
& -\frac{n}{n+1} \epsilon_{A_{0}\left(A_{1}\right.} \pi_{j-1 / 2}(g)_{\left.A_{2}, ., A_{n}\right) ;\left(B_{2}, . ., B_{n}\right.} \epsilon_{\left.B_{1}\right) B_{0}} . \tag{4.3}
\end{align*}
$$

Proof :
Elementary linear algebra.
We now multiply each of the two terms in (4.3) with $g_{B_{0}, A_{0}}^{-1}$ and sum over $A_{0}, B_{0}$. The result is

$$
\begin{align*}
& g_{B_{0}, A_{0}}^{-1} \pi_{j+1 / 2}(g)_{A_{0}, \ldots, A_{n} ; B_{0}, . ., B_{n}}=\frac{n+2}{n+1} \pi_{j}(g)_{A_{1}, \ldots, A_{n} ; B_{1}, ., B_{n}} \\
& g_{B_{0}, A_{0}}^{-1} \epsilon_{A_{0}\left(A_{1}\right.} \pi_{j-1 / 2}(g)_{\left.A_{2}, \ldots, A_{n}\right) ;\left(B_{2}, \ldots, B_{n}\right.} \epsilon_{\left.B_{1}\right) B_{0}}=-\pi_{j}(g)_{A_{1}, . ., A_{n} ; B_{1}, ., B_{n}} \tag{4.4}
\end{align*}
$$

Formulae (4.4) illustrate the computational reason for why the edge $s_{v}$ does not appear in a gauge invariant function $f$ after evaluating $\hat{L}(c) f$ on it. We now can write $h_{e_{3}} f=f_{+}+\frac{n_{3}}{n_{3}+1} f_{-}$where, according to (4.3), the vectors $f_{+}$and $f_{-}$respectively are proportional to the vectors $\mid j_{12}=j_{3}, j=1 / 2 ; j_{1}, j_{2}, j_{3}+1 / 2>$ and $\mid j_{12}=j_{3}, j=$ $1 / 2 ; j_{1}, j_{2}, j_{3}-1 / 2>$ respectively on the abstract angular momentum Hilbert space. That in both vectors $j_{12}$ still equals $j_{3}$ follows from the fact that we did not change the way the matrices are contracted in $f$ in multiplying it by $h_{e_{3}}$.
The space of states $V_{+}$with total angular momentum $j=1 / 2$ and spins $j_{1}, j_{2}, j_{3}^{\prime}=$ $j_{3}+1 / 2$ is two dimensional : it is spanned by $\mid j_{12}=j_{3}, j=1 / 2 ; j_{1}, j_{2}, j_{3}+1 / 2>$ and $\mid j_{12}=j_{3}+1, j=1 / 2 ; j_{1}, j_{2}, j_{3}+1 / 2>$ respectively. Likewise, the span of the space of states $V_{-}$with total angular momentum $j=1 / 2$ and spins $j_{1}, j_{2}, j_{3}^{\prime}=$ $j_{3}-1 / 2$ is given by $\mid j_{12}=j_{3}, j=1 / 2 ; j_{1}, j_{2}, j_{3}-1 / 2>$ and, provided that $j_{3} \geq 1$, $\mid j_{12}=j_{3}-1, j=1 / 2 ; j_{1}, j_{2}, j_{3}-1 / 2>$ respectively. The volume operator leaves these two spaces separately invariant and the operator $\hat{q}_{123}$, being antisymmetric on $\mathcal{H}$, reduces to an antisymmetric $2 \times 2$ matrix on $V_{+}$and to an anti-symmetric $2 \times 2$ matrix on $V_{-}$if $j_{3} \geq 1$ and to the zero matrix if $j_{3}<1$. Let the matrix elements different from zero of these matrices be denoted by $\pm \mu_{+}, \pm \mu_{-}$respectively, then, since we take the modulus of the square root of $\hat{q}_{123}$ it follows that $f_{ \pm}$are already eigenvectors of $\hat{V}$ with eigenvalues $\lambda_{ \pm}:=\frac{\sqrt{\left|\mu_{ \pm}\right|}}{4 \sqrt{2}}$. It follows then from (4.4) and $h^{-1} \hat{V}^{2} h=\left(h^{-1} \hat{V} h\right)^{2}$ that our spin-network state is eigenfunction of the length operator $\hat{L}_{v}(c)$ with eigenvalue $\frac{\ell_{p}}{2} \sqrt{\frac{1}{n_{3}+1}\left[\left(n_{3}+2\right) \lambda_{+}^{2}+n_{3} \lambda_{-}^{2}\right]}$.
It remains to compute the matrix elements $\mu_{ \pm}$of $\hat{q}_{123}=\left[J_{12}^{2}, J_{23}^{2}\right]$. We have

$$
\begin{aligned}
\mu_{+} & :=<j_{12}^{\prime}=j_{3}+1, j=\frac{1}{2} ; j_{1}, j_{2}, j_{3}^{\prime}=j_{3}+\frac{1}{2}\left|\hat{q}_{123}\right| j_{12}=j_{3}, j=\frac{1}{2} ; j_{1}, j_{2}, j_{3}^{\prime}=j_{3}+\frac{1}{2}> \\
& =\left[\left(j_{3}+1\right)\left(j_{3}+2\right)-j_{3}\left(j_{3}+1\right)\right]<j_{12}^{\prime}=j_{3}+1, j=\frac{1}{2}\left|J_{23}^{2}\right| j_{12}=j_{3}, j=\frac{1}{2}> \\
& =2\left(j_{3}+1\right) \sum_{j_{23}=j_{1} \pm \frac{1}{2}} j_{23}\left(j_{23}+1\right)<j_{12}^{\prime}=j_{3}+1, \left.j=\frac{1}{2} \right\rvert\, j_{23}, j=\frac{1}{2}>\times
\end{aligned}
$$

$$
\begin{equation*}
\times<j_{23}, \left.j=\frac{1}{2} \right\rvert\, j_{12}=j_{3}, j=\frac{1}{2}> \tag{4.5}
\end{equation*}
$$

where we have suppressed $j_{1}, j_{2}, j_{3}^{\prime}=j_{3}+\frac{1}{2}$. In the last step we have inserted a complete 1 in form of the coupling scheme $\left(j_{2}, j_{3}^{\prime}\right) \rightarrow j_{23},\left(j_{1}, j_{23}\right) \rightarrow j$ and realized that the only possible values for $j_{23}$ in order to couple to spin $j=1 / 2$ with $j_{1}$ are the ones displayed. Completely analogously we find

$$
\begin{align*}
\mu_{-}:= & <j_{12}^{\prime}=j_{3}-1, j=\frac{1}{2} ; j_{1}, j_{2}, j_{3}^{\prime}=j_{3}-\frac{1}{2}\left|\hat{q}_{123}\right| j_{1} 2=j_{3}, j=\frac{1}{2} ; j_{1}, j_{2}, j_{3}^{\prime}=j_{3}-\frac{1}{2}> \\
= & {\left[\left(j_{3}-1\right) j_{3}-j_{3}\left(j_{3}+1\right)\right]<j_{12}^{\prime}=j_{3}-1, j=\frac{1}{2}\left|J_{23}^{2}\right| j_{12}=j_{3}, j=\frac{1}{2}>} \\
= & -2 j_{3} \sum_{j_{23}=j_{1} \pm \frac{1}{2}} j_{23}\left(j_{23}+1\right)<j_{12}^{\prime}=j_{3}-1, \left.j=\frac{1}{2} \right\rvert\, j_{23}, j=\frac{1}{2}>\times \\
& \times<j_{23}, \left.j=\frac{1}{2} \right\rvert\, j_{12}=j_{3}, j=\frac{1}{2}> \tag{4.6}
\end{align*}
$$

We now use the formulae given in the appendix to compute the eight remaining matrix elements in terms of $6 j$-symbols and evaluate the latter by means of the Racah formula. The result is

$$
\begin{align*}
& \mu_{+}=\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} \\
& \text { where } a=j_{1}+\frac{1}{2}, b=j_{2}+\frac{1}{2}, c=j_{3}+1 \\
& \mu_{-}=-\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} \\
& \text { where } a=j_{1}+\frac{1}{2}, b=j_{2}+\frac{1}{2}, c=j_{3} . \tag{4.7}
\end{align*}
$$

The final result is thus given by the following theorem.
Theorem 4.1 The eigenvalue $\lambda$ of $\hat{L}_{v}(c)$ for a trivalent spin-network state $T_{\gamma}$ with vertex at $v$ such that $c$ and $\gamma$ share a segment incident at $v$ is given by $\left(j_{3} \in\left\{j_{1}+\right.\right.$ $\left.\left.j_{2}, j_{1}+j_{2}-1, . .,\left|j_{1}-j_{2}\right|\right\}\right)$

$$
\begin{align*}
& \lambda=\frac{\ell_{p}}{2 \sqrt{j_{3}+1 / 2}} \times \\
& \times \sqrt{\left(j_{3}+1\right) \sqrt{\left(j_{1}+j_{2}+j_{3}+2\right)\left(j_{1}+j_{2}-j_{3}\right)\left(j_{2}+j_{3}-j_{1}+1\right)\left(j_{3}+j_{1}-j_{2}+1\right)}} \\
& +j_{3} \sqrt{\left(j_{1}+j_{2}+j_{3}+1\right)\left(j_{1}+j_{2}-j_{3}+1\right)\left(j_{2}+j_{3}-j_{1}\right)\left(j_{3}+j_{1}-j_{2}\right)} \tag{4.8}
\end{align*}
$$

provided the edges $e_{1}, e_{2}, e_{3}$ of $\gamma$ incident at $v$ are linearly independent, otherwise $\lambda=0$.

The formula for $\lambda$ simplifies if we take $v$ to be a divalent vertex not sharing a segment with $c$. Then $j_{3}=0$ and necessarily $j_{1}=j_{2}=j_{0}$ and we find simply

$$
\lambda\left(j_{0}\right)=\ell_{p} \sqrt[4]{j_{0}\left(j_{0}+1\right)}
$$

which for large spin grows as $\sqrt{j_{0}}$. In other words, the warping of $\Sigma$ when the gravitational field is in the spin-network state labelled by $j_{0}$ grows with the spin $j_{0}$
since one and the same curve $c$ appears the longer the more spin $j_{0}$ we have in a neighbourhood of it.
The quantum of length is achieved for $j_{0}=1 / 2$ and given by

$$
\frac{1}{\sqrt{2}} \sqrt[4]{3} \ell_{p}
$$

The length can change only in packets of $\Delta L \approx \pm \frac{1}{2} \sqrt{\frac{1}{j_{0}}} \ell_{p}$ which approaches zero for large spin so that for high spin the length looks like a continuous operator.

### 4.2.2 Case B

As this is actually a four-valent problem the computations will be much more elaborate than in the previous case since we have to couple now four angular momenta which will necessarily invoke the $9 j$-symbol. Fortunately one can reduce the $3(n-1) j$-symbol to a polynomial in $6-j$ symbols, $n$ the valence of the vertex. The segment $e_{4}:=s_{v}$ carries spin $j_{4}=1 / 2$. First we note that if again $f$ is a spin-network state with vertex $v$ intersecting $c$ then we have $\left(\epsilon_{I J K}:=\epsilon\left(e_{I}, e_{J}, e_{K}\right)\right)$

$$
\begin{align*}
\hat{q}_{v} h_{e_{4}} f & =\left[\epsilon_{124} \hat{q}_{124}+\epsilon_{234} \hat{q}_{234}+\epsilon_{314} \hat{q}_{314}\right] h_{e_{4}} f+h_{e_{4}} \epsilon_{123} \hat{q}_{123} f \\
& =\left[\epsilon_{124}+\epsilon_{234}+\epsilon_{124}\right] \hat{q}_{124} h_{e_{4}} f \tag{4.9}
\end{align*}
$$

where in the second step we exploited gauge invariance of $f$, that is, in infinitesimal form $\left[X_{1}+X_{2}+X_{3}\right] f=0$.
The extended spin-network function $h_{e_{4}} f$ is represented on the abstract angular momentum Hilbert space by a vector proportional to $\psi:=\mid j_{12}=j_{3}, j_{123}=0, j=$ $1 / 2 ; j_{1}, j_{2}, j_{3}, j_{4}=1 / 2>$ where the notation means that the operator $\left[\left(J_{1}+J_{2}+J_{3}\right)^{i}\right]^{2}$ is diagonal as well, the coupling scheme being given by $\left(j_{1}, j_{2}\right) \rightarrow j_{12},\left(j_{12}, j_{3}\right) \rightarrow$ $j_{123},\left(j_{123}, j_{4}\right) \rightarrow j$.
The space of states with given values $j_{1}, j_{2}, j_{3}, j_{4}=j=1 / 2$ is easily seen to be threedimensional and is spanned by $\psi, \psi_{+}, \psi_{-}$where $\psi_{ \pm}=\mid j_{12}=j_{3} \pm 1, j_{123}=1, j=$ $1 / 2 ; j_{1}, j_{2}, j_{3}, j_{4}=1 / 2>$. This is because $j=1 / 2$ requires $j_{123}=0,1$ and $j_{123}=0$ enforces $j_{12}=j_{3}$ while $j_{123}=1$ enforces $j_{12}=j_{3} \pm 1$ by the usual Clebsh-Gordan decomposition into irreducibles $(j) \otimes\left(j^{\prime}\right)=\left(j+j^{\prime}\right) \oplus\left(j+j^{\prime}-1\right) \oplus . . \oplus\left(\left|j-j^{\prime}\right|\right)$. The task left to do is to compute the matrix elements of $q_{124}=\left[J_{12}^{2}, J_{24}^{2}\right]$ between $\psi, \psi_{ \pm}$. In the following computation we are going to insert a complete 1 in form of the basis corresponding to the coupling scheme $\left(j_{2}, j_{4}\right) \rightarrow j_{24},\left(j_{1}, j_{24}\right) \rightarrow j_{124},\left(j_{124}, j_{3}\right) \rightarrow j$ in order to evaluate $J_{24}^{2}$. We abbreviate $\mid j_{12}=j_{3}, j_{123}=0, j=1 / 2 ; j_{1}, j_{2}, j_{3}, j_{4}=$ $1 / 2>=: \mid j_{12}, j_{123}>$ etc. and have for any values of the various $j^{\prime} s$

$$
\begin{align*}
& <j_{12}^{\prime}, j_{123}^{\prime}\left|\hat{q}_{124}\right| j_{12}, j_{123}>=\left[j_{12}^{\prime}\left(j_{12}^{\prime}+1\right)-j_{12}\left(j_{12}+1\right)\right]<j_{12}^{\prime}, j_{123}^{\prime}\left|J_{24}^{2}\right| j_{12}, j_{123}> \\
= & {\left[j_{12}^{\prime}\left(j_{12}^{\prime}+1\right)-j_{12}\left(j_{12}+1\right)\right] \times } \\
& \times \sum_{j_{24}, j_{124}} j_{24}\left(j_{24}+1\right)<j_{12}^{\prime}, j_{123}^{\prime}\left|j_{24}, j_{124}><j_{24}, j_{124}\right| j_{12}, j_{123}> \tag{4.10}
\end{align*}
$$

where the allowed values for $j_{24}, j_{124}$ are $j_{124}=j_{3} \pm 1 / 2$ in order to couple with $j_{3}$ to resulting $j=1 / 2$ and $j_{24}=j_{2} \pm 1 / 2$ are the possible irreducible representations contained in $\left(j_{2}\right) \otimes\left(j_{4}\right)$. The next step is to reduce the matrix elements to
$6 j$-symbols. Note that the matrix elements are not of the standard textbook form $<j_{12}, j_{34} \mid j_{13}, j_{24}>$ so that we need to invent a new reduction of the $9 j$-symbol as a product of $6 j$-symbols. Upon inserting a complete 1 corresponding to a basis of yet another coupling scheme we have

$$
\begin{align*}
& <j_{12}, j_{123} \mid j_{24}, j_{124}> \\
\equiv & <j_{12}\left(j_{1}, j_{2}\right), j_{123}\left(j_{12}, j_{3}\right), j\left(j_{123}, j_{4}\right) \mid j_{24}\left(j_{2}, j_{4}\right), j_{124}\left(j_{1}, j_{24}\right), j\left(j_{124}, j_{3}\right)> \\
= & \sum_{j_{124}^{\prime}}<j_{12}\left(j_{1}, j_{2}\right), j_{123}\left(j_{12}, j_{3}\right), j\left(j_{123}, j_{4}\right)| | j_{12}\left(j_{1}, j_{2}\right), j_{124}^{\prime}\left(j_{12}, j_{4}\right), j\left(j_{124}, j_{3}\right)>\times \\
& \times<j_{12}\left(j_{1}, j_{2}\right), j_{124}^{\prime}\left(j_{12}, j_{4}\right), j\left(j_{124}, j_{3}\right) \mid j_{24}\left(j_{2}, j_{4}\right), j_{124}\left(j_{1}, j_{24}\right), j\left(j_{124}, j_{3}\right)> \\
= & <j_{123}\left(j_{12}, j_{3}\right), j\left(j_{123}, j_{4}\right) \mid j_{124}\left(j_{12}, j_{4}\right), j\left(j_{124}, j_{3}\right)>\times \\
& \times<j_{12}\left(j_{1}, j_{2}\right), j_{124}\left(j_{12}, j_{4}\right) \mid j_{24}\left(j_{2}, j_{4}\right), j_{124}\left(j_{1}, j_{24}\right)>. \tag{4.11}
\end{align*}
$$

In going from the third equality to the fourth equality, two things happened : a) because $j_{124}$ is diagonal on the vectors involved in the second scalar product factor in the third line, the sum reduces to only one term $j_{124}^{\prime}=j_{124}$, b) we used the Wigner-Eckart theorem to reduce the matrix elements 21], that is, we could get rid of the first entry of the vectors involved in the first scalar product factor and of the last entry in the second, both in the third line. Equation (4.11) is the form that allows to express everything in terms of $6 j$-symbols. Namely we have

$$
\begin{align*}
& <j_{123}\left(j_{12}, j_{3}\right), j\left(j_{123}, j_{4}\right) \mid j_{124}\left(j_{12}, j_{4}\right), j\left(j_{124}, j_{3}\right)>=(-1)^{j_{1}+j_{2}+j_{4}+j_{124}} \times\left\{\begin{array}{ccc}
j_{1} & j_{2} & j_{12} \\
j_{4} & j_{124} & j_{24}
\end{array}\right\} \\
& \times \sqrt{\left(2 j_{12}+1\right)\left(2 j_{124}+1\right)}  \tag{4.12}\\
& <j_{12}\left(j_{1}, j_{2}\right), j_{124}\left(j_{12}, j_{4}\right) \mid j_{24}\left(j_{2}, j_{4}\right), j_{124}\left(j_{1}, j_{24}\right)>=(-1)^{2\left(j_{3}+j_{12}\right)+j_{4}+j-j_{123}} \times \\
& \times \sqrt{\left(2 j_{123}+1\right)\left(2 j_{124}+1\right)}\left\{\begin{array}{ccc}
j_{3} & j_{12} & j_{123} \\
j_{4} & j & j_{124}
\end{array}\right\} . \tag{4.13}
\end{align*}
$$

In order to see this we do the following : In (4.12) set $j_{1}^{\prime}:=j_{1}, j_{2}^{\prime}:=j_{2}, j_{3}^{\prime}:=j_{4}, j^{\prime}:=$ $j_{124}, j_{12}^{\prime}:=j_{12}, j_{23}^{\prime}:=j_{24}$ and use the standard formula (A.1) for the $6 j-$ symbol given in the appendix in terms of the primed $j^{\prime} s$. In (4.13) we first set $j_{1}^{\prime}:=$ $j_{3}, j_{2}^{\prime}:=j_{12}, j_{3}^{\prime}:=j_{4}, j^{\prime}:=j, j_{12}^{\prime}:=j_{123}, j_{23}^{\prime}:=j_{124}$ and secondly recall the identity $\left|j_{12}\left(j_{2}, j_{1}\right), j>=(-1)^{j_{1}+j_{2}-j_{12}}\right| j_{12}\left(j_{1}, j_{2}\right), j>$ in order to apply the standard formula in terms of the primed $j^{\prime} s$.
As the whole expression becomes rather lengthy we refrain from writing the matrix elements (4.10) out explicitly in terms of $j_{1}, j_{2}, j_{3}$, rather we consider them as known through (4.10)-(4.14) and define

$$
\begin{equation*}
a:=<\psi\left|\hat{q}_{124}\right| \psi_{+}>, b:=<\psi\left|\hat{q}_{124}\right| \psi_{-}>, c:=<\psi_{+}\left|\hat{q}_{124}\right| \psi_{-}>. \tag{4.14}
\end{equation*}
$$

So $\hat{q}_{124}$ reduces to an antisymmetric $3 \times 3$ matrix with non-vanishing off-diagonal entries $\pm a, \pm b, \pm c$, eigenvalues $0, \pm i \mu$ where $\mu=\sqrt{a^{2}+b^{2}+c^{2}}$ and eigenvectors $\psi^{0}, \psi^{ \pm}$which can chosen to be

$$
\begin{equation*}
\psi^{0}=c \psi-b \psi_{+}+a \psi_{-}, \psi^{ \pm}=[\mp i b \mu-a c] \psi+[-\mp i c \mu+a b] \psi_{+}+\left[b^{2}+c^{2}\right] \psi_{-} \tag{4.15}
\end{equation*}
$$

and inverted

$$
\psi_{-}=\frac{1}{2 \mu^{2}}\left[\psi^{+}+\psi^{-}+2 a \psi^{0}\right]
$$

$$
\begin{align*}
-\left[b^{2}+c^{2}\right] \psi_{+} & =\frac{b}{\mu^{2}}\left[\left(b^{2}+c^{2}\right) \psi^{0}-\left(\psi^{+}+\psi^{-}\right) / 2\right]+\frac{i b}{2 \mu}\left[\psi^{+}-\psi^{-}\right] \\
+\left[b^{2}+c^{2}\right] \psi & =\frac{c}{\mu^{2}}\left[\left(b^{2}+c^{2}\right) \psi^{0}-\left(\psi^{+}+\psi^{-}\right) / 2\right]-\frac{i c}{2 \mu}\left[\psi^{+}-\psi^{-}\right] \tag{4.16}
\end{align*}
$$

Let now $\sigma:=\sqrt{\left|\epsilon_{124}+\epsilon_{234}+\epsilon_{124}\right| \frac{\mu}{32}}, \sigma$ being the eigenvalue of $\hat{V}$ on both $\psi^{ \pm}$while it vanishes on $\psi^{0}$. It is then straightforward to check that

$$
\begin{equation*}
\hat{V} \psi=\sigma\left[\psi-\frac{c}{\mu^{2}} \psi^{0}\right]=\sigma \frac{a^{2}+b^{2}}{\mu^{2}} \psi+\text { terms } \propto \psi_{ \pm} \tag{4.17}
\end{equation*}
$$

We now translate this result back into the representation $L_{2}\left(\overline{\mathcal{A} / \mathcal{G}}, d \mu_{0}\right)$, multiply by $h_{e_{4}}^{-1}$ and take the trace. The result must be a gauge invariant vector with no dependence on $e_{4}$, that is, $j=j_{4}=0$. It follows that $\operatorname{tr}\left(h_{e_{4}} \psi_{ \pm}\right)=0$ because in writing $h_{e_{4}} \psi_{ \pm}$in terms of extended spin-networks we cannot get $j=0$ since the contractor corresponding to $j_{123}=1$ is unchanged. Therefore
Theorem 4.2 The eigenvalue of $\hat{L}_{v}(c)$ on a trivalent spin-network state $T_{\gamma}$ with vertex $v$ such that $c$ and $\gamma$ do not share a segment incident at $v$ is given by

$$
\begin{equation*}
\hat{L}_{v}(c) T_{\gamma}=\ell_{p} \sqrt{\left|\epsilon_{124}+\epsilon_{234}+\epsilon_{124}\right| \mu} \frac{a^{2}+b^{2}}{2 \mu^{2}} T_{\gamma}, \mu:=\sqrt{a^{2}+b^{2}+c^{2}} \tag{4.18}
\end{equation*}
$$

where $a, b, c$ are given through (4.19)-(4.14).

## 5 Tube operator and weaves

The length operator has a strange feature unshared by the area and volume operators :
Ultimately, in a semiclassical approximation, one wishes to construct states which approximate a fixed classical 3-geometry $\left(\Sigma, q_{a b}\right)$. Such eigenstates have been called "weaves" in the literature (see for instance (25). Most of these eigenstates typically only involved linked, rather than intersecting loops, while the length and volume operators automatically annihilate those states, we need to construct more general weave states for which the name "lattice states" is more appropriate.
It is clear that a state that approximates such a geometry has to be defined on an (infinite, in case $\Sigma$ is not compact) graph filling all of $\Sigma$ in the following sense : consider a parameter $\delta$ of which we may think as a lattice spacing and which is to characterize the average distance (as measured by $q_{a b}$ ) between neighbouring vertices of the graph. In order to serve as a good approximation, the parameter needs to be small as compared to the scales of macroscopic objects. Consider any macroscopic volume or area in $\Sigma$. It follows that the graph necessarily intersects these objects provided that $\delta$ is small enough and it intersects it the more often the smaller $\delta$. Each intersection makes a contribution no matter whether the intersection point is a vertex or not. On the other hand, even for the most random distribution of vertices with mean separation $\delta$, even if $\delta$ is much smaller than the length of a given curve, the curve genuinely almost never intersects the graph in a vertex and so the quantum length of the curve would be always much smaller than its classical length.

This is obviously not what we want and so the length operator constructed cannot be directly associated with the length of a given classical curve. Following the old physical argument that an object always has to have a linear extension of at least one Planck length in order not to form a black hole, the conclusion is that a one-dimensional curve cannot correspond to a physical object. This motivates the following definition.

Definition 5.1 A tube $C$ with a curve $c$ as center corresponding to a classical metric $q_{a b}$ is a two parameter congruence of curves $C:(r, s) \in[0,1] \times[-\pi . \pi] \rightarrow C_{r, s}$ such that
i) $C_{0, s}=c \forall s \in[-\pi, \pi]$,
ii) the transversal extension of the tube as measured by

$$
\Delta:=\sup _{s, t} \int_{0}^{1} d r \sqrt{C_{, r}^{a} C_{, r}^{b} q_{a b}}(s, t)
$$

is much smaller than the length $L(c)$ of the central curve $c$.
So the picture is that we have a congruence of curves, all of which look like $c$ and which fill a cylinder with thickness $\Delta \ll L(c)$. We are now ready to define a tube operator.

Definition 5.2 The tube operator is given by

$$
\begin{equation*}
\hat{L}(C):=\sum_{(r, s) \in[0,1] \times[-\pi, \pi]} \hat{L}\left(C_{r, s}\right) . \tag{5.1}
\end{equation*}
$$

Notice that there is an uncountable sum involved in (5.1). In order to see that this makes sense we first of all notice that the classical volume of the region filled by the tube is given approximately by $\pi \Delta^{2} L(c)$, provided that $q_{a b}$ is slowly varying there. The density of vertices is given by $1 / \delta^{3}$ so that we have approximately $\pi L(c) \Delta^{2} / \delta^{3}$ vertices inside the region filled by the tube. For a genuine distribution of vertices, each curve involved in (5.1) will intersect at most one vertex. On the other hand, all these intersections should be assigned to the central curve only because we wish to measure the length of the curve $c$ only. This is the reason why we do not divide by the number of contributing curves.
It follows that only a finite number of curves contribute in (5.1) and so the tube operator is densely defined on $\mathcal{D}$ and it is trivial to see that it is positive semidefinite, symmetric and possesses self-adjoint extensions. Notice that since we have prescribed the tube to be a congruence, it follows that each vertex lies on at most one curve. Since the length operators $\hat{L}\left(s_{v}\right)$ commute for distinct vertices, it follows that all contributing length operators can be simultaneously diagonalized.
As an example assume that we wish to approximate the Euclidean metric $q_{a b}=\delta_{a b}$ so that it is everywhere constant and not varying at all. For simplicity we want to consider a weave built from a tri-valent lattice so that all length operators are automatically diagonal. To simplify life further, let us assume that all edges have equal spin $j_{0}$ which implies that $j_{0}$ is integral. Let $\lambda\left(j_{0}\right)$ be the corresponding eigenvalue in (4.18) (almost never will the lattice have a segment in common with
one of the contributing curves). Then it follows that we get for the eigenvalue of the tube operator

$$
\begin{equation*}
\Lambda(C)=\ell_{p} \frac{\pi \Delta^{2} L(c)}{\delta^{3}} \lambda\left(j_{0}\right)(1+o(\delta / L(c)) \tag{5.2}
\end{equation*}
$$

which we want to equal $L(c)$. Since we want the tube operator to be state-independent and since there is no other state-independent length scale in the problem, we are forced to choose $\Delta=\ell_{p}$ (which is also physically motivated as above; we have chosen a fixed factor of proportionality to equal unity). It then follows that $\delta=\ell_{p} \sqrt[3]{\pi \lambda\left(j_{0}\right)}$. Since $\lambda$ is a monotonously growing function of $j_{0}$ it follows that the lattice can be chosen the coarser the more spin it carries in order to still approximate the same geometry. The limit $\delta \rightarrow 0$ blows up, it is physically meaningless to consider lattices with mean distance of vertices smaller than Planck scale which, as already stated in the literature, hints at a discrete structure. Since we have chosen $\Delta=\ell_{p} \leq \delta\left(j_{0}\right)$ it follows as a consistency check that at most one vertex per unit length $\delta$ will contribute to $\Lambda(C)$ so that $\hat{L}(C)$ correctly measures a one-dimensional object.
The limit $\delta \rightarrow \infty$ gives a zero eigenvalue because $\Sigma$ becomes more and more empty. So we see that there are weave states $\Psi\left(q_{a b}:=\delta_{a b}, j_{0} ; \delta\left(j_{0}\right)\right)$ which approximate $q_{a b}=\delta_{a b}$ where we have considered $j_{0}$ as a free parameter. Notice that there is a large number of weave states corresponding to the various values of $j_{0}$ such that the same geometry is approximated (we cannot let $j_{0} \rightarrow \infty$ because of $\delta \ll L(c)$ ). Also observe that we use here weave states which necessarily involve intersecting and overlapping $\left(j_{0}>1 / 2\right)$, rather than only linked, loops [25].
On the other hand we can fix $\delta$, vary $j_{0}$ and ask which classical geometry is being approximated. Since one and the same curve appears the longer, according to (5.2), the more spin the lattice carries, we see that that the corresponding $q_{a b}$ must be warped as compared to $\delta_{a b}$. This hints at the fact that the spin of the lattice must have something to do with the curvature which in turn has to do with the local energy distribution of the gravitational field. Indeed, the spin characterizes the eigenvalue of the ADM Hamiltonian (7, 26).

## 6 Summary

- A satisfactory quantization of the length of a piecewise smooth curve was given. The length operator leaves the piecewise analytic graph of a cylindrical function invariant (in particular, analytic) therefore it is proper to allow the curve to be only piecewise smooth rather than piecewise analytic. It is gauge invariant and diffeomorphism covariantly defined.
- The length operator is an unbounded, self-adjoint, positive semi-definite operator with domain $\operatorname{Cyl}^{3}(\overline{\mathcal{A} / \mathcal{G}})$.
- The spectral analysis turns out to be tedious but straightforward. In particular, the methods displayed here reveal that the complete spectrum can be found by writing a suitable computer code. This also is true for the volume operator [23].
- The spectrum is entirely discrete, the quantum of the length being given by $\sqrt[4]{3} \ell_{p} /$. In particular, all spin-network states on graphs with valence not bigger than three are eigenvectors of the length operator with, in general, non-vanishing eigenvalue.
- In order approximate a 3-geometry a tube operator has to be constructed. One
can build intersecting and overlapping weave states which are such that they reproduce the classical length of the tubes given a classical geometry.


## Acknowledgements

This research project was supported in part by DOE-Grant DE-FG02-94ER25228 to Harvard University.

## A The $6 j$-symbol

The following can be found in any textbook on the recoupling theory of angular momenta (for instance [21]).
Consider the coupling of three angular momenta of some spin system consisting of three independent subsystems with angular momenta $j_{I}, I=1,2,3$. Since the three systems are independent of each other, the operators $J_{I}^{i}, J_{J}^{j}, I \neq J$ mutually commute. We are interested in states which are labelled by the quantum numbers of a maximal set of mutually commuting observables which includes the the square of the total angular momentum $J^{i}=J_{1}^{i}+J_{2}^{i}+J_{3}^{i}$ of this system. It is easy to see that the set consisting of $j_{1}, j_{2}, j_{3}, j, m$ (where $m$ is the eigenvalue of $J^{3}$ ) is insufficient because $j_{i}, m_{i}$ is a set consisting of six rather than five quantum numbers. The missing quantum number is any choice of $j_{I J}, I<J$ which labels the eigenvalue of the square of $J_{I J}:=J_{I}+J_{J}$. It is easy to see that $J, J_{I J}$ satisfy the angular momentum algebra.
Denote an orthonormal basis of states so constructed by $\mid j_{I J}, j ; j_{1}, j_{2}, j_{3}>$ where it is understood that one first couples $j_{I}, j_{J}$ to resulting spin $j_{I J}$ and then $j_{I J}, j_{K}$ to the resulting total spin $j$. Any choice of such a recoupling scheme leads to an orthonormal basis and thus the transformation between these bases must be unitary. The matrix elements of this transformation were explicitly computed by Racah. In particular we have

$$
\begin{align*}
& <j_{12}, j ; j_{1}, j_{2}, j_{3} \mid j_{23}, j ; j_{1}, j_{2}, j_{3}> \\
=: & (-1)^{j_{1}+j_{2}+j_{3}+j} \sqrt{\left(2 j_{12}+1\right)\left(2 j_{23}+1\right)}\left\{\begin{array}{ccc}
j_{1} & j_{2} & j_{12} \\
j_{3} & j & j_{23}
\end{array}\right\} \tag{A.1}
\end{align*}
$$

where the $2 \times 3$ matrix on the right hand side is the so-called $6 j$-symbol for which a closed formula exists [21].
There exists a choice of phases for the basis vectors such that all the $6 j$-symbols are real. With this choice they enjoy a large amount of symmetries of which we just need two :

1) it is invariant under an arbitrary permutation of its columns and

2 ) it is crossing-symmetric, that is

$$
\left\{\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{A.2}\\
j_{4} & j_{5} & j_{6}
\end{array}\right\}=\left\{\begin{array}{lll}
j_{1} & j_{6} & j_{5} \\
j_{4} & j_{3} & j_{2}
\end{array}\right\} .
$$

For the purposes of this paper it is sufficient to table the value of the $6 j$-symbol for the special case that one of $j_{I}, j_{I J}, j$ takes the value $1 / 2$. Then we have the special
values

$$
\begin{align*}
& \left\{\begin{array}{ccc}
a & b & c \\
\frac{1}{2} & c-\frac{1}{2} & b+\frac{1}{2}
\end{array}\right\}=(-1)^{a+b+c} \sqrt{\frac{(a+c-b)(a+b-c+1)}{(2 b+1)(2 b+2)(2 c)(2 c+1)}}  \tag{A.3}\\
& \left\{\begin{array}{ccc}
a & b & c \\
\frac{1}{2} & c-\frac{1}{2} & b-\frac{1}{2}
\end{array}\right\}=(-1)^{a+b+c} \sqrt{\frac{(a+b+c+1)(b+c-a)}{(2 b)(2 b+1)(2 c)(2 c+1)}} \tag{A.4}
\end{align*}
$$

This is all one needs to know in order to verify the eigenvalue calculations of the present paper. It is clear how to generalize the recoupling theory for any number of spins.

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