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A Limit Theorem for Stochastic Acceleration

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Abstract. We consider the motion of a particle in a weak mean zero random force field F, which depends on the position, $x(t)$, and the velocity, $v(t)=\dot{x}(t)$. The equation of motion is $\ddot{x}(t)=\varepsilon F(x(t), v(t), \omega)$, where $x(\cdot)$ and $v(\cdot)$ take values in $\mathbb{R}^d, d\geqq 3$, and ω ranges over some probability space. We show, under suitable mixing and moment conditions on F, that as $\varepsilon\to 0, v^\varepsilon(t)\equiv v(t/\varepsilon^2)$ converges weakly to a diffusion Markov process $v(t)$, and $\varepsilon^2 x^\varepsilon(t)$ converges weakly to $\int_0^t v(s)ds + x$, where $x=\lim \varepsilon^2 x^\varepsilon(0)$.

1. Introduction

For simplicity we do not discuss the general situation in this section, but restrict ourselves to force fields which depend on position only.

Let $F(x), x\in\mathbb{R}^d$, be a random vector field, a random force field, which is stationary and has mean zero. Let $x(t)$ be the coordinate of a particle of unit mass moving through this force field. The equation of motion is

$$\ddot{x}=F(x). \tag{1.1}$$

with given initial position and velocity. Suppose that the force is weak and weakly correlated for points that are far apart. Then one expects that after a long time the velocity $\dot{x}$ will behave like a diffusion Markov process and the position x like the integral of this diffusion process.

To be more specific, suppose that the root mean square of the force field F is proportional to ε so that we may replace (1.1) by

$$\ddot{x}=\varepsilon F(x) \tag{1.2}$$

in which $F(x)$ is of order one. Rescaling of time t into t/ε^2 and putting $\dot{x}(t/\varepsilon^2)=v^\varepsilon(t), x(t/\varepsilon^2)=x^\varepsilon(t)$ leads from (1.1) to the system

$$\frac{dx^\varepsilon(t)}{dt}=\frac{1}{\varepsilon^2}v^\varepsilon(t)$$

$$\frac{dv^\varepsilon(t)}{dt}=\frac{1}{\varepsilon}F(x^\varepsilon(t)) \tag{1.3}$$

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It is proved in the following sections that under suitable conditions on F, v^ε converges weakly as $\varepsilon \to 0$ to a diffusion Markov process $v(t)$ whose generator is given explicitly. Moreover, $\varepsilon^2 x^\varepsilon(t)$ converges weakly to $\int_0^t v(s)ds + x$, where $x = \lim \varepsilon^2 x^\varepsilon(0)$, as $\varepsilon \to 0$.

The Eq. (1.2) describes for instance the motion of a charged particle in an electromagnetic field, and several authors have obtained formulas for the limit process by perturbation methods or similar procedures [1–5]. We now give such a formal derivation of the relevant results for (1.3). We note that the method used in [6] for the much simpler problem than (1.3)

$$\frac{dx^\varepsilon(t)}{dt} = \frac{1}{\varepsilon^2} v + \frac{1}{\varepsilon} F(x^\varepsilon(t)),$$

does not work well in the present situation.

Let $f(v)$ be a bounded and smooth function on $\mathbb{R}^d$ and let $u^\varepsilon(t,x,v) = f(v^\varepsilon(t;x,v))$ where $x^\varepsilon(t;x,v)$, $v^\varepsilon(t;x,v)$ is the solution of (1.3) with $x^\varepsilon(0) = x$, $v^\varepsilon(0) = v$. As a function of t, x and v, u^ε satisfies (the adjoint) Liouville equation

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{\varepsilon^2} v \cdot \frac{\partial u^\varepsilon}{\partial x} + \frac{1}{\varepsilon} F(x) \cdot \frac{\partial u^\varepsilon}{\partial v}, \quad t > 0,$$
$$u^\varepsilon(0,x,v) = f(v), \tag{1.4}$$

Here $\partial/\partial x$ and $\partial/\partial v$ denote the x and v gradient operators and $\cdot$ stands for dot product in $\mathbb{R}^d$. We now attempt to solve (1.4) by a formal series expansion $u^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$. Inserting this into (1.4) and collecting terms leads to the equations

$$v \cdot \frac{\partial u_0}{\partial x} = 0 \tag{1.5}$$

$$v \cdot \frac{\partial u_1}{\partial x} + F(x) \cdot \frac{\partial u_0}{\partial v} = 0 \tag{1.6}$$

$$v \cdot \frac{\partial u_2}{\partial x} + F(x) \cdot \frac{\partial u_1}{\partial v} - \frac{\partial u_0}{\partial t} = 0, \text{ et.} \tag{1.7}$$

From (1.5) we conclude that $u_0 = u_0(t,v)$ and $u_0(0,v) = f(v)$ (to satisfy (1.4)), but u_0 is otherwise undetermined at this stage. We consider (1.6) and note that we can write the random function u_1 in the form

$$u_1(t,x,v) = \chi(x,v) \cdot \frac{\partial u_0}{\partial v} \tag{1.8}$$

where $\chi(x,v)$ satisfies

$$v \cdot \frac{\partial \chi}{\partial x} + F(x) = 0. \tag{1.9}$$

One may write formally $\chi(x,v) = \int_0^\infty F(x + vt)dt$ but of course this expression does not make sense. We retain it anyway with the understanding that some convergence factor has been introduced $\left(\text{like } \int_0^\infty e^{-\beta t} F(x+vt)dt\right)$.

Now we use this in (1.7) and demand as usual that the expectation of $F\cdot\frac{\partial u_1}{\partial v}-\frac{\partial u_0}{\partial t}$ be zero. This gives a diffusion equation that determines $u_0(t,v)$. Specifically $Eu_0(t,v)$ satisfies

$$\frac{\partial Eu_0}{\partial t}=\mathscr{L}Eu_0, \quad t>0, \quad u_0(0,v)=f(v) \tag{1.10}$$

where $\mathscr{L}$ is given by

$$\begin{aligned}\mathscr{L}&=\int_0^\infty E\left\{F(x)\cdot\frac{\partial}{\partial v}\left(F(x+vt)\cdot\frac{\partial}{\partial v}\right)\right\}dt\\ &=\sum_{j,k=1}^{d}\frac{\partial}{\partial v_j}\left(A_{jk}(v)\frac{\partial}{\partial v_k}\right)\end{aligned} \tag{1.11}$$

with

$$A_{jk}(v)=\int_0^\infty E\{F_j(x)F_k(x+vt)\}dt. \tag{1.12}$$

When the correlations in the force field die out rapidly enough, the diffusion coefficients $A_{jk}(v)$ are well defined if $v\neq 0$ but they are necessarily singular at $v=0$. (Note that $\mathscr{L}$ is not always self-adjoint if F depends on v as well; see (2.3) and (2.4) below.)

The problem then is to show that $v^\varepsilon(t)$ converges weakly to the diffusion generated by $\mathscr{L}$ of (1.11) under some suitable hypotheses. The theorem of the next section gives such conditions for convergence. It is discussed further there. Some specific examples are given in Sect. 4.

It is of interest to point out some special cases of (1.11) and (1.12) here.

Let

$$R_{jk}(x)=E\{F_j(x+y)F_k(y)\}, \quad j,k=1,2,\dots,d, \tag{1.13}$$

be the covariance of the force field F. It is assumed that it decays rapidly with x; in fact much stronger asymptotic independence assumptions are introduced in the next section. Let us assume also that the symmetry condition

$$R_{jk}(x)=R_{jk}(-x) \tag{1.14}$$

holds. Then (1.12) may be written in the form

$$A_{jk}(v)=a_{jk}(v)\equiv\frac{1}{2}\int_{-\infty}^{\infty}R_{jk}(vt)dt. \tag{1.15}$$

If we introduce the power spectral density $\hat{R}_{jk}(l)$, then

$$R_{jk}(x)=\int_{\mathbb{R}^d}e^{il.x}\hat{R}_{jk}(l)dl, \tag{1.16}$$

and (1.15) becomes

$$a_{jk}(v) = \pi \int_{\mathbb{R}^d} \delta(l \cdot v) \hat{R}_{jk}(l) dl \tag{1.17}$$

where $\delta(\alpha)$ is the delta function with unit mass at zero. The expression (1.17) for the diffusion coefficients is useful when $F(x)$ is the gradient of a potential $V(x)$. If we set

$$R(x-y) = E\{V(x)V(y)\} \tag{1.18}$$

then (1.17) yields

$$\begin{aligned} a_{jk}(v) &= \pi \int_{\mathbb{R}^d} \delta(l \cdot v) l_j l_k \hat{R}(l) dl \\ &= \sum_{p,q} \left(\delta_{pj} - \frac{v_p v_j}{|v|^2}\right)\left(\delta_{qk} - \frac{v_q v_k}{|v|^2}\right) \pi \int_{\mathbb{R}^d} \delta(l \cdot v) l_p l_q \hat{R}(l) dl. \end{aligned}$$

From this we see that in the potential case the limiting diffusion operator is degenerate. The limit diffusion process is concentrated on the sphere $|v| = |v_0|$ where $v_0 \neq 0$ is the starting velocity. (Since $a(v) \cdot v = 0$ and $L(v^2) = 0 \cdot$) (2.6) is automatic when $|v|$ is constant, but unfortunately, our theorem as stated does not allow $a_{ij}(v)$ to become singular, and hence does not apply without modification to the above case. One such modification of the theorem is given in Remark 5 of Section 4. The conclusion of the theorem remains valid if (4.2) and (4.3) hold, even when $a(\cdot)$ becomes singular. This comment also applies to other cases where $|v(t)|$ remains constant (e.g. when $F(x, v)$ is always perpendicular to v, such as when $F(x, v)$ is of the form $F(x, v) = v \wedge \Gamma(x)$).

2. Statement of Theorem

Throughout $(\Omega, \mathscr{F}, \mathscr{P})$ denotes our basic probability space. On this space $F(x, v, \omega)$: $\mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a random field with the following properties:

(I) F is jointly measurable with respect to $\mathscr{B} \times \mathscr{B} \times \mathscr{F}$ where $\mathscr{B}$ is the Borel field in $\mathbb{R}^d$. As a function of (x, v), $F(\cdot, \omega)$ is almost surely in $C^2(\mathbb{R}^d \times \mathbb{R}^d)$.

(II) F is strictly stationary in x, i.e. for any $x_i, v_i \in \mathbb{R}^d$ the joint distribution of $F(x_1 + h, v_1), \ldots, F(x_k + h, v_k)$ is independent of $h \in \mathbb{R}^d$. Equivalently, the process $\{F(x, \cdot, \omega)\}_{x \in \mathbb{R}^d}$ is stationary in x. In addition

$$E\{F(x, v)\} = 0, \quad x, v \in \mathbb{R}^d. \tag{2.1}$$

(III) For $\Lambda \subset \mathbb{R}^d$, set

$$\begin{aligned} \mathscr{G}_\Lambda &= \sigma\{F(x, v, \cdot) : x \in \Lambda, v \in \mathbb{R}^d\} \\ &= \text{sigma field generated by } F(x, v, \cdot), x \in \Lambda, v \in \mathbb{R}^d. \end{aligned}$$

For $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ define

$$\alpha(\Lambda_1, \Lambda_2) = \sup_{A \in \mathscr{G}_{\Lambda_1}, B \in \mathscr{G}_{\Lambda_2}} |P(B) - P(B \mid A)|.$$

Also, set

$$\beta(\rho) = \sup\{\alpha(\Lambda_1, \Lambda_2) : \Lambda_1, \Lambda_2 \in \mathscr{B} \text{ with } d(\Lambda_1, \Lambda_2) \geqq \rho\}.$$

Here

$$d(A_1, A_2) = \inf\{|x_1 - x_2| : x_i \in A_i\}.$$

Assume that

$$\int_0^\infty \{\beta(t)\}^{1/24} dt < \infty. \tag{2.2}$$

Note that (2.2) is in a sense a measure of the rate at which $F(x_1, \cdot)$ and $F(x_2, \cdot)$ become independent when $|x_2 - x_1| \to \infty$.

(IV) For some constant C_0 and all $v_0 \in \mathbb{R}^d$, $0 \leqq |\beta| \leqq 2$ and $r = 16d + 64$

$$E\left\{ \sup_{|x| \leqq 1, |v - v_0| \leqq 1} |D^\beta F(x, v)|^r \right\} \leqq C_0.$$

As usual β stands here for a multi-index and D^β for the corresponding partial derivative. Thus $D^\beta F(x, v)$ can be any mixed derivative of F.

(V) Let

$$a_{ij}(v) = \int_{-\infty}^{+\infty} E\{F_i(0, v) F_j(tv, v)\} dt, \tag{2.3}$$

$$\begin{aligned} b_i(v) &= \sum_j \int_0^\infty E\left\{ F_j(0, v) \frac{d}{dv_j} F_i(tv, v) \right\} dt \\ &= \sum_j \int_0^\infty \left[tE\left\{ F_j(0, v) \frac{\partial}{\partial x_j} F_i(tv, v) \right\} \right. \\ &\quad \left. + E\left\{ F_j(0, v) \frac{\partial}{\partial v_j} F_i(tv, v) \right\} \right] dt. \end{aligned} \tag{2.4}$$

Here $(\partial/\partial v_j) F(tv, v) = [(\partial/\partial v_j) F(x, v)]_{x = tv}$. The integrals in (2.3) and (2.4) can be shown to be absolutely convergent on the set $\{v \neq 0\}$ and to be bounded as $|v| \to \infty$ by means of (III) and (IV) (use Theorem 17.23 of [7] or Lemma 20.1 of [8]). Assume that $a_{ij}(v)$ is strictly positive definite on $\{v \neq 0\}$ and that $a_{ij}(\cdot)$ and $b_i(\cdot)$ are C^∞ functions on $\{v \neq 0\}$.

For any $f \in C^2(\mathbb{R}^d)$ define

$$\mathscr{L} f(v) = \frac{1}{2} \sum_{i,j} a_{ij}(v) \frac{\partial^2}{\partial v_i \partial v_j} f(v) + \sum_i b_i(v) \frac{\partial f}{\partial v_i}, \quad v \neq 0. \tag{2.5}$$

Let V_t be a diffusion with generator $\mathscr{L}$ and starting-point $v_0 \neq 0$ (see Remark 1 below). Assume finally that for each $v_0 \neq 0$ and $T < \infty$

$$\lim_{M \to \infty} P^{v_0} \left\{ |V_t| \leqq \frac{1}{M} \text{ for some } t \leqq T \right\} = 0. \tag{2.6}$$

Last, let $\{v^\varepsilon(t), y^\varepsilon(t)\} = \{v^\varepsilon(t,\omega), y^\varepsilon(t,\omega)\}$ be the solution of the equations

$$\frac{dy^\varepsilon}{dt} = \frac{1}{\varepsilon^2} v^\varepsilon(t)$$

$$\frac{dv^\varepsilon}{dt} = \frac{1}{\varepsilon} F(y^\varepsilon(t), v^\varepsilon(t))$$

$$v^\varepsilon(0) = v_0 \neq 0, \; y^\varepsilon(0) = y_0. \tag{2.7}$$

These solutions exist and are unique with probability one by the argument in step (ii) of [6]. Denote by Q^ε the probability measure on $C = C([0,\infty);\mathbb{R}^d)$ induced by $\{v^\varepsilon(t)\}_{t\geqq 0}$.

Theorem. *If $d \geqq 3$, $v_0 \neq 0$ and $F(\cdot)$ satisfies conditions (I)–(V) above, then Q^ε converges weakly on C as $\varepsilon \downarrow 0$ to the measure Q corresponding to the diffusion process in $\mathbb{R}^d$ with generator $\mathscr{L}$ and initial point v_0 (i.e., $Q(v(0) = v_0) = 1$).*

Corollary. *Under the conditions of the theorem the measure R^ε induced by $(v^\varepsilon(\cdot), \varepsilon^2 y^\varepsilon(\cdot))$ on $C([0,\infty);\mathbb{R}^d \times \mathbb{R}^d)$ converges weakly to the unique measure R which is concentrated on the set*

$$\left\{ X, Y : Y_i(t) = \int_0^t X_i(\sigma) d\sigma, X(0) = v_0 \right\}$$

and whose marginal distribution of $X(\cdot)$ coincides with Q.

(Here $X_1(t), \ldots, X_d(t), Y_1(t), \ldots, Y_d(t)$ are the coordinate functions on $C([0,\infty);\mathbb{R}^d \times \mathbb{R}^d)$.

Remark 1. The diffusion V_t on $\mathbb{R}^d \backslash \{0\}$ can be constructed by "patching together" local diffusions. The local diffusions can be obtained as solutions of suitable Ito equations (see [9], Ch. 4.3) or by semigroup theory (see [10]). It is also possible to define the diffusion $V_t^{(n)}$ which has generator $\mathscr{L}$ on

$$C_n = \left\{ v \in \mathbb{R}^d : |v| > \frac{1}{n} \right\},$$

and is killed at time $\tau_n = \inf\{t : V_t^{(n)} \notin C_n\}$. For $m \geqq n$, $V_t^{(m)}$ up until time τ_n is equivalent to $V_t^{(n)}$ ([11], Corollary in Chap. 5.24), and V_t can be viewed as a limit of the $V_t^{(n)}$.

Remark 2. In our most important examples (see Remark 3) the coefficients $a_{ij}(v)$ and $v_i(v)$ are singular at the origin so that one should not replace (V) by the simpler condition $a_{ij}(\cdot), b_i(\cdot) \in C^\infty(\mathbb{R}^d)$.

In Remark 6, Sect. 4, we shall discuss a replacement for the condition $d \geqq 3$ and $a_{ij}(\cdot), b_i(\cdot) \in C^\infty(\mathbb{R}^d \backslash \{0\})$. We shall also give some sufficient conditions for (2.6). For the definition of the spaces $C([0,\infty);\mathbb{R}^d)$, $D([0,\infty);\mathbb{R}^d)$ and weak convergence on these spaces see [12] and [13].

Remark 3. Note that under R the process $\{X(t), Y(t)\}_{t\geqq 0}$ of the corollary is a singular diffusion; the Y-part has zero diffusion coefficients. By itself the Y-part is not Markovian, let alone a diffusion.

3. Proof of Theorem

The basic outline of the proof is the same as for Theorem 3 of [6]. We first introduce a truncated process (in step (i)). The truncation will be removed only in the last step. The second step proves the basic mixing lemma which is used in step (iii) to show tightness of the family of measures (indexed by ε) induced by the truncated processes. The remaining steps identify the limit process as the solution of a certain martingle problem.

Step (i)

In contrast to [6] we need here not one, but several cutoff functions. These will depend on parameters η, δ, M, N, which remain fixed until step (v). We shall not exhibit these parameters explicitly in the notation before step (v); it is understood, though, that all constants C below may depend on these parameters, the dimension, d, and the length of the time interval, T, but not on ε.

As will become apparent it is best to define the cutoff functions as nonanticipatory functionals which depend in addition on a variable which ranges over $\mathbb{R}^d$. We begin with the velocity cutoff. Let $D = D([0, \infty); \mathbb{R}^d)$ and $\eta > 0$, and for $X(\cdot) \in D$ set

$$X_k = \begin{cases} \dfrac{X(k\eta)}{|X(k\eta)|} & \text{if } X(k\eta) \neq 0 \\ (1, 0, \ldots, 0) & \text{if } X(k\eta) = 0. \end{cases}$$

In addition let $\psi_0 : \mathbb{R}^d \times S^{d-1} \times S^{d-1} \to [0, 1]$ be a C^∞ function (S^{d-1} is the unit ball in $\mathbb{R}^d$) such that

$$\psi_0(u, x_1, x_2) = 0 \quad \text{if} \quad |u| \leqq \frac{1}{2M} \quad \text{or} \quad |u| \geqq 2M$$

$$\text{or} \quad (u, x_1) \leqq \frac{1}{2N} \quad \text{or} \quad (u, x_2) \leqq \frac{1}{2N}; \tag{3.1}$$

$$\psi_0(u, x_1, x_2) = 1 \quad \text{if} \quad \frac{1}{M} \leqq |u| \leqq M \quad \text{and}$$

$$(u, x_1) \geqq \frac{1}{N} \quad \text{and} \quad (u, x_2) \geqq \frac{1}{N}. \tag{3.2}$$

Throughout we take M so large that

$$\frac{1}{M} \leqq |v_0| \leqq M.$$

Now define $\Psi : [0, \infty) \times D \times \mathbb{R}^d$ by

$$\Psi(t, X, w) = \begin{cases} \psi_0(w, X_0, X_0) & \text{if} \quad 0 \leqq t < \eta \\ \psi_0(w, X_{k-1}, X_k) & \text{if} \quad k\eta \leqq t < (k+1)\eta, k \geqq 1 \end{cases} \tag{3.3}$$

To prevent the y^ε path at any given time to come too close to a value taken on before another cutoff function is needed. It will be seen in step (ii) how this guarantees a certain amount of independence between the present and the "distant past" for the truncated process, and thereby allows us to prove the mixing lemma. We construct a function $\phi_k : D \times \mathbb{R}^d \to [0, 1]$ which is smooth in its second argument, uniformly in the first argument and k. The principal requirement for ϕ_k if $k \geqq 1$ is that for fixed $X(\cdot) \in D$,

$$\begin{aligned} \phi_k(X, z) &= 0 \quad \text{if} \quad \inf_{0 \leqq u \leqq (k-1)\eta} \left| z - \int_0^u X(t)dt \right| \leqq \delta, \\ \phi_k(X, z) &= 1 \quad \text{if} \quad \inf_{0 \leqq u \leqq (k-1)\eta} \left| z - \int_0^u X(t)dt \right| \geqq 2\delta. \end{aligned} \tag{3.4}$$

To construct such a function we take

$$\chi_k(X, z) = \tilde{\chi}\left(\inf_{0 \leqq u \leqq (k-1)\eta} \left| z - \int_0^u X(t)dt \right| \right)$$

where $\tilde{\chi}$ is continuous, $0 \leqq \tilde{\chi} \leqq 1$ and

$$\tilde{\chi}(y) = \begin{cases} 0 & \text{if} \quad |y| \leqq \frac{5}{4}\delta \\ 1 & \text{if} \quad |y| \geqq \frac{7}{4}\delta . \end{cases}$$

Also we take for $\Delta(\cdot)$ a nonnegative function $C^\infty(\mathbb{R}^d)$ with support in $\{|z| \leqq \delta/4\}$ and such that

$$\int_{\mathbb{R}^d} \Delta(z)dz = 1 .$$

Then

$$\phi_k(X, z) = \int_{\mathbb{R}^d} \Delta(z - x)\chi_k(X, x)dx \tag{3.5}$$

satisfies (3.4). Finally we define $\Phi = \Phi_\varepsilon : [0, \infty) \times D \times \mathbb{R}^d$ by

$$\begin{aligned} \Phi(t, X, z) &\equiv 1 & &\text{if } 0 \leqq t < \eta, \\ \Phi(t, X, z) &= \phi_k(X, \varepsilon^2(z - y_0)) & &\text{if } k\eta \leqq t < (k+1)\eta, k \geqq 1 . \end{aligned} \tag{3.6}$$

Lastly, we set

$$G(t, X, z, w) = G_\varepsilon(t, X, z, w) = \Psi(t, X, w)\Phi_\varepsilon(t, X, z)F(z, w) \tag{3.7}$$

and we define our truncated process $u(\cdot)$, $z(\cdot)$ as the solution of

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{\varepsilon^2} w(t) \\ \frac{dw}{dt} &= \frac{1}{\varepsilon} G(t, w(\cdot), z(t), w(t)) \\ w(0) &= v_0, \quad z(0) = y_0 . \end{aligned} \tag{3.8}$$

As in [6] this means that z and w are continuous functions which satisfy

$$z(t) = y_0 + \frac{1}{\varepsilon^2}\int_0^t w(\sigma)d\sigma,$$

$$w(t) = v_0 + \frac{1}{\varepsilon}\int_0^t G(\sigma, w(\cdot), z(\sigma), w(\sigma))d\sigma. \tag{3.9}$$

Note that G is continuously differentiable in its last two arguments and for $k\eta \leqq t < (k+1)\eta$ depends on $w(\cdot)$ only through the values of $w(u)$ on $u \leqq k\eta$. In particular, for $t < \eta$, G does not depend on its second argument and (3.9) has w.p. 1 a unique solution on $t \leqq \eta$ by the argument of step (ii) of [6]. Once a solution has been found for $t \leqq k\eta$, the dependence of G on its second argument is determined up to time $(k+1)\eta$ and by step (ii) of [6] one then obtains w.p. 1 a unique solution for $t \leqq (k+1)\eta$.

Of course $w(\cdot)$ and $z(\cdot)$ depend on ε. When necessary we shall indicate this by writing $w^\varepsilon(t)$ and $z^\varepsilon(t)$. In particular we denote by R^ε the measure induced on $C([0,\infty);\mathbb{R}^d)$ or $D = D([0,\infty);\mathbb{R}^d)$ by $w^\varepsilon(\cdot)$. Towards the end we shall write $R^\varepsilon(\cdot\,; M, N, \eta, \delta)$ to indicate the dependence of R^ε on M, N, η, δ. For brevity we shall write

$$G(t, w, z) = G(t, w(\cdot), w, z)$$

for $w, z \in \mathbb{R}^d$ and $w(\cdot)$ the solution of (3.8). Similarly

$$\Psi(t, w) = \Psi(t, w(\cdot), w), \quad \Phi(t, z) = \Phi(t, w(\cdot), z).$$

Before turning to the proof of tightness of the family of measures $\{R^\varepsilon(\;): 0 < \varepsilon \leqq 1\}$ we need some simple observations. First, $G(t, w, z)$ is constant in t over each of the intervals $[k\eta, (k+1)\eta)$. Second, for every T there exists a constant $C_1 = C_1(T, M, \eta, \delta)$ such that

$$\left|\left(\frac{\partial}{\partial z}\right)^\beta \Phi(t, X, z)\right| \leqq C_1 \varepsilon^{2|\beta|}, \left|\left(\frac{\partial}{\partial w}\right)^\beta \Psi(t, X, w)\right| \leqq C_1 \tag{3.10}$$

for all $X \in D$, $w, z \in \mathbb{R}^d$, $0 \leqq t \leqq T$, $0 < \varepsilon \leqq 1$ and $|\beta| \leqq 1$. Formula (3.10) is obvious for Ψ from (3.3); for Φ it follows from (3.5) and (3.6). Indeed, for $k\eta \leqq t < (k+1)\eta$,

$$\begin{aligned}\left|\left(\frac{\partial}{\partial z}\right)^\beta \Phi(t, X, z)\right| &= \varepsilon^{2|\beta|}\left|\int_{\mathbb{R}^d} D^\beta A(\varepsilon^2(z - y_0) - x)\chi_k(X, x)dx\right| \\ &\leqq \varepsilon^{2|\beta|}\int_{\mathbb{R}^d} |D^\beta A(x)|dx.\end{aligned}$$

Lastly, for any z of the form $z(\xi) + \dfrac{\sigma - \xi}{\varepsilon^2} w(\xi)$, $\Phi_\varepsilon(t, z)$ does not depend explicitly on ε, but only through $\{w(u): 0 \leqq u \leqq \xi \vee ((k-1)\eta)^+\}$, when $k\eta \leqq t < (k+1)\eta$.

Indeed for such t, the above z and $u \leqq ((k-1)\eta)^+$

$$\varepsilon^2\left\{z(\xi) + \frac{\sigma-\xi}{\varepsilon^2} w(\xi) - y_0\right\} - \int_0^u w(t)dt$$
$$= \int_u^\xi w(\lambda)d\lambda + (\sigma-\xi)w(\xi).$$

Finally, we have the following simple

Lemma 1.

$$\frac{1}{2M} \leqq |w(t)| \leqq 2M \quad \textit{for all } t \geqq 0. \tag{3.11}$$

$$(w(t), w_{k-1}) \geqq \frac{1}{2N} \quad \textit{and } (w(t), w_k) \geqq \frac{1}{2N} \quad \textit{for } k\eta \leqq t < (k+1)\eta, \tag{3.12}$$

where $w_j = \dfrac{w(j\eta)}{|w(j\eta)|}$, $(w_{-1} = w_0)$.

$$|z(t) - z(s)| \leqq \frac{2M}{\varepsilon^2}|t-s|, \quad t,s \geqq 0; \tag{3.13}$$

$$P\left\{|w(t)-w(s)| \geqq |t-s|\varepsilon^{-9/8} \quad \textit{for some } 0 \leqq s,t \leqq T\right\} \leqq C_2\varepsilon^8 \tag{3.14}$$

Proof. Formulas (3.11) and (3.12) are easily proved by induction on k. If they hold at $t = k\eta$, then they must hold up till $(k+1)\eta$ because $\Psi(t, w(t))$ vanishes as soon as (3.11) or (3.12) fails. Formula (3.13) is immediate from (3.9) and (3.11). Lastly, for (3.14) observe that

$$|w(t)-w(s)| = \frac{1}{\varepsilon}\left|\int_s^t G(\lambda, z(\lambda), w(\lambda))d\lambda\right|$$
$$\leqq \frac{|t-s|}{\varepsilon} \sup_{|\lambda| \leqq T} |F(z(\lambda), w(\lambda))|$$

so that by (3.11), (3.13)

$$\leqq P\{|w(t)-w(s)| \geqq |t-s|\varepsilon^{-9/8} \text{ for some } 0 \leqq s,t \leqq T\}$$
$$\leqq P\{\sup_{|\lambda| \leqq T} |F(z(\lambda), w(\lambda))| \geqq \varepsilon^{-1/8}\}$$
$$\leqq P\left\{\sup_{|z| \leqq (2M/\varepsilon^2)T + |y_0|, |w| \leqq 2M} |F(z)| \geqq \varepsilon^{-1/8}\right\}. \tag{3.15}$$

Formula (3.14) now follows from the fact that the set $\{(z,w): |z| \leqq (2M/\varepsilon^2)T + |y_0|, |w| \leqq 2M\}$ can be covered by at most $C_3((T+1)/\varepsilon^2)^d$ cubes of edge-length one, and for any z_0, w_0

$$P\left\{\sup_{|z-z_0|\leqq 1, |w-w_0|\leqq 1} |F(z,w)| \geqq \varepsilon^{-1/8}\right\}$$
$$\leqq \varepsilon^{r/8} E\left\{\sup_{|z-z_0|\leqq 1, |w-w_0|\leqq 1} |F(z,w)|^r\right\} \leqq \varepsilon^{r/8} C_0$$

(by (II) and (IV)). Thus, the left-hand side of (3.14) is at most

$$C_3\left(\frac{T+1}{\varepsilon^2}\right)^d \varepsilon^{r/8} C_0 = C_0 C_3 (T+1)^d \varepsilon^{r/8-2d}$$
$$= C_0 C_3 (T+1)^d \varepsilon^8. \tag{3.16}$$

Step (ii)

This is devoted to the fundamental mixing Lemma 4 and some of its consequences. The preparatory Lemma 3 gives a bound for expectations along the path of z which will be used frequently. Both lemmas rely on the possibility of "predicting" $z(\sigma)$ by the linear function $z(\xi) + ((\sigma - \xi)/\varepsilon^2) w(\xi)$, which depends only on the path up until time ξ. A crucial role is also played by the measure theoretical Lemma 2 which follows directly from the definition of the mixing coefficient β. For convenience we extend the definition of β by setting

$$\beta(\rho) = 2 \quad \text{for} \quad \rho \leqq 0$$

We also replace $\beta(\cdot)$ by its left continuous modification. This can always be done without invalidating (III) because $\beta(\cdot)$ is nonincreasing. We need a further convention. For $\theta = (\theta', \theta'') \in \mathbb{R}^d \times \mathbb{R}^d$, $\tau_\theta F$ will denote the random field whose value at (z, w) is given by

$$\tau_\theta F(z,w) = F(z+\theta', w+\theta'')$$

If h is a function of the $F(z,w)$ which depends only on $\{F(z,u): z \in \Lambda, w \in \mathbb{R}^d\}$ and such that $h(F)$ is $\mathscr{G}_\Lambda$ measurable for some $\Lambda \subset \mathbb{R}^d$, then we see immediately that $h(\tau_\theta F)$ is $\mathscr{G}_{\Lambda+\theta'}$ measurable. In the next lemma we shall take θ itself also random. Lastly, we set

$$\mathscr{F}_t = \sigma\{z(u), w(u), F(z(u), w(u)): u \leqq t\} \tag{3.17}$$

Lemma 2. *Let X be an $\mathscr{F}_t$ measurable random variable with $E\{|X|\} < \infty$ and let $g_i = (g_i', g_i'')$ be $\mathbb{R}^d \times \mathbb{R}^d$ valued random variables, measurable with respect to $\mathscr{F}_t$ and such that*

$$\min\{|g_i' - z(u)| : u \leqq t\} \geqq \rho \tag{3.18}$$

a.e. on the set $\{X \neq 0\}$. Lastly, let h_i be Borel functions of $\{F(z,w): z \in \Delta_i, w \in \mathbb{R}^d\}$ for Borel sets $\Delta_i \subset \mathbb{R}^d$ with

$$0 \in \Delta_i, \text{ diameter } \Delta_i \leqq \kappa \tag{3.19}$$

and

$$|h_i(F)| \leqq A \text{ everywhere.} \tag{3.20}$$

For $\theta_i \in \mathbb{R}^d \times \mathbb{R}^d$ *set*

$$U_i(\theta) = E\{h_i(\tau_\theta F)\}, \quad V(\theta_1, \theta_2) = E\{h_1(\theta_1)h_2(\theta_2)\}.$$

Then

$$|E\{Xh_1(\tau_{g_1}F)\} - E\{XU_1(g_1)\}| \leqq 2A\beta(\rho - \kappa)E\{|X|\}, \tag{3.21}$$

and

$$|E\{Xh_1(\tau_{g_1}F)h_2(\tau_{g_2}F)\} - E\{XV(g_1, g_2)\}| \leqq 2A^2\beta(\rho - \kappa)E\{|X|\}. \tag{3.22}$$

If (3.18) is replaced by

$$\min\{|g_2' - z(u)| : u \leqq t\} \geqq \rho \quad and \quad |g_2' - g_1'| \geqq \rho \tag{3.23}$$

a.e. on the set $\{X \neq 0\}$, *then*

$$\begin{aligned}&|E\{Xh_1(\tau_{g_1}F)h_2(\tau_{g_2}F)\} - E\{Xh_1(\tau_{g_1}F)U_2(g_2)\}| \\ &\quad \leqq 2A^2\beta(\rho - 2\kappa)E\{|X|\}.\end{aligned} \tag{3.24}$$

Proof. We only prove (3.21). First we change g_1, g_2 on the set $X = 0$ such that (3.18) holds everywhere. Since $\{X = 0\} \in \mathscr{F}_t$ we can do this in such a way that the modified g_i are still $\mathscr{F}_t$ measurable. Moreover this modification does not affect (3.21). We may also assume $\rho - \kappa > 0$ since we took $\beta(\rho) = 2$ for $\rho \leqq 0$. Now take $0 < \tau < (\rho - \kappa)/2$ and let $C_1, C_2, \ldots$ be a sequence of disjoint cubes whose union is all of $\mathbb{R}^d$ and such that diameter $(C_i) \leqq \tau$. Let

$$D_i = \{z : d(z, C_i) \geqq \rho - \tau\}$$
$$E_i = \{z : d(z, C_i) > \rho - 2\tau\}$$
$$I_i = \text{indicator function of } \{g_1' \in C_i\},$$

and last,

$$R = \{z(u) : u \leqq t\}.$$

R is the (random) range of $z(\cdot)$ up until time t, and it follows from (3.18) that if $g_1' \in C_i$, then R must be contained in D_i. Consequently

$$E\{Xh_1(\tau_{g_1}F)\} = \sum_i E\{Xh_1(\tau_{g_1}F)I_i\} = \sum_i E\{Xh_1(\tau_{g_1}F)I_i; R \subset D_i\}. \tag{3.25}$$

Now E_i is an open neighborhood of D_i, and we proceed to show that for any $\mathscr{F}_t$ measurable random variable Y one has

$$YI[R \subset D_i] \text{ is } \mathscr{G}_{E_i} \text{ measurable.} \tag{3.26}$$

To verify (3.26) it suffices to consider only Y's of the form

$$Y = \prod_{j=1}^{l} K_j(z(u_j), w(u_j), F(z(u_j), w(u_j)))$$

with $0 \leqq u_j \leqq t$ and $K_j : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ bounded Borel functions (e.g. by [14], Theorem I.20). Now let $\zeta:\mathbb{R}^d \to [0, 1]$ be a smooth function such that

$$\zeta(z)=\begin{cases} 1 & \text{if} \quad d(z, C_i) \geqq \rho - \dfrac{3\tau}{2} \\ 0 & \text{if} \quad z \notin E_i, \end{cases}$$

and set

$$G^*(t, w(\cdot), z, w) = G(t, w(\cdot), z, w)\zeta(z).$$

Also, let $z^*(\cdot), w^*(\cdot)$ be the solution of

$$z^*(t) = y_0 + \frac{1}{\varepsilon^2}\int_0^t w^*(\sigma)d\sigma$$

$$w^*(t) = v_0 + \frac{1}{\varepsilon}\int_0^t G^*(\sigma, w^*(\cdot), z^*(\sigma), w^*(\sigma))d\sigma \tag{3.27}$$

One can obtain z^*, w^* by the usual iteration procedure, i.e. $z^*, u^* = \lim_{n\to\infty} (z^{(n)}, u^{(n)})$, where $z^{(0)}(t) \equiv y_0$, $w^{(0)}(t) \equiv v_0$ and

$$z^{(n+1)}(t) = y_0 + \frac{1}{\varepsilon^2}\int_0^t w^{(n)}(\sigma)d\sigma,$$

$$w^{(n+1)}(t) = v_0 + \frac{1}{\varepsilon}\int_0^t G^*(\sigma, w^{(n)}(\cdot), z^{(n)}(\sigma), w^{(n)}(\sigma))d\sigma. \tag{3.28}$$

For fixed z, w and $t < \eta$, $G^*(t, w(\cdot), z, w)$ is clearly $\mathscr{G}_{E_i}$ measurable and hence by (3.28) so are $z^{(1)}(t), u^{(1)}(t)$. It then follows by induction on n from (3.28) that $(z^{(n)}(t), w^{(n)}(t))$ and also $z^*(t), u^*(t)$ are $\mathscr{G}_{E_i}$ measurable for all $t < \eta$. This remains valid for $t = \eta$ by continuity and the argument can now be repeated for $\eta \leqq t < 2\eta$ etc. It follows that $(z^*(t), u^*(t))$ is $\mathscr{G}_{E_i}$ measurable for all t. However, it is also clear that $z^*(t), u^*(t)$ coincides with $z(t), u(t)$ for all $t \leqq S$, where $S = \inf\{v \geqq 0: d(z(v), C_i) \geqq \rho - \frac{3}{2}\tau\}$. In particular this holds until the first time z^* leaves D_i and

$$YI[R \subset D_i] = Y^*I[R^* \subset D_i] \tag{3.29}$$

where Y^* and R^* are defined by replacing $z(\cdot), w(\cdot)$ by $z^*(\cdot), w^*(\cdot)$ in the definition of Y and R. Since the right-hand side of (3.29) is $\mathscr{G}_{E_i}$ measurable, this implies (3.26).

Now set

$$F_i = \{z : d(z, C_i) \leqq \kappa\}$$

and $\mathscr{B}_i$ = the collection of Borel sets of $C_i \times \mathbb{R}^d$. Then the map from $C_i \times \mathbb{R}^d \times \Omega$ into $\mathbb{R}$ given by $(\theta, \omega) \to h_1(\tau_\theta F)$ is $\mathscr{B}_i \times \mathscr{G}_{F_i}$ measurable, because for fixed $\theta = (\theta', \theta'') \in C_i \times \mathbb{R}^d$, $\theta' + \Delta_i \subset F_i$ and hence $h_1(\tau_\theta F)$ is $\mathscr{G}_{F_i}$ measurable in ω. Moreover, for fixed ω, $\tau_\theta F$ is continuous in θ. We now combine this with the fact that for any Borel set $B \subset C_i \times \mathbb{R}^d$, $XI_iI[g_1 \in B]I[R \subset D_i]$ is $\mathscr{G}_{E_i}$ measurable (by (3.26)) to conclude that

$$XI_iI[R \subset D_i]h_1(\tau_{g_1}F) \text{ is } \mathscr{G}_{E_i \cup F_i} \text{ measurable.} \tag{3.30}$$

In addition

$$d(E_i, F_i) \geqq d(C_i, E_i) - \kappa \geqq \rho - 2\tau - \kappa,$$

so that

$$|P(AB) - P(A)P(B)| \leqq P(A)\beta(\rho - 2\tau - \kappa) \tag{3.31}$$

whenever $A \in \mathscr{G}_{E_i}$, $B \in \mathscr{G}_{F_i}$. Let Q be the probability measure on $\mathscr{G}_{E_i \cup F_i}$ which is defined by $Q(AB) = P(A)P(B)$ whenever $A \in \mathscr{G}_{E_i}$, $B \in \mathscr{G}_{F_i}$. Q is well defined since $\mathscr{G}_{E_i \cup F_i}$ is generated by such sets AB. We can then rephrase (3.31) as

$$|P(AB) - Q(AB)| \leqq P(A)\beta(\rho - 2\tau - \kappa). \tag{3.32}$$

Now let Z be $\mathscr{G}_{E_i}$ measurable and Γ be $\mathscr{G}_{E_i \cup F_i}$ measurable. Following [8], p. 171 we shall show that (3.32) implies

$$|\int Z\Gamma dP - \int Z\Gamma dQ| \leqq 2\beta(\rho - 2\tau - \kappa) \sup |\Gamma| E\{|Z|\}. \tag{3.33}$$

Indeed it suffices to prove (3.33) if Z and Γ are of the form

$$Z = \Sigma z_k I_{A_k}, \quad \Gamma = \Sigma \gamma_{k,l} I_{A_k \cap B_l}$$

with $A_k \in \mathscr{G}_{E_i}$ and $B_l \in \mathscr{G}_{F_i}$. But for such Z and Γ,

$$\begin{aligned} &|\int Z\Gamma dP - \int Z\Gamma dQ| \\ &\quad \leqq \sum_k |z_k| P(A_k) \sum_l |\gamma_{k,l}| \, |P(B_l | A_k) - P(B_l)| \\ &\quad \leqq E\{|Z|\} \sup |\Gamma| \max_k \sum_l |P(B_l | A_k) - P(B_l)| \\ &\quad \leqq 2\beta(\rho - 2\tau - \kappa) \sup |\Gamma| E\{|Z|\}. \end{aligned}$$

The last inequality is just (20.27) on p. 171 of [8].

We can apply (3.33) to

$$Z\ XI_i I[R \subset D_i], \quad \Gamma = I_i I[R \subset D_i] h_1(\tau_{g_1} F),$$

for which

$$\int Z\Gamma dQ = E\{XI[R \subset D_i] I_i U_1(g_1)\}.$$

As a result (recall (3.20))

$$\begin{aligned} &|E\{XI_i I[R \subset D_i] h_1(\tau_{g_1} F)\} - E\{XI_i I[R \subset D_i] U_1(g_1)\}| \\ &\quad \leqq 2\beta(\rho - 2\tau - \kappa) A E\{|X| I_i\}. \end{aligned}$$

Taking into account (3.25) we obtain after summation over i

$$|E\{Xh_1(\tau_{g_1} F)\} - E\{XU_1(g_1)\}| \leqq 2A\beta(\rho - 2\tau - \kappa) E\{|X|\}.$$

Formula (3.21) follows by taking the limit as $\tau \downarrow 0$. □

We define

$$L(\xi, \sigma) = z(\xi) + \frac{\sigma - \xi}{\varepsilon^2} w(\xi), \tag{3.34}$$

$$r(s, \xi, \sigma) = sz(\sigma) + (1 - s)L(\xi, \sigma). \tag{3.35}$$

Lemma 3. *Let*

$$H(t,z,w)=D^{\beta}G(t,z,w) \quad \text{or} \quad H=D^{\beta}F(z,w)$$

for some $|\beta|\leqq 2$ *and* D^{β} *only involving derivatives with respect to* z *and* w. *Then for each fixed* T, M *there exists a constant* C_4 *such that for all*

$$0\leqq\sigma\leqq T,(\sigma-\varepsilon^{3/2})^{+}\leqq\xi\leqq\sigma,0\leqq s\leqq 1,0<\varepsilon\leqq 1. \tag{3.36}$$

one has

$$E\{\sup_{|w|\leqq 2M}|H(\sigma,r(s,\xi,\sigma),w)|^{8}\}\leqq C_4^8. \tag{3.37}$$

Proof. We only consider an H of the form

$$H(t,z,w)=D^{\beta_1}\Psi(t,w)D^{\beta_2}\Phi(t,z)D^{\beta_3}F(z,w)$$

where D^{β_1} involves only w-derivatives and D^{β_2} only z derivatives. $D^{\beta}G(t,z,w)$ is a finite sum of such terms, and the same estimates can be used if $H=D^{\beta}F(t,w)$.

We shall also restrict ourselves to

$$0<\varepsilon\leqq\eta\wedge(4N)^{-8}\wedge T^{-16d(r(r-8-4d))^{-1}} \tag{3.38}$$

($r=16d+64$ again), since (3.37) is immediate from (II) and (IV) for ε bounded away from zero (and hence $r(s,\xi,\sigma)$ bounded above, on account of (3.11) and (3.13)).

Let

$$K^{A}(z,w)=\begin{cases}D^{\beta_3}F(z,w) & \text{if } |D^{\beta_3}F(z,w)|\leqq A\\ 0 & \text{if } |D^{\beta_3}F(z,w)|>A,\end{cases} \tag{3.39}$$

and

$$\begin{aligned}H^{A}(t,z,w)&=D^{\beta_1}\Psi(t,w)D^{\beta_2}\Phi(t,z)K^{A}(z,w),\\ \tilde{H}^{A}(t,z,w)&=H(t,z,w)-H^{A}(t,z,w).\end{aligned} \tag{3.40}$$

We begin by estimating the error introduced by replacing H by H^A in (3.37). Since $D^{\beta_1}\Psi(t,w)$ and $D^{\beta_2}\Phi(t,z)$ are uniformly bounded in $t\leqq T, w, z\in\mathbb{R}^d$ (cf. (3.10)) and

$$|r(s,\xi,\sigma)|\leqq\frac{2M}{\varepsilon^2}\sigma+|y_0| \text{ (cf. (3.11) and (3.13))},$$

this error is

$$\begin{aligned}&E\{\sup_{|w|\leqq 2M}|\tilde{H}^{A}(\sigma,r(s,\xi,\sigma),w|^{8}\}\\ &\leqq E\{\sup_{|w|\leqq 2M}|D^{\beta_3}F(r(s,\xi,\sigma),w)-K^{A}(r(s,\xi,\sigma),w\}|^{8}\}\\ &\leqq A^{8-r}E\{\sup_{\substack{|z|\leqq 2M\varepsilon^{-2}\sigma+|y_0|\\|w|\leqq 2M}}|D^{\beta_3}F(z,w)|^{r}\}\\ &\leqq C_5A^{8-r}\left(\frac{\sigma}{\varepsilon^2}+1\right)^{d} \text{ (by (IV))}.\end{aligned}$$

Next we shall replace $r(s, \xi, \sigma)$ by

$$x(s, \tau, \xi, \sigma) = sL(\tau, \sigma) + (1-s)L(\xi \wedge \tau, \sigma)$$
$$= s\left[z(\tau) + \frac{\sigma - \tau}{\varepsilon^2} w(\tau)\right] + (1-s)\left[z(\xi \wedge \tau) + \frac{\sigma - \xi \wedge \tau}{\varepsilon^2} w(\xi \wedge \tau)\right]$$

where τ will be chosen later in such a way that

$$(\sigma - \eta)^+ \leqq \tau \leqq \xi \leqq \sigma \quad \text{or} \quad (\sigma - \eta)^+ \leqq \tau \leqq \sigma - \varepsilon^{7/4}. \tag{3.41}$$

We claim that

$$P\{|r(s, \xi, \sigma) - x(s, \tau, \xi, \sigma)| \geqq 1\} \leqq C_6 \varepsilon^{-d-2r} (\sigma - \tau)^{2r} \tag{3.42}$$

Indeed, by (3.9), (3.11) and (3.13),

$$|z(\sigma) - L(\tau, \sigma)| = \left|\frac{1}{\varepsilon^2} \int_\tau^\sigma [w(\lambda) - w(\tau)] d\lambda\right|$$
$$= \left|\frac{1}{\varepsilon^3} \int_\tau^\sigma d\rho G(\rho, z(\rho), w(\rho))\right|$$
$$\leqq \frac{1}{\varepsilon^3} (\sigma - \tau)^2 \sup_{\substack{|z| \leqq 2M\varepsilon^{-2}T + |y_0| \\ |w| \leqq 2M}} |F(z, w)|. \tag{3.43}$$

Therefore, as in (3.15), (3.16)

$$P\{|z(\sigma) - L(\tau, \omega)| \geqq 1\}$$
$$\leqq P\left\{\sup_{\substack{|z| \leqq 2M\varepsilon^{-1}T + |y_0| \\ |w| \leqq 2M}} |F(z, w)| \geqq \varepsilon^3 (\sigma - \tau)^{-2}\right\}$$
$$\leqq C_6 \varepsilon^{-2d-3r} (\sigma - \tau)^{2r}. \tag{3.44}$$

Similarly, for $\tau \leqq \xi$,

$$P\{|L(\xi \wedge \tau, \sigma) - L(\xi, \sigma)| \geqq 1\}$$
$$\leqq P\left\{\left|z(\xi) - z(\tau) - \frac{\xi - \tau}{\varepsilon^2} w(\tau)\right| \geqq \frac{1}{2}\right\} + P\left\{\frac{\sigma - \xi}{\varepsilon^2} |w(\xi) - w(\tau)| \geqq \frac{1}{2}\right\}$$
$$\leqq C_6 \varepsilon^{-2d-3r} (\sigma - \tau)^{2r}. \tag{3.45}$$

For $\tau > \xi$ the first member of (3.45) vanishes. This proves (3.42), since the left-hand side of (3.42) is bounded by the sum of (3.44) and (3.45). From (3.42) we deduce

$$E\{\sup_{|w| \leqq 2M} |H^A(\sigma, r(s, \xi, \sigma), w)|^8\}$$
$$\leqq E\left\{\sup_{|y| \leqq 1, |w| \leqq 2M} |H^A(\sigma, x(s, \tau, \xi, \sigma) + y, w)|^8\right\}$$
$$+ A^8 P\{|r(s, \xi, \sigma) - x(s, \tau, \xi, \sigma)| \geqq 1\}$$
$$\leqq C_7 E\left\{\sup_{|y| \leqq 1} |D^{\beta_2}\Phi(\sigma, x(s, \tau, \xi, \sigma) + y)|^8\right.$$
$$\left.\sup_{|y| \leqq 1, |w| \leqq 2M} |K^A(x(s, \tau, \xi, \sigma) + y, w)|^8\right\}$$
$$+ C_6 A^8 \varepsilon^{-2d-3r} (\sigma - \tau)^{2r}. \tag{3.46}$$

Now notice that $x(s,\tau,\xi,\sigma)$ is $\mathscr{F}_\tau$-measurable. If we further choose k such that $k\eta \leqq \sigma < (k+1)\eta$, then also $\sup_{|y|\leqq 1} D^{\beta_2}|\phi(\sigma, x(s,\tau,\xi,\sigma)+y)|$ is $\mathscr{F}_\tau$-measurable because $((k-1)\eta)^+ \leqq \tau$ by (3.41). We shall now apply Lemma 2 to estimate the first term in the last member of (3.46). We choose $t=\tau$,

$$\tilde{X} = \sup_{|y|\leqq 1} |D^{\beta_2}\phi(\sigma, x(s,t,\xi,\sigma)+y)|^8,$$
$$g_1' = x(s,\tau,\xi,\sigma), \quad g_1''=0, \quad h_1 = \sup_{|y|\leqq 1, |w|\leqq 2M} |K^A(y,w)|^8. \tag{3.47}$$

To apply (3.21) we need a lower bound for

$$\min\{|x(s,\tau,\xi,\sigma)-z(u)| : u \leqq \tau\} = \rho' \wedge \rho'',$$

where

$$\rho' = \min\{|x(s,\tau,\xi,\sigma)-z(u)| : u \leqq (k-1)\eta\},$$
$$\rho'' = \min\{|x(s,\tau,\xi,\sigma)-z(u)| : (k-1)\eta \leqq u \leqq \tau\}.$$

By definition of ϕ,

$$\rho' \geqq \frac{\delta}{\varepsilon^2} - 1 \text{ on the set } \{\tilde{X} \neq 0\},$$

and we merely have to worry about ρ''. Again by (3.9)

$$\varepsilon^2\left[z(\xi\wedge\tau) + \frac{\sigma-\xi\wedge\tau}{\varepsilon^2} w(\xi\wedge\tau) - z(u)\right]$$
$$= \int_u^{\xi\wedge\tau} w(\lambda)d\lambda + (\sigma-\xi\wedge\tau)w(\xi\wedge\tau). \tag{3.48}$$

By (3.41), (3.36) and (3.38), $\xi\wedge\tau \geqq ((k-1)\eta)^+$ so that for $((k-1)\eta)^+ \leqq u \leqq \xi\wedge\tau$, the inner product of (3.48) with $w_{k-1} = |w((k-1)\eta)|^{-1} w((k-1)\eta)(|w(0)|^{-1}w(0)$ if $k=0$) is by virtue of (3.12) at least

$$\int_u^{\xi\wedge\tau} \frac{1}{2N} d\lambda + (\sigma-\xi\wedge\tau)\frac{1}{2N} = (\sigma-u)\frac{1}{2N} \geqq (\sigma-\tau)\frac{1}{2N}.$$

A fortiori, if $\tau \leqq \xi$

$$\inf_{(k-1)\eta \leqq u \leqq \tau} \left| z(\xi\wedge\tau) + \frac{\sigma-\xi\wedge\tau}{\varepsilon^2} w(\xi\wedge\tau) - z(u)\right|$$
$$\geqq \frac{1}{2N\varepsilon^2}(\sigma-\tau). \tag{3.49}$$

The same estimate holds if $(\xi\wedge\tau)$ is replaced by τ. Unfortunately (3.49) does not necessarily hold for $\xi < u$. This can occur for some $(k-1)\eta \leqq \xi < u \leqq \tau$ if $\xi < \tau$. In that case we can only conclude that the inner product of (3.48) with w_{k-1} is no less than

$$(\sigma-u)(w(\xi\wedge\tau), w_{k-1}) - \left| \int_u^{\xi\wedge\tau} |w(\lambda) - w(\xi\wedge\tau)| d\lambda \right|$$
$$\geqq (\sigma-u)\frac{1}{2N} - (\sigma-\xi) \sup_{\xi\leqq\lambda\leqq\tau} |w(\lambda)-w(\xi)|. \tag{3.50}$$

Consider the event (which belongs to $\mathscr{F}_\tau$)

$$\left\{\sup_{\xi \leqq \lambda \leqq \tau} |w(\lambda) - w(\xi)| \leqq (\sigma - \xi)\varepsilon^{-9/8}\right\}, \tag{3.51}$$

and denote its indicator function by J. Then on the set $\{J \neq 0\}$, i.e., when the event (3.51) occurs, the right-hand side of (3.50) is at least

$$(\sigma - \tau)\frac{1}{2N} - (\sigma - \xi)^2\varepsilon^{-9/8} \geqq (\sigma - \tau)\frac{1}{4N}$$

(by (3.36), (3.38), and (3.41)). Together with the above estimates this shows that

$$\rho' \wedge \rho'' \geqq \left(\frac{\delta}{\varepsilon^2} - 1\right) \wedge \frac{1}{4N\varepsilon^2}(\sigma - \tau) \text{ on the set } \{\tilde{X} J \neq 0).$$

We can now apply (3.21) with $X = \tilde{X} J$ and g_1 and h_1 as in (3.47). We then obtain

$$\begin{aligned}
&|E\{\tilde{X} J \sup_{|y| \leqq 1, |w| \leqq 2M} |K^A(x(s, \tau, \xi, \sigma) + y, w)|^8\} \\
&\leqq E\left\{|\tilde{X}|\left[U_1(x(s, \tau, \xi, \sigma), 0) + 2A^8\beta\left(\frac{\delta}{\varepsilon^2} - 3\right) + 2A^8\beta\left(\frac{\sigma - \tau}{4N\varepsilon^2} - 2\right)\right]\right\},
\end{aligned}$$

where

$$E\{|\tilde{X}|\} \leqq (E\{|\tilde{X}|^2\})^{1/2} \leqq \sup_z (E\{|D^{\beta_2}\phi(\sigma, z)|^{16}\})^{1/2} \leqq C_8 < \infty,$$

and (by (IV))

$$\begin{aligned}
U_1(x(s, \tau, \xi, \sigma), 0) &= E\left\{\sup_{|y| \leqq 1, |w| \leqq 2M} |K^A(z + y, w)|^8\right\}_{z = x(s, \tau, \xi, \sigma)} \\
&\leqq E\left\{\sup_{|y| \leqq 1, |w| \leqq 2M} |D^{\beta_3}F(y, w)|^8\right\} \leqq C_9.
\end{aligned}$$

Of course, by Schwarz' inequality we also have

$$\begin{aligned}
&|E\{\tilde{X}(1 - J) \sup_{|y| \leqq 1, |w| \leqq 2M} |K^A(x(s, \tau, \xi, \sigma) + y, w)|^8\} \\
&\leqq (E\{\tilde{X}^2\})^{1/2}(P\{J = 0\})^{1/2}A^8 \\
&\leqq C_8 C_2^{1/2} A^8 \varepsilon^4 \quad \text{(by (3.14) and (3.51))}.
\end{aligned}$$

Combining all these estimates we finally obtain that the first term on the right-hand side of (3.46) is at most

$$C_9 + C_8 A^8\left[\beta\left(\frac{\delta}{\varepsilon^2} - 3\right) + =\left(\frac{\sigma - \tau}{4N\varepsilon^2} - 2\right) + C_2^{1/2}\varepsilon^4\right]. \tag{3.52}$$

The left-hand side of (3.37) is therefore bounded by (3.52) plus

$$C_5 A^{8-r}\left(\frac{\sigma}{\varepsilon^2} + 1\right)^d + C_6 A^8 \varepsilon^{-2d-3r}(\sigma - \tau)^{2r}. \tag{3.53}$$

It remains to choose A and τ so that (3.52) and (3.53) are both bounded. We may assume that

$$\frac{\sigma}{\varepsilon^2} \geqq 16N \tag{3.54}$$

since, if (3.54) fails, then by (3.13)

$$|r(s,\xi,\sigma)| \leqq 2M \cdot 16N + |y_0|,$$

and (3.37) is immediate from (IV). If (3.54) holds we choose

$$A = \left(\frac{\sigma}{\varepsilon^2}\right)^{d(r-8)^{-1}}$$

and

$$\sigma - \tau = \min\left\{\sigma, \varepsilon^{(2d+3r)/2r}\left(\frac{\varepsilon^2}{\sigma}\right)^{8d(2r(r-8))^{-1}}\right\} \tag{3.55}$$

Note that the second expression in (3.55) is at most $\varepsilon \leqq \eta$ (by virtue of (3.38) and (3.54)) so that the requirement $\tau \geqq (\sigma - \eta)^+$ of (3.41) is satisfied by (3.55). If we take into account that $\sigma \leqq T$ we immediately see that under (3.54), (3.52) plus (3.53) is bounded by

$$C_{10}\left\{1 + \left(\frac{\sigma}{\varepsilon^2}\right)^{8d(r-8)^{-1}}\left[\beta\left(\frac{\delta}{\varepsilon^2} - 3\right) + \beta\left(\frac{\sigma - \tau}{4N\varepsilon^2} - 2\right)\right]\right\} \tag{3.56}$$

Now observe that

$$\beta(u) = o(u^{-24}),\ u \to \infty,\ \text{whence}\ \beta(u) \leqq C_{11}(u+1)^{-24} \tag{3.57}$$

by virtue of (2.2) and the fact that $\beta(\cdot)$ is nonincreasing. Therefore (recall $r = 16d + 64$) also

$$\left(\frac{\sigma}{\varepsilon^2}\right)^{8d(r-8)^{-1}}\beta\left(\frac{\delta}{\varepsilon^2} - 3\right) \leqq C_{12}.$$

If the min in (3.55) equals σ, then we take $\tau = 0$ (which always satisfies (3.41) since $\xi \geqq 0$) and the remaining term in (3.56) becomes

$$C_{10}\left(\frac{\sigma}{\varepsilon^2}\right)^{8d(r-8)^{-1}}\beta\left(\frac{\sigma}{4N\varepsilon^2} - 2\right) \leqq C_{13}$$

as above. If the min in (3.55) equals

$$\begin{aligned}\varepsilon^{3/2+d/(r-8)}\sigma^{-8d(2r(r-8))^{-1}} &\geqq \varepsilon^{3/2+d/(r-8)}T^{-8d(2r(\xi-8))^{-1}}\\ &\geqq \varepsilon^{7/4},\end{aligned} \tag{3.58}$$

(use (3.38) for the last inequality) then $\sigma - \tau$ is given by the left-hand side of (3.58). In this case,

$$\left(\frac{\sigma}{\varepsilon^2}\right)^{8d(r-8)^{-1}}\beta\left(\frac{\sigma - \tau_0}{4N\varepsilon^2} - 2\right) \leqq T^{8d(r-8)^{-1}}\varepsilon^{-16d(r-8)^{-1}}\beta\left(\frac{1}{4N\varepsilon^{1/4}} - 2\right)$$

which (again by (3.57)) is uniformly bounded in $\varepsilon \in (0,1]$. The proof is complete.

Corollary 1. *For* $0 \leqq \tau \leqq \sigma \leqq T$, $p \leqq 8$, $|\beta| \leqq 2$,

$$E|D^\beta G(\sigma, z(\sigma), w(\sigma))|^p \leqq C_4^p \tag{3.59}$$

and

$$\left(E\left\{\left|z(\sigma) - z(\tau) - \frac{\sigma - \tau}{\varepsilon^2} w(\tau)\right|^p\right\}\right)^{1/p} \leqq C_4 \frac{(\sigma - \tau)^2}{\varepsilon^3}. \tag{3.60}$$

Proof. Formula (3.59) is immediate from (3.37) with $s = 1$, $\xi = \sigma$ (so that $r(s, \xi, \sigma) = z(\sigma)$), and (3.11) and Jensen's inequality (3.60) now follows from Holder's inequality and

$$z(\sigma) - z(\tau) - \frac{\sigma - \tau}{\varepsilon^2} w(\tau) = \frac{1}{\varepsilon^3} \int_\tau^\sigma d\lambda \int_\tau^\lambda d\rho G(\rho, z(\rho), w(\rho))$$

(see (3.9)).

Lemma 4. *Let*

$$\begin{aligned}
&0 \leqq \sigma_1, \sigma_2 \leqq T, \quad k\eta \leqq \sigma_1 \leqq \sigma_2 < (k+1)\eta, \\
&0 \leqq \lambda_1, \lambda_2, \lambda_3 \leqq \sigma_1, \quad (\sigma_2 - \varepsilon^{3/2})^+ \leqq \xi_2 \leqq \sigma_2, \\
&(\sigma_1 - \varepsilon^{25/16})^+ \leqq \xi_1, \xi_2 \leqq \sigma_1
\end{aligned} \tag{3.61}$$

(*notice* σ_1 *in both sides of the last inequality*) *and set*

$$v = \max\{k\eta, \lambda_1, \lambda_2, \lambda_3, \xi_1, \xi_2\}$$

Let ζ *be an* $\mathscr{F}_{\lambda_3}$ *measurable random variable with* $E\zeta^2 < \infty$ *and*

$$H_1(t, z, w) = D^{\beta_1}\Psi(t, w) D^{\beta_2}\Phi(t, z) D^{\beta_3}F(z, w),$$

and

$$H_2(t, z, w) = D^{\beta_4}\Psi(t, w) D^{\beta_5}\Phi(t, z) D^{\beta_6}F(z, w),$$

where all derivatives are with respect to w or z. Set

$$V(z_1, w_1, z_2, w_2) = E\{D^{\beta_3}F(z_1, w_1) D^{\beta_6}F(z_2, w_2)\}.$$

Then (*see* (3.34) *for* $L(\cdot,\cdot)$)

$$\begin{aligned}
&|E\{\zeta H_1(\sigma_1, L(\xi_1, \sigma_1), w(\lambda_1)) H_2(\sigma_2, L(\xi_2, \sigma_2), w(\lambda_2))\} \\
&- E\{\zeta D^{\beta_1}\Psi(\sigma_1, w(\lambda_1)) D^{\beta_4}\psi(\sigma_2, w(\lambda_2)) D^{\beta_2}\Phi(\sigma_1, L(\xi_1, \sigma_1)) \\
&\cdot D^{\beta_5}\Phi(\sigma_2, L(\xi_2, \sigma_2)) V(L(\xi_1, \sigma_1), w(\lambda_1), L(\xi_2, \sigma_2), w(\lambda_2))\}| \\
&\quad \leqq C_{14}(E\{\zeta^2\})^{1/2}\left[\beta^{1/3}\left(\frac{\delta}{\varepsilon^2}\right) + \beta^{1/3}\left(\frac{\sigma_1 - v}{2N\varepsilon^2} - 1\right) + \varepsilon^2\right]
\end{aligned} \tag{3.62}$$

Also, with

$$\begin{aligned}
&U(z_1, w_1) = E\{D^{\beta_3}F(z_1, w_1)\}, \quad \pi = \max\{k\eta, \lambda_1, \lambda_3, \xi_1\} \\
&|E\{\zeta H_1(\sigma_1, L(\xi_1, \sigma_1), w(\lambda_1))\} \\
&- E\{\zeta D^{\beta_1}\Psi(\sigma_1, w(\lambda_1)) D^{\beta_2}\Phi(\sigma_1, L(\xi_1, \sigma_1)) U(L(\xi_1, \sigma_1), w(\lambda_1))\}|
\end{aligned}$$

$$\leqq C_{14}(E\{\zeta^2\})^{1/2}\left[\beta^{1/2}\left(\frac{\delta}{\varepsilon^2}\right)+\beta^{1/2}\left(\frac{\sigma_1-\pi}{2N\varepsilon^2}-1\right)+\varepsilon^2\right] \tag{3.63}$$

C_{14} is independent of ζ, σ_1, ξ_j, λ_l and ε.

Proof. We only prove (3.62). Define K_i^A, H_i^A and $\tilde{H}_i^A$ as (3.39), (3.40) and set

$$V^A(z_1, w_1, z_2, w_2) = E\{K_1^A(z_1, w_1)K_2^A(z_2, w_2)\}.$$

We first replace H_i by H_i^A and V by V^A in the left-hand side of (3.62). The error introduced by the replacement is at most

$$\begin{aligned}&E\{|\zeta\tilde{H}_1^A(\sigma_1, L(\xi_1, \sigma_1), w(\lambda_1))H_2(\sigma_2, L(\xi_2, \sigma_2), w(\lambda_2))|\}\\&+E\{|\zeta H_1^A(\sigma_1, L(\xi_1, \sigma_1), w(\lambda_1))\tilde{H}_2^A(\sigma_2, L(\xi_2, \sigma_2), w(\lambda_2))|\}\\&+C_1^4E\{|\zeta||V(L(\xi_1, \sigma_1), w(\lambda_1), L(\xi_2, \sigma_2), w(\lambda_2))\\&\qquad-V^A(L(\xi_1, \sigma_1), w(\lambda_1), L(\xi_2, \sigma_2), w(\lambda_2))|\}\end{aligned} \tag{3.64}$$

We estimate the first term of (3.64); it is easily seen that the same bound applies to the other terms. By using (3.10) and (3.37) twice we see that the first term in (3.64) is at most

$$\begin{aligned}&C_1^2E^{1/2}\{\zeta^2\}E^{1/4}\{|H_2(\sigma_2, L(\xi_2, \sigma_2), w(\lambda_2)|^4\}.\\&E^{1/4}\{|D^{\beta_3}F(L(\xi_1, \sigma_1), w(\lambda_1)|^4I[|D^{\beta_3}F(L(\xi_1, \sigma_1), w(\lambda_1))|>A]\}\\&\quad\leqq C_1^2C_4E^{1/2}\{\zeta^2\}A^{-1}E^{1/4}\{|D^{\beta_3}F(L(\xi_1, \sigma_1), w(\lambda_1)|^8\}\\&\quad\leqq C_1^2C_4^3A^{-1}E^{1/2}\{\zeta^2\}.\end{aligned}$$

From here on the proof is very similar to Lemma 3. We shall apply (3.22). We make the following choices:

$$\begin{aligned}&\tilde{D}=\zeta D^{\beta_1}\Psi(\sigma_1, w(\lambda_1))D^{\beta_4}\Psi(\sigma_2, w(\lambda_2))\\&D^{\beta_2}\Phi(\sigma_1, L(\xi_1, \sigma_1))D^{\beta_5}\Phi(\sigma_2, L(\xi_2, \sigma_2)),\\&g_i=(L(\xi_i, \sigma_i), w(\lambda_i)),\\&h_1=D^{\beta_3}F(0,0)I[|D^{\beta_3}F(0,0)|\leqq A],\quad h_2=D^{\beta_6}F(0,0)I[|D^{\beta_6}F(0,0)|\leqq A].\end{aligned} \tag{3.65}$$

Also we define

$$\begin{aligned}&\rho_i'=\min\{|L(\xi_i, \sigma_i)-z(u)|:0\leqq u\leqq(k-1)\eta\},\\&\rho_i''=\min\{|L(\xi_i, \sigma_i)-z(u)|:(k-1)\eta\leqq u\leqq v\},\\&J_i=\text{indicator function of}\left\{\sup_{\xi_i\leqq\lambda\leqq v}|w(\lambda)-w(\xi_i)|\leqq(v-\xi_i)\varepsilon^{-9/8}\right\}.\end{aligned}$$

One easily checks that $\tilde{X}$ is $\mathscr{F}_v$-measurable. Indeed $\Psi(\sigma, w(\lambda_i))$ depends only on $w(\lambda_i)$ and $w((k-1)\eta)$ and $w(k\eta)$, and for the Φ factors we already checked the appropriate measurability just before (3.47). As in Lemma 3

$$\rho_1'\geqq\frac{\delta}{\varepsilon^2}\text{ on the set }\{D^{\beta_2}\Phi(\sigma_1, L(\xi_1, \sigma_1))\neq 0\},$$

and similarly for ρ_2'. Also as in Lemma 3 (cf. (3.48)–(3.50)), on the set $\{J_i \neq 0\}$,

$$\begin{aligned}\varepsilon^2 \rho_i'' &\geqq (\sigma_i - v)\frac{1}{2N} - (v - \xi_i) \sup_{\xi_i \leqq \lambda \leqq v} |w(\lambda) - w(\xi_i)| \\ &\geqq f(\sigma_i - v)\frac{1}{2N} - (\sigma_1 - \xi_i)^2 \varepsilon^{-9/8} \\ &\geqq (\sigma_i - v)\frac{1}{2N} - \varepsilon^2 \quad \text{(by (3.61))}.\end{aligned}$$

Thus

$$\begin{aligned}&\min_{i=1,2} \min_{u \leqq v} \{|L(\xi_i, \sigma_i) - z(u)|\} \\ &= \rho_1' \wedge \rho_2' \wedge \rho_1'' \wedge \rho_2'' \geqq \frac{\delta}{\varepsilon^2} \wedge \left(\frac{1}{2N\varepsilon^2}(\sigma_1 - v) - 1\right)\end{aligned} \tag{3.66}$$

on the set $\{X \neq 0\}$, where

$$X = \tilde{X} J_1 J_2.$$

Moreover, by (3.14)

$$P\{J_i = 0\} \leqq C_2 \varepsilon^8, \quad i = 1, 2.$$

Now the left-hand side of (3.62) with H_i and V replaced by H_i^A, respectively, V^A, equals

$$\begin{aligned}&|E\{\tilde{X} h_1(\tau_{g_1} F) h_2(\tau_{g_2} F)\} - E\{\tilde{X} V^A(g_1, g_2)\}| \\ &\quad \leqq |E\{\tilde{X} h_1(\tau_{g_1} F) h_2(\tau_{g_2} F)\} - E\{X V^A(g_1, g_2)\}| \\ &\qquad + E\{|\tilde{X}|[(1 - J_1) + (1 - J_2)]|h_1(\tau_{g_1} F) h_2(\tau_{g_2} F)|\} \\ &\qquad + E\{|\tilde{X}|[(1 - J_1) + (1 - J_2)]|V^A(g_1, g_2)|\}\end{aligned} \tag{3.67}$$

The first term on the right-hand side of (3.67) is at most

$$\begin{aligned}&2A^2 E\{|X|\}\left[\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_1 - v}{2N\varepsilon^2} - 1\right)\right] \\ &\quad \leqq 2C_1^4 A^2 E^{1/2}\{\zeta^2\}\left[\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_1 - v}{2N\varepsilon^2} - 1\right)\right]\end{aligned}$$

on account of (3.22), (3.66) and (3.10).

As for the second term in the right-hand side of (3.67) notice that

$$\begin{aligned}&E\{|\tilde{X}|(1 - J_i)|h_1(\tau_{g_1} F) h_2(\tau_{g_2} F)| \\ &\quad = E\{|\zeta|(1 - J_i)|H_1^A(\sigma_1, L(\xi_1, \sigma_1), w(\lambda_1)) H_2^A(\sigma_2, L(\xi_2, \sigma_2), w(\lambda_2))|\}. \\ &\quad \leqq E^{1/2}\{|\zeta|^2\} E^{1/4}\{(1 - J_i)^4\} \\ &\quad \cdot (E\{|H_1(\sigma_1, L(\xi_1, \sigma_1), w(\lambda_1))|^8\} E\{|H_2(\sigma_2, L(\xi_2, \sigma_2), w(\lambda_2))|^8\})^{1/8}\end{aligned} \tag{3.68}$$

By (3.37),

$$E^{1/8}\{|H_i(\sigma_i, L(\xi_i, \sigma_i), w(\lambda_i))|^8\} \leqq C_4$$

so that (3.68) is at most

$$C_4^2 E^{1/2}\{(\zeta|^2\}(P\{J_i = 0\})^{1/4} \leqq C_4^2 C_2^{1/4} E^{1/2}\{|\zeta|^2\}\varepsilon^2. \tag{3.69}$$

By (IV), $|V^A(z_1, w_1, z_2, w_2)| \leqq C_0$ and hence by (3.10) the third term on the right-hand side of (3.67) is bounded by

$$2C_0 C_1^4 E^{1/2}\{\zeta^2\} C_2^{1/2}\varepsilon^4.$$

Collecting all contribution we find that the left-hand side of (3.62) is bounded by

$$C_{15}E^{1/2}\{\zeta^2\}\left[A^{-1} + A^2\left\{\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_1 - v}{2N\varepsilon^2} - 1\right)\right\} + \varepsilon^2\right].$$

Formula (3.62) now follows if we take

$$A = \left\{\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_1 - v}{2N\varepsilon^2} - 1\right)\right\}^{-1/3}. \qquad \square$$

Remark 4. For the sequel we set

$$\gamma(u) = \left\{\beta\left(\frac{u}{2N} - 1\right)\right\}^{1/3} \tag{3.70}$$

We then have (see (3.57))

$$\gamma(u) = o(u^{-8}),\ \gamma(u) \leqq C_{11}^{1/3}\left(\frac{u}{2N} + 1\right)^{-8} \text{ and } \int_0^\infty u\gamma(u)du < \infty, \tag{3.71}$$

and from (3.61) it follows that we may replace the right-hand sides of (3.62) and (3.63) by

$$C_{16}E^{1/2}\{\zeta^2\}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_1 - v}{\varepsilon^2}\right)\right\} \text{ respectively } C_{16}E^{1/2}\{\zeta^2\}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_1 - \pi}{\varepsilon^2}\right)\right\}. \tag{3.72}$$

Lemma 5. *If the hypotheses of Lemma* 4 *hold, as well as* $\xi_1 = \xi_2 = \xi$ *and* $E\{F(z,w)\} \equiv 0$, *then the left-hand side of* (3.62) *is also bounded by*

$$C_{17}E^{1/2}\{\zeta^2\}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_1 - v}{\varepsilon^2}\right)\right\}^{1/2}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_2 - \sigma_1}{\varepsilon^2}\right)\right\}^{1/2} \tag{3.73}$$

Proof. We shall show that the left-hand side of (3.62) is also bounded by

$$C_{18}E^{1/2}\{\zeta^2\}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_2 - \sigma_1}{\varepsilon^2}\right)\right\}. \tag{3.74}$$

Formula (3.73) then follows by taking the geometric mean of the bounds (3.71) and (3.74). Formula (3.74) is proved in the same way as (3.63). We choose $\tilde{X}, g_i, h_i$ and J_i as in Lemma 4 (see (3.65) and the preceding lines) and again we take $X =$

$\bar{X}J_1J_2$. This time we begin the estimate of (3.62) with the term

$$\begin{aligned}
&|E\{\zeta H_1^A(\sigma_1 L(\xi_1,\sigma_1), w(\lambda_1)(H_2^A(\sigma_2, L(\xi_2,\sigma_2), w(\lambda_2))\}| \\
&\quad = |E\{\tilde{X}h_1(\tau_{g_1}F)h_2(\tau_{g_2}F)\}| \\
&\quad \leqq |E\{Xh_1(\tau_{g_1}F)h_2(\tau_{g_2}F)\}| \\
&\qquad + E\{|\tilde{X}|[(1-J_1)+(1+J_2)]|h_1(\tau_{g_1}F)h_2(\tau_{g_2}F)|\} \qquad (3.75)
\end{aligned}$$

In the present situation

$$\begin{aligned}
U_2(\theta) &= E\{h_2(\tau_\theta F)\} = E\{D^{\beta_6}F(\theta)I[|D^{\beta_6}F(0)| \leqq A]\} \\
&= E\{D^{\beta_6}F(\theta)I[|D^{\beta_6}F(\theta)| > A]\},
\end{aligned}$$

because (2.1) implies

$$E\{D^{\beta_6}F(0)\} = 0. \qquad (3.76)$$

Thus

$$|U_2(\theta)| \leqq C_0 A^{-1}.$$

In addition, by (3.12)

$$\begin{aligned}
|g_2' - g_1'| &= |L(\xi,\sigma_2) - L(\xi,\sigma_1)| = \frac{1}{\varepsilon^2}|\sigma_2-\sigma_1||w(\xi)| \\
&\geqq \frac{1}{2N\varepsilon^2}(\sigma_2-\sigma_1). \qquad (3.77)
\end{aligned}$$

Therefore, by (3.24) and the lower bound for $\rho_2' \wedge \rho_2''$ in Lemma 4,

$$\begin{aligned}
&|E\{Xh_1(\tau_{g_1}F)h_2(\tau_{g_2}F)\}| \\
&\quad \leqq |E\{Xh_1(\tau_{g_1}F)U_2(g_2)\}| + 2A^2E\{|X|\}\left\{\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_2-\sigma_1}{2N\varepsilon^2} - 1\right)\right\} \\
&\quad \leqq C_{19}E^{1/2}\{\zeta^2\}\left[A^{-1} + A^2\left\{\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_2-\sigma_1}{2N\varepsilon^2} - 1\right)\right\}\right]
\end{aligned}$$

The second term on the right-hand side of (3.75) is at most

$$2C_4^2C_2^{1/4}E^{1/2}\{\zeta^2\}\varepsilon^2,$$

as in (3.68), (3.69). We combine this with the estimate for (3.64), and as before we take

$$A = \left\{\beta\left(\frac{\delta}{\varepsilon^2}\right) + \beta\left(\frac{\sigma_2-\sigma_1}{2N\varepsilon^2} - 1\right)\right\}^{-1/3}.$$

There results

$$\begin{aligned}
&|E\{\zeta H_1(\sigma_1, L(\xi,\sigma_1), w(\lambda_1))H_2(\sigma_2, L(\xi,\sigma_2), u(\lambda_2))\}| \\
&\quad \leqq C_{20}E^{1/2}\{\zeta^2\}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_2-\sigma_1}{\varepsilon^2}\right)\right\}. \qquad (3.78)
\end{aligned}$$

Lastly, it follows from (3.76), (IV) and Theorem 17.2.3 of [7] or Lemma 20.1 of [8] that

$$|V(z_1, w_1, z_2, w_2)| = |E\{D^{\beta_3}F(z_1, w_1)D^{\beta_6}F(z_2, w_2)\}$$
$$\leqq 2C_0^2\beta^{1/2}(|z_1 - z_2|). \tag{3.79}$$

For $z_i = g_i' = L(\xi, \sigma_i)$ we obtain from this and (3.77)

$$|E\{\zeta V(L(\xi, \sigma_1), w(\lambda_1), L(\xi, \sigma_2), w(\lambda_2))\}|$$
$$\leqq 2C_0^2\beta^{1/2}(|z_1 - z_2|)E\{|\zeta|\}$$
$$\leqq 2C_0^2E^{1/2}\{\zeta^2\}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma_2 - \sigma_1}{\varepsilon^2}\right)\right\}. \tag{3.80}$$

This, together with (3.78) implies (3.74). □

Step (iii)

In this step we prove that the family of measures $\{R^\varepsilon(\cdot):0<\varepsilon\leqq 1\} = \{R^\varepsilon(\cdot\,;M,N,\eta,\delta):0<\varepsilon\leqq 1\}$ introduced in step (i) is tight in D.

As pointed out in step (iii) of [6] it certainly suffices for this to prove

$$E\{\zeta|w(u) - w(t)|^2\} \leqq C_{21}(u - t)E^{1/4}\{\zeta^4\} \tag{3.81}$$

For ζ $\mathscr{F}_t$ measurable and $0 \leqq t \leqq u \leqq T$. To prove (3.81) it suffices to restrict oneself to

$$k\eta \leqq t \leqq u < (k+1)\eta \tag{3.82}$$

for some $k \leqq T/\eta$. For once (3.81) has been proved for such t, u, then it also holds by continuity for $k\eta \leqq t \leqq u \leqq (k+1)\eta$, whereas for $(k+1)\eta \leqq u < (k+2)\eta$, (3.81) (with k replaced by $(k+1)$) gives

$$E(\zeta|w(u) - w(k+1)|^2\} \leqq C_{21}(u - (k+1)\eta)E^{1/4}\{\zeta^4\}$$

so that (3.81) under (3.82) for each $k \leqq T/\eta$ implies that also for $k\eta \leqq t \leqq (k+1)\eta \leqq u < (k+2)\eta$,

$$E\{\zeta|w(u) - w(t)|^2\} \leqq 2E\{\zeta|w((k+1)\eta) - w(t)|^2\}$$
$$+ 2E\{\zeta|w(u) - w((k+1)\eta)|^2\} \leqq 2C_{21}(u - t)E^{1/4}\{\zeta^4\}.$$

For $t \leqq (k+1)\eta$ and $u \geqq (k+2)\eta$ we would have $u - t \geqq \eta$ and then (3.81) is trivial with $C_{21} = 4M\eta^{-1}$ since $|w(u) - w(t)| \leqq 4M$ by (3.11).

The proof of (3.81) under (3.82) is very similar to step (iii) of [6]. We shall use the summation convention and write

$$D_{2j}G(\sigma, z(\sigma), w(\sigma)) \text{ for } \frac{\partial}{\partial z_j}G(\sigma, z(\sigma), u(\sigma))$$

and similarly for D_{3j}. We take

$$\xi = \xi(\sigma) = \max\{t, \sigma - \varepsilon^{7/4}\}. \tag{3.83}$$

Then, by (3.9)

$$\begin{aligned}|w(u)-w(t)|^2 &= \frac{2}{\varepsilon}\int_t^u G_i(\sigma, z(\sigma), w(\sigma))(w_i(\sigma)-w_i(t))d\sigma \\ &= \frac{2}{\varepsilon}\int_t^u G_i(\sigma, z(\sigma), w(\xi))(w_i(\xi)-w_i(t))d\sigma \\ &\quad + \frac{2}{\varepsilon^2}\int_t^u d\sigma \int_\xi^\sigma [D_{3j}G_i(\sigma, z(\sigma), w(\lambda))(w_i(\lambda)-w_i(t)) \\ &\quad + \delta_{i,j}G_i(\sigma, z(\sigma), w(\lambda))]G_j(\lambda, z(\lambda), w(\lambda))d\lambda \\ &= I_1 + I_2.\end{aligned}$$

In I_1 we replace $z(\sigma)$ by $L(\xi,\sigma)$. More precisely, we write (in the notation of (3.34), (3.35))

$$\begin{aligned}I_1 &= \frac{2}{\varepsilon}\int_t^u G_i(\sigma, L(\xi,\sigma), w(\xi))(w_i(\xi)-w_i(t))d\sigma \\ &\quad + \frac{2}{\varepsilon}\int_t^u d\sigma \int_0^1 D_{2j}G_i(\sigma, r(s,\xi,\sigma), w(\xi))(w_i(\xi)-w_i(t)) \\ &\quad \cdot [z_j(\sigma)-L_j(\xi,\sigma)]ds \\ &= J_1 + J_2.\end{aligned}$$

By (3.9) again

$$\begin{aligned}J_1 &= \frac{2}{\varepsilon^2}\int_t^u d\sigma \int_t^\xi d\lambda [D_{3j}G_i(\sigma, L(\xi,\sigma), w(\lambda))(w_i(\lambda)-w_i(t)) \\ &\quad + \delta_{ij}G_i(\sigma, L(\xi,\sigma), w(\lambda))]G_j(\lambda, z(\lambda), w(\lambda)).\end{aligned}$$

Moreover, by (3.63) and Remark 4 with ζ replaced by

$$\zeta(w_i(\lambda)-w_i(t))G_j(\lambda, z(\lambda), w(\lambda))$$

we have for fixed i and j and $\lambda \leqq \xi$,

$$\begin{aligned}&|E\{\zeta D_{3j}G_i(\sigma, L(\xi,\sigma), w(\lambda))(w_i(\lambda)-w_i(t))G_j(\lambda, z(\lambda), w(\lambda))\}| \\ &\quad \leqq C_{16}E^{1/2}\{|\zeta(w_i(\lambda)-w_i(t))G_j(\lambda, z(\lambda), w(\lambda))|^2\} \\ &\quad \cdot\left\{\varepsilon^2+\gamma\left(\frac{\sigma-\xi}{\varepsilon^2}\right)\right\} \leqq C_{16}C_4E^{1/4}\{\zeta^4\}\left\{\varepsilon^2+\gamma\left(\frac{\sigma-\xi}{\varepsilon^2}\right)\right\}\end{aligned}$$

(recall that by (3.82) and (3.83) $\xi \geqq t \geqq k\eta$ and that (3.76) holds; in the last step we also used (3.59)). The other term in J_1 is handled similarly, so that

$$\begin{aligned}|E\{\zeta J_1\}| &\leqq \frac{C_{22}}{\varepsilon^2}E^{1/4}\{\zeta^4\}\int_t^u d\sigma \int_t^\xi d\lambda \left\{\varepsilon^2+\gamma\left(\frac{\sigma-\xi}{\varepsilon^2}\right)\right\} \\ &\leqq C_{22}E^{1/4}\{\zeta^4\}(u-t)\left\{T+\int_0^\infty \gamma(t)dt + T\varepsilon^{-2}\gamma(\varepsilon^{-1/4})\right\} \\ &\leqq C_{23}E^{1/4}\{\zeta^4\}(u-t) \quad \text{(see (3.83) and (3.71)).}\end{aligned}$$

Next we replace $r(s,\xi,\sigma)$ in J_2 also by $L(\xi,\sigma)$. Note that for $0 \leqq v \leqq 1$,

$$vr(s,\xi,\sigma)+(1-v)L(\xi,\sigma) = r(vs,\xi,\sigma)$$

(see (3.34), (3.35)), so that

$$J_2 = \frac{2}{\varepsilon}\int_t^u d\sigma \int_0^1 D_{2j}G_i(\sigma, L(\xi,\sigma), w(\xi))(w_i(\xi) - w_i(t))$$
$$\cdot[z_j(\sigma) - L_j(\xi,\sigma)]ds$$
$$+\frac{2}{\varepsilon}\int_t^u d\sigma \int_0^1 ds \int_0^1 dv D_{2l}D_{2j}G_i(\sigma, r(vs,\xi,\sigma), w(\xi))(w_i(\xi) - w_i(t))$$
$$\cdot[z_j(\sigma) - L_j(\xi,\sigma)]s[z_l(\sigma) - L_l(\xi,\sigma)]$$
$$= K_1 + K_2.$$

As in (3.43)

$$\zeta K_1 = \frac{2}{\varepsilon^4}\int_t^u d\sigma D_{2j}G_i(\sigma, L(\xi,\sigma), w(\xi))(w_i(\xi) - w_i(t))$$
$$\cdot\int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho G_j(\rho, z(\rho), w(\rho))\zeta.$$

Again by (3.63) and Remark 4, this time with ζ replaced by

$$\zeta G_j(\rho, z(\rho), w(\rho))(w_i(\xi) - w_i(t))$$

we find

$$|E\{\zeta K_1\}| \leqq \frac{2C_{16}}{\varepsilon^4} E^{1/4}\{\zeta^4\}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \left\{\varepsilon^2 + \gamma\left(\frac{\sigma-\rho}{p2}\right)\right\}$$
$$= 2C_{16}E^{1/4}\{\zeta^4\}\int_t^u d\sigma \int_\xi^\sigma d\rho \varepsilon^{-4}(\sigma-\rho)\left\{\varepsilon^2 + \gamma\left(\frac{\sigma-\rho}{\varepsilon^2}\right)\right\}$$
$$\leqq C_{23}E^{1/4}\{\zeta^4\}(u-t) \quad \text{(again use (3.83) and (3.71))}.$$

Finally ζK_2 can be estimated directly by (3.11), Holder's inequality, (3.60), Lemma 3 and (3.83),

$$|E\{\zeta K_2\}| \leqq \frac{4M}{\varepsilon} E^{1/4}\{\zeta^4\}\int_t^u d\sigma \int_0^1 ds \int_0^1 dv E^{1/2}\{|z(\sigma) - L(\xi,\sigma)|^4\}$$
$$\cdot E^{1/4}\{|D_{2l}D_{2j}G_i(\sigma, r(vs,\xi,\sigma), w(\xi))|^4\}$$
$$\leqq 4MC_4E^{1/4}\{\zeta^4\}\int_t^u d\sigma \frac{(\sigma-\xi)^4}{\varepsilon^7} \leqq C_{23}E^{1/4}\{\zeta^4\}(u-t).$$

Thus all pieces of ζI_1 are of the order claimed in (3.81). ζI_2 is handled in the same way. For brevity denote the expression in square brackets in I_2 by $K_j(\sigma, z(\sigma), w(\lambda))$. Then

$$\zeta I_2 = \frac{2}{\varepsilon^2}\int_t^u d\sigma \int_\xi^\sigma d\lambda \zeta K_j(\sigma, L(\xi,\sigma), w(\lambda))G_j(\lambda, z(\lambda), w(\lambda))$$
$$+\frac{2}{\varepsilon^2}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_0^1 ds \zeta D_{2,l}K_j(\sigma, r(s,\xi,\sigma), w(\lambda))$$
$$\cdot G_j(\lambda, z(\lambda), w(\lambda))[z_l(\sigma) - L_l(\xi,\sigma)].$$

The first part of ζI_2 can again be estimated by (3.63) (compare ζJ_1) whereas the second integral is at most

$$C_{24}\frac{1}{\varepsilon^2}E^{1/4}\{\zeta^4\}\int_t^u d\sigma\int_\xi^\sigma d\lambda\frac{(\sigma-\xi)^2}{\varepsilon^3}$$
$$\leqq C_{24}\varepsilon^{1/4}E^{1/4}\{\zeta^4\}(u-t),$$

on account of (3.37), (3.60) and (3.83).

This completes the proof of (3.61) and hence the tightness of $\{R^\varepsilon:0<\varepsilon\leqq 1\}$.

Step (iv)

It follows from step (iii) that any sequence $\varepsilon_n\downarrow 0$ can be refined such that

$$R^{\varepsilon_n}\left(\cdot\,;M,N,\frac{1}{p},\frac{1}{q}\right)\Rightarrow R\left(\cdot\,;M,N,\frac{1}{p},\frac{1}{q}\right)\text{ on }D \tag{3.84}$$

as $n\to\infty$ for all quadruples of integers $M,N,p,q\geqq 1$, with $R(\cdot\,;M,N,1/p,1/q)$ some probability measure on D. We denote by $X(t)$ the t-coordinate function on D, and the corresponding σ-fields of subsets of D are given by

$$\mathcal{M}_v^u=\sigma\text{-field generated by }\{X_t:u\leqq t\leqq v\}.$$

The first step towards proving the convergence of Q^ε will be to show that

$$f(X_t)-\int_0^t(L^{M,N,p,q}f)(\sigma,X)d\sigma \tag{3.85}$$

is an $(R(\cdot\,;M,N,1/p,1/q),\mathcal{M}_t^0)$ martingale for any C^∞ function $f:\mathbb{R}^d\to\mathbb{R}$ with compact support and the following definition of $L^{M,N,p,q}$: Take $\eta=1/p$ and $\delta=1/q$ and let

$$\Psi_*(t,X)=\Psi_*^{M,N,p}(t,X)=\Psi(t,X,X(t))$$

where Ψ is as in (3.3) for given M,N and $\eta=1/p$. Also take

$$\Phi_*(t,X)=\Phi_*^{p,q}(t,X)=\phi_k\left(X,\int_0^t X(u)du\right)$$

for $k/p\leqq t<(k+1)/p$, where ϕ_k is as in (3.5) for $\eta=1/p$, $\delta=1/q$. One easily checks (compare the observation just before Lemma 1)

$$\Phi_*(t,w(\cdot))=\Phi(t,w(\cdot),z(t)). \tag{3.86}$$

Further we define for $v\in\mathbb{R}^d$ the following coefficients:

$$a_{ij}(v)=\int_{-\infty}^{+\infty}E\{F_i(0,v)F_j(tv,v)\}dt,$$

$$c_{ij}(\sigma,X,v)=c_{ij}^{M,N,p}(\sigma,X,v)$$
$$=\Psi(\sigma,X,v)\int_0^\infty tE\left\{F_j(0,v)\frac{\partial}{\partial x_j}F_i(tv,v)\right\}dt$$

$\left(\text{of course } \frac{\partial}{\partial x_j} F(tv, v) = \left[\frac{\partial}{\partial x_j} F(x, v)\right]_{x=tv} \text{ here}\right),$

$$d_{ij}(\sigma, X, v) = d_{ij}^{M,N,p}(\sigma, X, v)$$
$$= \left(\frac{\partial}{\partial v_j} \Psi(\sigma, X, v)\right) \int_0^\infty E\{F_j(0, v) F_i(tv, v)\} dt,$$

$$e_{ij}(\sigma, X, v) = e_{ij}^{M,N,p}(\sigma, X, v)$$
$$= \Psi(\sigma, X, v) \int_0^\infty E\left\{F_j(0, v) \frac{\partial}{\partial v_j} F_i(tv, v)\right\} dt,$$

$\left(\frac{\partial}{\partial v_j} F_i(tv, v) \text{ has the same meaning as in (2.4)}\right),$

$$b_i(\sigma, X, v) = b_i^{M,N,p}(\sigma, X, v)$$
$$= \sum_{j=1}^{d} \{c_{ij}(\sigma, X, v) + d_{ij}(\sigma, X, v) + e_{ij}(\sigma, X, v)\}.$$

Finally,

$$(L^{M,N,p,q} f)(\sigma, X) = \Psi_*(\sigma, X) \Phi_*^2(\sigma, X) \left\{\frac{1}{2} \sum_{i,j} \Psi_*(\sigma, X) a_{ij}(X(\sigma)) \frac{\partial^2}{\partial v_i \partial v_j} f(X(\sigma)) + \sum_i b_i^{M,N,p}(\sigma, X, X(\sigma)) \frac{\partial}{\partial v_i} f(X(\sigma))\right\}. \quad (3.87)$$

As shown in step (v) of [6], to prove that (3.85) is an $(R(\cdot; M, N, 1/p, 1/q), \mathscr{M}_t^0)$ martingale it suffices to show that

$$\lim_{n\to\infty} E\{[f(w^{\varepsilon_n}(u)) - f(w^{\varepsilon_n}(t))] \zeta(w^{\varepsilon_n}(\cdot))\}$$
$$= E^{M,N,p,q}\left\{\int_t^u \zeta(X(\cdot))(L^{M,N,p,q} f)(\sigma, X) d\sigma\right\} \quad (3.88)$$

for $t \leqq u$ and $\zeta(X(\cdot))$ a bounded continuous function of $X(t_1), X(t_2), \ldots, X(t_m)$ for any $0 \leqq t_1 < t_2 < \ldots < t_m \leqq t$. It is clear that it suffices to prove (3.88) for $k\eta \leqq t \leqq u < (k+1)\eta$ for some k. Indeed, if it is true for such pairs t, u, then by continuity it also holds for $k\eta \leqq t \leqq u \leqq (k+1)\eta$ and the general case can then be obtained by iteration. E.g. if $k\eta \leqq t < (k+1)\eta \leqq u < (k+2)\eta$ we merely have to write the left-hand side of (3.88) as

$$\lim E\{[f(w(u)) - f(w(k+1)\eta)]\zeta\}$$
$$+ \lim E\{f(w((k+1)\eta) - f(w(t))]\zeta\}$$

and to apply (3.88) to each of these limits separately.

From now on we assume $k\eta \leqq t \leqq u < (k+1)\eta$. As in [6] we rewrite the left-hand side of (3.88) by means of (3.9):

$$\begin{aligned}
&[f(w(u))-f(w(t))] \\
&=\frac{1}{\varepsilon}\int_t^u \frac{\partial f}{\partial w_i}(w(\sigma))G_i(\sigma, z(\sigma), w(\sigma))d\sigma \\
&=\frac{1}{\varepsilon}\int_t^u \frac{\partial f}{\partial w_i}(w(\xi))G_i(\sigma, z(\sigma), w(\xi))d\sigma \\
&\quad+\frac{1}{\varepsilon^2}\int_t^u d\sigma\int_\xi^\sigma d\lambda\left[\frac{\partial^2 f}{\partial w_j \partial w_i}(w(\lambda))G_i(\sigma, z(\sigma), w(\lambda))\right. \\
&\quad\left.+\frac{\partial f}{\partial w_i}(w(\lambda))D_{3,j}G_i(\sigma, z(\sigma), w(\lambda))\right]G_j(\lambda, z(\lambda), w(\lambda)) \\
&=I_1+I_2,
\end{aligned}$$

where we take

$$\xi=\xi(\sigma)=\max\{t, \sigma-\varepsilon^{15/8}\}. \tag{3.89}$$

We shall only analyze I_1. I_2 can be handled in the same way. As in the previous step we start with replacing $z(\sigma)$ by $L(\xi, \sigma)$, i.e., we rewrite I_1 as

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_t^u \frac{\partial f}{\partial w_i}(w(\xi))G_i(\sigma, L(\xi, \sigma), w(\xi))d\sigma \\
&+\frac{1}{\varepsilon}\int_t^u d\sigma\int_0^1 ds\,\frac{\partial f}{\partial w_i}(w(\xi))D_{2,j}G_i(\sigma, r(s, \xi, \sigma), w(\xi)) \\
&\cdot[z_j(\sigma)-L_j(\xi, \sigma)]=J_1+J_2.
\end{aligned}$$

By (3.63) and Remark 5 with ζ replaced by $\zeta(\partial f/\partial w_i)(w(\xi))$,

$$\begin{aligned}
|E\{\zeta J_1\}| \leqq C_{16}\sup\left|\frac{\partial f}{\partial w_i}\right| E^{1/2}\{\zeta^2\} \\
\cdot\frac{1}{\varepsilon}\int_t^u\left\{\varepsilon^2+\gamma\left(\frac{\sigma-\xi}{\varepsilon^2}\right)\right\}d\sigma,
\end{aligned}$$

so that $E\{\zeta J_1\}\to 0$ as $\varepsilon\downarrow 0$ (see (3.71) and (3.89)). As in the analysis of J_2 in step (iii) we replace $r(s, \xi, \sigma)$ again by $L(\xi, \sigma)$ in J_2:

$$\begin{aligned}
J_2&=\frac{1}{\varepsilon}\int_t^u d\sigma\,\frac{\partial f}{\partial w_i}(w(\xi))D_{2,j}G_i(\sigma, L(\xi, \sigma), w(\xi)) \\
&\quad\cdot[z_j(\sigma)-L_j(\xi, \sigma)] \\
&\quad+\frac{1}{\varepsilon}\int_t^u d\sigma\int_0^1 ds\int_0^1 dv\,\frac{\partial f}{\partial w_i}(w(\xi))D_{2,l}D_{2,j}G(\sigma, r(vs, \xi, \sigma), w(\xi)) \\
&\quad\cdot[z_j(\sigma)-L_j(\xi, \sigma)]s[z_l(\sigma)-L_l(\xi, \sigma)] \\
&=K_1+K_2.
\end{aligned}$$

As in the estimate of ζK_2 in step (iii)

$$|E\{\zeta K_2\}| \leqq C_{25}E^{1/4}\{\zeta^4\}\frac{1}{\varepsilon}\int_t^u d\sigma\,\frac{(\sigma-\xi)^4}{\varepsilon^6}\to 0$$

as $\varepsilon \downarrow 0$ (use (3.89)). K_1 is rewritten by means of (3.43) as

$$\frac{1}{\varepsilon^4}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \frac{\partial f}{\partial w_i}(w(\xi))D_{2,j}G_i(\sigma, L(\xi, \sigma), w(\xi))$$
$$\cdot G_j(\rho, z(\rho), w(\rho))$$
$$= \frac{1}{\varepsilon^4}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \frac{\partial f}{\partial w_i}(w(\xi))D_{2,j}G_i(\sigma, L(\xi, \sigma), w(\xi))$$
$$\cdot G_j(\rho, z(\rho), w(\xi))$$
$$+ \frac{1}{\varepsilon^5}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \frac{\partial f}{\partial w_i}(w(\xi))D_{2,j}G_i(\sigma, L(\xi, \sigma), w(\xi))$$
$$\cdot \int_\xi^\rho d\tau D_{3,l}G_j(\rho, z(\rho), w(\tau))G_l(\tau, z(\tau), w(\tau))$$
$$= L_1 + L_2.$$

By Holder's inequality and Lemma 3,

$$|E\{\zeta L_2\}| = O\left(\frac{1}{\varepsilon^5}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \int_\xi^\rho d\tau\right) = O\left(\int_t^u d\sigma \frac{(\sigma-\xi)^3}{\varepsilon^5}\right).$$
$$= o(1) \quad \text{as } \varepsilon \downarrow 0.$$

Similarly we may in L_1 replace $G_j(\rho, z(\rho), w(\xi))$ by $G_j(\rho, L(\xi, \rho), w(\xi))$. So far we have shown

$$E\{\zeta I_1\} = o(1) + \frac{1}{\varepsilon^4}E\left\{\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \frac{\partial f}{\partial w_i}(w(\xi))\right.$$
$$\left.\cdot D_{2,j}G_i(\sigma, L(\xi, \sigma), w(\xi))G_j(\rho, L(\xi, \rho), w(\xi))\zeta\right\}.$$

We now apply (for the first time) Lemma 5. We obtain with (we still use the summation conventions)

$$V_i(y, w_1, z, w_2) = E\left\{F_j(y, w_1)\frac{\partial}{\partial z_j}F_i(z, w_2)\right\},$$
$$W_{ij}(y, w_1, z, w_2) = E\{F_j(y, w_1)F_i(z, w_2)\},$$

that

$$E\{\zeta I_1\} = o(1) + \frac{1}{\varepsilon^4}E\left\{\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \frac{\partial f}{\partial w_i}(w(\xi))\zeta\Psi(\rho, w(\xi))\right.$$
$$\cdot \Psi(\sigma, w(\xi))\Phi(\rho, L(\xi, \rho))\Big[\Phi(\sigma, L(\xi, \sigma))V_i(L(\xi, \rho), w(\xi), L(\xi, \sigma), w(\xi))$$
$$+ \frac{\partial \Phi}{\partial z_j}(\sigma, L(\xi, \sigma)W_{ij}(L(\xi, \rho), w(\xi), L(\xi, \sigma), w(\xi))\Big]$$
$$+ O\left(\frac{1}{\varepsilon^4}\int_t^u d\sigma \int_\xi^\sigma d\lambda \int_\xi^\lambda d\rho \left\{\varepsilon^2 + \gamma\left(\frac{\rho-\xi}{\varepsilon^2}\right)\right\}^{1/2}\left\{\varepsilon^2 + \gamma\left(\frac{\sigma-\rho}{\varepsilon^2}\right)\right\}^{1/2}\right).$$

The last error term is again $o(1)$ by (3.71) and (3.89). Also, by (3.10), (3.79) and (3.77)

$$\left|\frac{\partial \phi}{\partial z_j}(\sigma, L(\xi,\sigma))W_{ij}(L(\xi,\rho), w(\xi), L(\xi,\sigma), w(\xi))\right| \leqq C_{26}\varepsilon^2\left\{\beta\left(\frac{\sigma-\rho}{2N\varepsilon^2}\right)\right\}^{1/2}.$$

and this term too can be dropped. This almost gives us the required form. We still observe that for $k\eta \leqq t \leqq \xi \leqq \rho \leqq \sigma \leqq u < (k+1)\eta$

$$\Psi(\rho, w(\xi)) = \Psi(\sigma, w(\xi)) = \Psi(\xi, w(\xi)).$$

Similarly, by (3.10) and (3.11),

$$\begin{aligned}
&|\Phi(\rho, L(\xi,\rho)) - \Phi(\xi, z(\xi))| = |\Phi(\xi, L(\xi,\rho) - \Phi(\xi, z(\xi))| \\
&\quad \leqq C_1\varepsilon^2 |L(\xi,\rho) - z(\xi)| = C_1(\rho-\xi)|w(\xi)| \\
&\quad \leqq 2C_1 M(\rho-\xi),
\end{aligned}$$

and the same inequality holds when ρ is replaced by σ throughout. From these observations, (3.79) and (3.10) we see that

$$\begin{aligned}
E\{\zeta I_1\} = o(1) + \int_t^u d\sigma E\bigg\{&\frac{\partial f}{\partial w_i}(w(\xi))\zeta\Psi^2(\xi, w(\xi))\phi^2(\xi, z(\xi)) \\
&\cdot\frac{1}{\varepsilon^4}\int_\xi^\sigma d\rho(\sigma-\rho)V_i(L(\xi,\rho), w(\xi), L(\xi,\sigma), w(\xi))\bigg\}.
\end{aligned}$$

Last, if we take into account the stationarity of F (see (II)),

$$\begin{aligned}
&\Psi(\xi, w(\xi))\frac{1}{\varepsilon^4}\int_\xi^\sigma d\rho(\sigma-\rho)V_i(L(\xi,\rho), w(\xi), L(\xi,\sigma), w(\xi)) \\
&= \Psi(\xi, w(\xi)) \int_0^{\varepsilon^{-2}(\sigma-\xi)} tE\left\{F_j(0,w)\frac{\partial}{\partial z_j}F_i(tw, w)\right\}_{w=w(\xi)} dt \\
&= \sum_j c_{ij}(\xi, w(\cdot), w(\xi)) + o\left(\int_{\varepsilon^{-2}(\sigma-\xi)}^{\infty} t\beta^{1/2}(t|w(\xi)|)dt\right) \quad \text{(by (3.79))} \\
&= \sum_j c_{ij}(\xi, w(\cdot), w(\xi)) + o(1) \quad \text{(by (3.89), (3.57) and (3.11))}.
\end{aligned}$$

Thus

$$\begin{aligned}
E\{\zeta I_1\} &= o(1) + \int_t^u d\sigma E\bigg\{\frac{\partial f}{\partial w_i}(w(\xi))\zeta\Psi(\xi, w(\xi))\Phi^2(\xi, z(\xi)) \\
&\qquad \cdot \sum_j c_{i,j}(\xi, w(\cdot), w(\xi)). \\
&= o(1) + \int_t^u d\xi E\bigg\{\frac{\partial f}{\partial w_i}(w(\xi))\zeta\Psi(\xi, w(\xi))\Phi^2(\xi, z(\xi))\sum_j c_{i,j}(\xi, w(\cdot), w(\xi)).
\end{aligned}$$

The last equality results merely from the fact that $\sigma = \xi + \varepsilon^{15/8}$ except on an interval of length $\varepsilon^{15/8}$. Finally, we view $w(\cdot)$ as an element of D. Then the expectation in the last integral can be written as an integral over D with respect to the measure $R^{\varepsilon}(\cdot\,; M, N, 1/p, 1/q)$ and integrand

$$\frac{\partial f}{\partial w_i}(X(\xi))\zeta(X(\cdot))\Psi_*(\xi, X(\cdot))\Phi_*^2(\xi, X(\cdot))\sum_j c_{ij}(\xi, X(\cdot), X(\xi)) \tag{3.90}$$

(This is where we use (3.86)). As observed already in [6] it follows from (3.81) and (3.84) that for each fixed σ, $X(\cdot)$ is continuous at σ with $R(\cdot\,; M, N, 1/p, 1/q)$ probability one. From this one easily derives that (3.90) is a continuous function of $X(\cdot)$ a.e. on D with respect to $R(\cdot\,; M, N, 1/p, 1/q)$ (in the usual J_1 topology). Thus, if we write $E^{M,N,p,q}$ for the expectation operator with respect to $R(\cdot\,; M, N, 1/p, 1/q)$ and let $\varepsilon \to 0$ through the sequence ε_n, then (3.84) yields

$$\lim_{n\to\infty} E\{\zeta I_1\} = \int_t^u d\xi E^{M,N,p,q}\{\zeta(X(\cdot))\Psi_*(\xi, X(\cdot)\} \cdot \Phi_*^2(\xi, X(\cdot))\sum_j c_{ij}(\xi, X(\cdot), X(\xi))\frac{\partial f}{\partial w_i}(X(\xi)).$$

which is one of the terms of the right-hand side of (3.88). The other terms come from $E\{\zeta I_2\}$. This proves the claimed martingale property for (3.85).

Step (v)

We complete the proof of our theorem in this step by removing the truncations in N, η, δ and M (in this order). It is convenient to return first to the space $C = C([0, \infty); \mathbb{R}^d)$. We assumed that the convergence in (3.84) takes place on D. However, by definition $R^{\varepsilon}(\cdot\,; M, N, 1/p, 1/q)$ is concentrated on C, and if we can show that $R(\cdot\,; M, N, 1/p, 1/q)$ is also concentrated on C then (3.84) implies

$$R^{\varepsilon_n}\left(\cdot\,; M, N, \frac{1}{p}, \frac{1}{q}\right) \Rightarrow R\left(\cdot\,; M, N, \frac{1}{p}, \frac{1}{q}\right) \text{ on } C \tag{3.91}$$

(see [8], p. 151). But the fact that $R(\cdot\,; M, N, 1/p, 1/q)$ is concentrated on C follows immediately from the martingale property of (3.85), Theorem 2.1 (especially inequality (2.1)) of [15] and the fact that

$$\begin{aligned}&\psi_*(\sigma, X)\Phi_*^2(\sigma, X)\{|\psi_*(\sigma, X)a_{ij}(X(\sigma))| + |b_i^{M,N,p}(X(\sigma))|\}\\ &\leqq \sup_{1/2M \leqq |v| \leqq 2M} \{|a_{ij}(v)| + |b_i^{M,N,p}(v)|\} < \infty\end{aligned}$$

(because $\Psi_*(\sigma, X) = 0$ for $|X(\sigma)| \notin [1/2M, 2M]$; see (3.1)).

From now on we can assume (3.91) and all further manipulations take place on C. With a slight abuse of notation we also use X_t for the t-coordinate function on C and $\mathscr{M}_v^u$ for $\sigma\{X_t : u \leqq t \leqq v\}$. Formula (3.85) still is an $(R(\cdot\,; M, N, 1/p, 1/q), M_t^0)$ martingale, even with the convention that R is a measure on $\mathscr{M}_\infty^0$ and $\mathscr{M}_t^0$, $\mathscr{M}_\infty^0$ σ-fields in C. We now define $\{\mathscr{M}_t^0\}$ stopping times, S, T, U and V as follows:

$$S = S(X(\cdot); N, p) = \lim_{n\to\infty} S_n,$$

where X_k is as in step (i) with $\eta = 1/p$ and

$$S_n = \inf\left\{ t \geqq 0\text{: for some } k \geqq 0, \frac{k}{p} \leqq t < \frac{k+1}{p}\right.$$

and

$$\left.(X_{k-1}, X(t)) < \frac{1}{N} + \frac{1}{n} \text{ or } (X_k, X_t) < \frac{1}{N} + \frac{1}{n}\right\}.$$

S_n is an $\{\mathscr{M}^0_{t+}\}$ stopping time and for t not of the form k/p

$$\{S \leqq t\} = \bigcap_n \{S_n < t\} \in \mathscr{M}^0_t.$$

For $t = k/p$.

$$\{S \leqq t\} = \left(\bigcap_n \{S_n < t\}\right) \cup \left((X(t), X_k) \leqq \frac{1}{N}\right) \in \mathscr{M}^0_t$$

so that indeed S is an $\mathscr{M}^0_t$ stopping time. Similarly,

$$T = T(X(\cdot);M)) = \lim_{n\to\infty} T_n,$$

$$T_n = \inf\left\{t \geqq 0: |X(t)| < \frac{1}{M} + \frac{1}{n} \text{ or } |X(t)| > M - \frac{1}{n}\right\},$$

$$U = U(X(\cdot); p, q) = \lim_{n\to\infty} U_n,$$

$$U_n = \inf\left\{ t \geqq 0\text{: for some } k \geqq 1 \text{ and } u \leqq (k-1)p, k/p \leqq t < (k+1)p \quad \text{and} \right.$$

$$\left. \left|\int_u^t X(v)dv\right| < \frac{1}{q} + \frac{1}{n}\right\}$$

Lastly,

$$V = S \wedge T \wedge U.$$

From these definitions it follows that

$$\Psi_*(\sigma, X) = 1 \quad \text{and} \quad \frac{\partial}{\partial w_j}\Psi(\sigma, X, X(\sigma)) = 0 \quad \text{for } \sigma < S \vee T$$

and

$$\Phi_*(\sigma, X) = 1 \qquad\qquad \text{for } \sigma < U.$$

Thus

$$\begin{aligned} &f(X_{t\wedge V}) - \int_0^{t\wedge V} (L^{M,N,p,q}f)(\sigma, X)d\sigma \\ &= f(X_{t\wedge V}) - \int_0^{t\wedge V} \mathscr{L}f(X(\sigma))d\sigma \\ &= f(X_{t\wedge V}) - \int_0^{t\wedge V} \tilde{\mathscr{L}}^M f(X(\sigma))d\sigma, \end{aligned} \tag{3.92}$$

where $\mathscr{L}$ is given by (2.5) and

$$\tilde{\mathscr{L}}^M f(v) = \frac{1}{2}\sum_{i,j} \tilde{a}_{ij}^M(v) \frac{\partial^2}{\partial v_i \partial v_j} f(v) + \sum_i \tilde{b}_i^M(v) \frac{\partial f}{\partial v_i}(v)$$

for any choice of the coefficients $\tilde{a}_{ij}^M(v), \tilde{b}_i^M(v)$ which agrees with the $a_{ij}(v)$ and $b_i(v)$ of (2.3), (2.4) on the set $\left\{\frac{1}{M} \leqq |v| \leqq M\right\}$ and which makes the $\tilde{a}_{ij}^M$ and $\tilde{b}_i^M$ twice continuously differentiable on $\mathbb{R}^d$ and $\tilde{a}_{ij}^M$ strictly positive definite.

Now let $\tilde{Q}^M$ be the measure corresponding to the diffusion process $\tilde{\mathscr{L}}^M$ and initial point v_0, and let Q be as in the theorem. Then (3.92) is an $(R(\cdot;M,N,p,q), \mathscr{M}_t^0)$ martingale, by virtue of the optional sampling theorem (cf. [14], Theorem VI.7 and 13) and the martingale property of (3.85). (3.92) is also a $(Q, \mathscr{M}_t^0)$ and a $(\tilde{Q}^M, \mathscr{M}_t^0)$ martingale by Dynkin's formula ([11] Corollary 5.1). Moreover, by (3.91)

$$Q(X(0)=0) = \tilde{Q}^M(X(0)=v_0) = R(X(0)=v_0;M,N,p,q) = 1 \tag{3.93}$$

We claim that this implies

$$\tilde{Q}^M(B) = Q(B) = R(B;M,N,p,q) \text{ whenever } B \in \mathscr{M}^0_{V(M,N,p,q)}; \tag{3.94}$$

(see [16] for the definition of the last σ-field). Indeed if ν is any of the measures $\tilde{Q}^M, Q$ or $R(\cdot;M,N,p,q)$ then one can use Lemma 3.6 and Theorem 3.4 of [16] and the martingale property of (3.92) to construct a measure μ on $\mathscr{M}_\infty^0$ which agrees with ν on $\mathscr{M}_V^0$ and such that

$$f(X_t) - \int_0^t \tilde{\mathscr{L}}^M f(X(\sigma))d\sigma$$

is a $(\mu, \mathscr{M}_t^0)$ martingale and (by (3.93)).

$$\mu(X(0)=0) = 1.$$

By the uniqueness theorem 6.2 of [16] such a μ is unique. Thus all three choices for ν give rise to the same μ which implies (3.94).

We apply (3.94) to

$$B = B(M,N,p,q) = \{S(N,p) \wedge U(p,q) \leqq T(M) \wedge t\}$$

for fixed t. This gives

$$R(B(M,N,p,q);M,N,p,q) = Q(B(M,N,p,q)). \tag{3.95}$$

We shall show in Lemma 6, Section 4 that as a consequence of assumption V, for each fixed p,

$$\lim_{q\to\infty} U(p,q) = \infty \text{ a.e. } [Q]. \tag{3.96}$$

This of course implies

$$\lim_{q\to\infty} Q\{U(p,q) \leqq T \wedge t\} = 0.$$

In addition, for $N \geqq 2M$

$$\lim_{p\to\infty} Q\{S(N,p) \leqq T(M) \wedge t\} = 0$$

because $|X(\sigma)| \geqq 1/M$ for all $\sigma \leqq T(M)$ and $X(\cdot)$ is continuous. It follows that the right-hand side of (3.95) can be made small for any $N \geqq 2M$ by choosing first p, then q large.

It is now easy to remove the cutoffs. Fix t and any bounded positive continuous functional f on C which is measurable with respect to $\mathscr{M}_t^0$. Let $\varepsilon_n \to 0$, and if necessary refine the sequence such that

$$\lim_{n\to\infty} E\{f(v^{\varepsilon_n}(\cdot)); T(v^{\varepsilon_n}, M) > t\}$$

exists for all integers M, and such that (3.91) holds for all integers M, N, p and q. Here we view v^ε as an element of C and $T(v^\varepsilon, M)$ is the value of $T(M)$ evaluated at v^ε. Fix M and $\alpha > 0$ and choose N, p, q such that

$$Q(\bar{B}(M-1, N-1, p, q-1)) \leqq Q(B(M, N-2, p, q-2)) \leqq \alpha$$

and

$$Q(B(M, N, p, q)) \leqq \alpha$$

($\bar{B}$ denotes the closure of B in C.) Finally note that $v^\varepsilon(t) = w^\varepsilon(t; M, N, p, q)$ for all

$$t \leqq V(v^\varepsilon; M, N, p, q) = V(w^\varepsilon; M, N, p, q) \tag{3.97}$$

because the Eqs. (2.7) for v^ε and (3.8) for w^ε coincide up until this time V. This, together with (3.91) and (3.94), implies that

$$\begin{aligned}
&\limsup_{n\to\infty} E\{f(v^{\varepsilon_n}(\cdot)); V(v^{\varepsilon_n}; M-1, N-1, p, q-1) > t\} \\
&\quad = \limsup_{n\to\infty} E\{f(w^{\varepsilon_n}(\cdot)); V(w^{\varepsilon_n}; M-1, N-1, p, q-1) > t\} \\
&\quad \leqq E^{M,N,p,q}\{f(X(\cdot)); V(X; M, N, p, q) > t\} \\
&\quad = \int f(X(\cdot)) I[V(X; M, N, p, q) > t] dQ.
\end{aligned}$$

Moreover, by our choice of N, p, q, the last member here differs from

$$\int f(X(\cdot)) I[T(X, M) > t] dQ$$

by at most

$$\sup|f| Q(B(M, N, p, q)) \leqq \alpha \sup|f|.$$

Finally,

$$\begin{aligned}
&\limsup_{n\to\infty} |E\{f(v^{\varepsilon_n}(\cdot)); T(v^{\varepsilon_n}, M-1) > t\} \\
&- E\{f(v^{\varepsilon_n}(\cdot)); V(v^{\varepsilon_n}; M-1, N-1, p, q-1) > t\}| \\
&\quad \leqq \sup|f| \limsup_{n\to\infty} P\{S(v^{\varepsilon_n}; N-1, p) \wedge U(v^n; p, q-1) \\
&\qquad\qquad \leqq T(v^{\varepsilon_n}; (M-1) \wedge t)\},
\end{aligned} \tag{3.98}$$

and (again by (3.97), (3.91) and (3.94)), the lim sup in the right-hand side of (3.98) equals

$$\limsup_{n\to\infty} P\{S(w^{\varepsilon_n};N-1,p)\wedge U(w^{\varepsilon_n};p,q-1)\leqq T(w^{\varepsilon_n};(M-1)\wedge t)\}$$
$$=\limsup_{n\to\infty} \mathrm{R}^{\varepsilon_n}(B(M-1,N-1,p,q-1);M,N,p,q)$$
$$\leqq R(\bar{B}(M-1,N-1,p,q-1);M,N,p,q)$$
$$=Q(\bar{B}(M-1,N-1,p,q-1))\leqq\alpha.$$

Since α is arbitrary these estimates show that for each M, t,

$$\limsup_{n\to\infty} E\{f(v^{\varepsilon_n}(\cdot));T(v^{\varepsilon_n},M-1)>t\}$$
$$\leqq\int f(X(\cdot))I[T(X,M)>t]dQ. \tag{3.99}$$

Finally, we take the limit as $M\to\infty$. From (3.99) we see that

$$\limsup_{n\to\infty} P\{T(v^{\varepsilon_n},M-1)>t\}\leqq Q\{T(X,M)>t\}$$

and this can be made as small as desired by taking M large, on account of assumption (2.6) and the boundedness of $a_{ij}(\cdot)$ and $b_i(\cdot)$ away from the origin (see [15], formula (2.1)). Thus

$$\limsup_{n\to\infty} E\{f(v^{\varepsilon_n}(\cdot))\}\leqq\int f(X(\cdot))dQ.$$

Since any sequence $\varepsilon_n\to 0$ contains a subsequence to which this applies we have proved

$$\limsup_{\varepsilon\downarrow 0} E\{f(v^{\varepsilon}(\cdot))\}\leqq\int f(X(\cdot))dQ.$$

As shown in [8], p. 13, and [6], Sect. 3, Step (vi), this implies that the measures induced by $v^{\varepsilon}(\cdot)$ on C converge weakly to Q on C. The proof is complete.

4. Properties of Diffusions and Examples

We begin this section with Lemma 6 which immediately implies the relation (3.96) which we needed in Step (iv) of Sect. 3. The proof of this lemma is entirely independent of Sect. 3.

Lemma 6. *Let V_t be a diffusion with generator $\mathscr{L}$ as in (2.3)–(2.5) and initial point $v_0\neq 0$. Assume that $(a_{ij}(\cdot))$ is symmetric and strictly positive definite on $\mathbb{R}^d\setminus\{0\}$ and*

$$a_{ij}(\cdot),\, b_i(\cdot)\in C^{\infty}(\mathbb{R}^d\setminus\{0\}). \tag{4.1}$$

If $d\geqq 3$, $v_0\neq 0$ and (2.6) holds, then for all $\eta>0$, $T<\infty$,

$$\lim_{\delta\downarrow 0} P^{v_0}\left\{\left|\int_s^t V_\sigma d\sigma\right|<\delta \text{ for some } 0\leqq s,t\leqq T \text{ with } |t-s|\geqq\eta\right\}=0. \tag{4.2}$$

Remark 5. A check of the proof in the previous section shows that the only place where the smoothness of the a_{ij} and b_i and the strict positive definiteness of a_{ij} played a role was in the proof of (3.94), and more precisely for the uniqueness of the measure μ in that proof. For this uniqueness we only need

$$(a_{ij}(\cdot)) \text{ is nonnegative definite and twice continuously differentiable on } \mathbb{R}\backslash\{0\}; \tag{4.3}$$

(see [17], Theorem 2.3 and Remark 2.1; note that the $b_i(v)$ of (2.4) is uniformly Lipschitz continuous and bounded on any set $\{v\in\mathbb{R}^d: |v| \geqq \delta\}$, $\delta > 0$, by (II)–(IV)). The stronger requirement (4.1) and the requirement $d \geqq 3$ are used only to prove Lemma 6. Therefore, our theorem and corollary will remain valid in any dimension if in (V) we only require (4.3) of the coefficients and (2.6), provided we add (4.2) as a separate hypothesis. Actually, it is likely that even for Lemma 6 only a finite number of derivatives of $a_{ij}(\cdot)$ and $b_i(\cdot)$ are needed, but we have not pursued this.

The proof of Lemma 6 will come after we discuss a sufficient condition for (2.6), and (III) and explicit examples. Formula (2.6) can often be verified by means of the following criterion of Khasminskii's [18] (see also [9], Chap. 4.5).

Lemma 7. *Let $\mathscr{L}$ and V_t be as in (V). For $v \neq 0$ and $\Gamma = (\Gamma_{ij})$ a nonsingular (constant) $d \times d$ matrix define (Γ^t = transpose of Γ) $A(v) = A(v,\Gamma) = \sum_{i,j} (\Gamma a(v) \Gamma^t)_{i,j} (\Gamma v)_i (\Gamma v)_j$,*

$$\begin{aligned} A^+(r,\Gamma) &= \max_{|\Gamma v| = r} A(v,\Gamma), \\ B_-(r) &= B_-(r,\Gamma) \\ &= \min_{|\Gamma v| = r} \frac{1}{A(v,\Gamma)} \Big[2\sum_i (\Gamma b(v))_i (\Gamma v)_i + \sum_i (\Gamma a(v) \Gamma^t)_{i,i}\Big], \end{aligned}$$

and

$$C_-(r,\Gamma) = \exp\left\{ -\int_r^1 s B_-(s,\Gamma)\, ds \right\}.$$

If there exists a nonsingular Γ such that

$$\int_0^1 \frac{t\,dt}{C_-(t,\Gamma)} \int_t^1 \frac{C_-(s,\Gamma)}{A_+(s,\Gamma)}\, ds = \infty, \tag{4.4}$$

then (2.6) *holds.*

Remark 6. Take $d \geqq 3$ and

$$\mathscr{L} = \sum_{i,j} \frac{\partial}{\partial v_i} \frac{a_{ij}}{|v|} \frac{\partial}{\partial v_j}$$

for some constant symmetric positive definite matrix α, as discussed in Remark 4. Then (4.4) holds for $\Gamma = \alpha^{-1/2}$ (the positive symmetric square root), so that also (2.6) holds in this case. In general the criterion of Lemma 7 is only helpful if for some Γ the diffusion $\{\Gamma V_t\}_{t \geqq 0}$ is close to being radially symmetric (see the examples in [9], Chap. 4.5).

Lemma 7 too will be proved later. First we discuss two situations which guarantee the mixing conditions (III). (a) (III) holds for what we called the *trivial Gaussian examples* in [6]. These are Gaussian fields $\{F(x,v)\}_{x,v\in\mathbb{R}^d\times\mathbb{R}^d}$ with mean zero, stationary in x, and such that the correlation function

$$r_{ij}(y,v_1,v_2)=E\{F_i(x,v_1)F_j(x+y,v_2)\} \tag{4.5}$$

vanishes for all i,j,v_1,v_2 as soon as $|y|\geqq L$ where $L<\infty$ is some constant. In this case $\mathscr{G}_{\Lambda_1}$ and $\mathscr{G}_{\Lambda_2}$ are actually independent as soon as $d(\Lambda_1,\Lambda_2)>L$. Thus $\beta(\rho)=0$ for $\rho>L$ and (2.2) certainly holds. (It is not hard to show that for these stationary Gaussian fields it is actually necessary for (2.2) that $\beta(\rho)$ vanishes for large ρ). One can obtain such examples by taking ($\bar{\rho}$ is the complex conjugate of ρ)

$$r_{ij}(y,v_1,v_2)=\sum_k\int_{\mathbb{R}^d}\rho_{i,k}(y+z,v_1)\bar{\rho}_{j,k}(z,v_2)dz \tag{4.6}$$

for any measurable complex matrix valued function ρ on $\mathbb{R}^d\times\mathbb{R}^d$ with $\rho(z,v)=0$ for $|z|\geqq L/2$, as long as

$$\int_{\mathbb{R}^d}|\rho_{ii}(z)|^2dz<\infty \text{ for all } i.$$

It is easy to see that (4.6) is indeed positive definite, hence a correlation function, and has support in $|y|\leqq L$. This leads to the following *explicit example: The Theorem and Corollary apply if $d\geqq 3$ and $\{F(x,v)\}$ is a mean zero Gaussian field independent of v and stationary in x, with correlation function given by (4.5) and (4.6) with*

$$\rho(y,v)=\rho^*(y)=\rho^*(|y|)$$

(depending on $|y|$ only), whenever $\rho^*\in C^2(\mathbb{R}^d)$ and the following properties hold:

$$\rho^*(z)=0 \quad \text{for} \quad |z|\geqq\tfrac{1}{2}L. \tag{4.7}$$

$$\int_{\mathbb{R}^d}\left|\left(\frac{\partial}{\partial z_l}\right)^2\rho_{ii}^*(z+u)-\left(\frac{\partial}{\partial z_l}\right)^2\rho_{ii}^*(z)\right|dz\leqq K\left(\log\frac{1}{|\eta|}\right)^\alpha \tag{4.8}$$

for some $K<\infty,\alpha>1$ and all $|u|\leqq 1$, $1\leqq i,l\leqq d$, and finally,

$$\alpha_{ij}=\sum_k\int_0^\infty dt\int_{\mathbb{R}^d}\rho_{ik}^*(te+z)\overline{\rho_{j,k}^*}(z)dz$$

$$\text{is nonsingular for } e=(1,0,\ldots,0). \tag{4.9}$$

In this case

$$\mathscr{L}=\sum_{i,j}\alpha_{ij}\frac{\partial}{\partial v_i}\frac{1}{|v|}\frac{\partial}{\partial v_j}. \tag{4.10}$$

See below for a proof.

(b) (III) also holds for the following special case of the "Poisson blobs" of [6]: Let P_ρ be a Poisson point process on $\mathbb{R}^d$ with intensity ρ, i.e. $N(B)\equiv$ number of points of P_ρ in the Borel set $B\subset\mathbb{R}^d$ has a Poisson distribution with mean $\rho|B|$

($|B|$ denotes the Lebesgue measure of B). Moreover, if $B_1, \ldots, B_k$ are disjoint, then $N(B_1), \ldots, N(B_k)$ are independent. In addition, let $H^0, H^1, H^2, \ldots$ be independent identically distributed random fields, independent of P_ρ and with the following properties:

Each $H^{(n)}$ is indexed by $\mathbb{R}^d$, i.e. $H^{(n)} = H^{(n)}(x, \omega)$, $x \in \mathbb{R}^d$, ω in our basic probability space. $H^{(0)}(x) = 0$ for $|x| \geqq L$ for some constant L.
Let

$$K(x) = \sum_{n=1}^{\infty} H^{(n)}(x + p_n) \tag{4.11}$$

where $p_1, p_2, \ldots$ are the (random) points of P_ρ, and let f be some deterministic function from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$. Then the field

$$F(x, v) = f(K(x), v) \tag{4.12}$$

satisfies the mixing condition (III). Again we can make this into an explicit example. Assume that $H^{(0)}$ also has the following properties:

The joint distribution of $H^{(0)}(x_1)$ and $H^{(0)}(x_2)$ is the same as that of $H^{(0)}(Ox_1)$ and $H^{(0)}(Ox_2)$ for any $x_1, x_2 \in \mathbb{R}^d$ and orthogonal matrix O.

$$E\{H^{(0)}(x)\} = 0, \quad x \in \mathbb{R}^d \tag{4.13}$$

$H^{(0)}$ is twice continuously differentiable and

$$E\left\{\max_{|x| \leqq L} |D^\beta H^{(0)}(x)|^p\right\} \leqq c_0, \tag{4.14}$$

for $|\beta| \leqq 2$, $p \leqq 16d + 64$.
The $d \times d$ matrix with entries

$$\alpha_{ij} = \rho \int_0^\infty dt \int_{\mathbb{R}^d} E\{H_i^{(0)}(z) H_j^{(0)}(te + z)\} dz,$$

is nonsingular ($e = (1, 0, 0, \ldots, 0)$).
Then our Theorem and Corollary apply if we take $d \geqq 3$ *and*

$$F(x, v) = \sum_n H^{(n)}(x + p_n) \tag{4.15}$$

(independent of v; corresponding to $f(K, v) = K$ in (4.12)).
In this case

$$\mathscr{L} = \sum_{i,j} \alpha_{i,j} \frac{\partial}{\partial v_i} \frac{1}{|v|} \frac{\partial}{\partial v_j}. \tag{4.16}$$

We turn to the proofs.

Proof of Lemma 6. We are indebted to Daniel W. Stroock for pointing out how to use hypoellipticity in the first part of this proof. Let

$$Z_t = \int_0^t V_\sigma d\sigma.$$

Then $(V_t, Z_t)_{t \geqq 0}$ is a singular diffusion with generator

$$\mathscr{L} + \sum_{i=1}^{d} v_i \frac{\partial}{\partial z_i}$$

Since the coefficients of $\mathscr{L}$ may be badly behaved at the origin we first replace $\mathscr{L}$ by

$$\tilde{\mathscr{L}}^M = \frac{1}{2} \sum_{i,j} \tilde{a}_{ij}^M(v) \frac{\partial^2}{\partial v_i \partial v_j} + \sum_i \tilde{b}_i^M(v) \frac{\partial}{\partial v_i}$$

for any choice of the coefficients $\tilde{a}_{ij}^M(x)$, $\tilde{b}_i^M(x)$ which agrees with the $a_{ij}(v)$ and $b_i(v)$ of $\mathscr{L}$ on the set $\left\{\frac{1}{M} \leqq |v| \leqq M\right\}$ and which makes $\tilde{a}_{ij}^M$ and $\tilde{b}_i^M$ infinitely differentiable on $\mathbb{R}^d$ and $\tilde{a}_{ij}^M$ symmetric and strictly positive definite on $\mathbb{R}^d$. $(\tilde{V}_t^M, \tilde{Z}_t^M)$ will be the corresponding process replacing (Z_t, V_t). For the time being we shall suppress the superscript M. Let

$$\begin{aligned} G^* = & \frac{1}{2} \sum \tilde{a}_{ij}(v) \frac{\partial^2}{\partial v_i \partial v_j} + \sum \left(\frac{\partial}{\partial v_j} \tilde{a}_{ij}(v) \right) \frac{\partial}{\partial v_i} \\ & - \sum \tilde{b}_i(v) \frac{\partial}{\partial v_i} - \sum v_i \frac{\partial}{\partial z_i} + \frac{1}{2} \sum \frac{\partial^2}{\partial v_i \partial v_j} \tilde{a}_{ij}(v) - \sum_i \frac{\partial}{\partial v_i} \tilde{b}_i(v) \end{aligned}$$

be the formal adjoint of the generator

$$G = \frac{1}{2} \sum \tilde{a}_{ij}(v) \frac{\partial^2}{\partial v_i \partial v_j} + \sum \tilde{b}_i(v) \frac{\partial}{\partial v_i} + \sum v_i \frac{\partial}{\partial z_i}$$

of the process $\{\tilde{V}_t, \tilde{Z}_t\}_{t \geqq 0}$. For fixed initial point (w, y), let

$$U(t, dv, dz \,|\, w, y) = P\{\tilde{V}_t \in dv, \tilde{Z}_t \in dz \,|\, V_0 = w, Z_0 = y\}$$

be the distribution of $(\tilde{V}_t, \tilde{Z}_t)$. Then the formal density u of U is a distribution solution of the equation

$$\left(-\frac{\partial}{\partial t} + G^* \right) u = 0 \text{ on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{4.17}$$

More precisely, for any function $j \in C_0^\infty((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d)$ (C_0^∞ denotes the C^∞ functions with compact support)

$$\int_0^\infty dt \int_{(v,z) \in \mathbb{R}^d \times \mathbb{R}^d} U(t, dv, dz) \left(\frac{\partial}{\partial t} + G \right) j(t, v, z) = 0$$

(see [9], p. 61). We now show that $-\partial/\partial t + G^*$ is hypoelliptic by means of Hörmander's theorem 1.1 in [19]. For this purpose let $(c_{ij}(v))$ be the positive symmetric C^∞ square root of $\tilde{a}_{ij}(v)$ (see [9], p. 83) and define the linear differential operators

$$X_i(v) = \sum_j c_{ij}(v) \frac{\partial}{\partial v_j}, \quad 1 \leqq i \leqq d,$$

$$X_0(v) = -\frac{1}{2} \sum_{i,j,k} c_{ik} \left(\frac{\partial}{\partial v_k} c_{ij} \right) \frac{\partial}{\partial v_j} - \frac{\partial}{\partial t} + \sum_{i,j} \left(\frac{\partial}{\partial v_j} \tilde{a}_{ij} \right) \frac{\partial}{\partial v_i} - \sum_i \tilde{b}_i \frac{\partial}{\partial v_i} - \sum_i v_i \frac{\partial}{\partial z_i}.$$

Then

$$-\frac{\partial}{\partial t} + G^* = \frac{1}{2} \sum X_i^2 + X_0 + \frac{1}{2} \sum \frac{\partial^2}{\partial v_i \partial v_j} \tilde{a}_{ij}(v) - \sum_i \frac{\partial}{\partial v_i} \tilde{b}_i(v)$$

A simple calculation verifies that each of the operators $\partial/\partial t, \partial/\partial v_i$ and $\partial/\partial z_j$ is a linear combination of the $X_i(v)$ or $[X_i(v), X_j(v)], 0 \leqq i, j \leqq d$. This is clear for the $\partial/\partial v_i$, because $c(v) = \tilde{a}^{1/2}(v)$ is nonsingular. Moreover

$$[X_i, X_0] = -\sum_k c_{i,k}(v) \frac{\partial}{\partial z_k} + \text{linear combination of the } \frac{\partial}{\partial v_i}$$

Thus also the $\partial/\partial z_k$ and hence $\partial/\partial t$ are in the Lie algebra generated by the X_i. By Theorem 1.1 of [17] this guarantees the hypoellipticity of $-\partial/\partial t + G^*$ and we conclude from (4.17) that we can write

$$U(t, dv, dz | w, y) = u(t, v, z | w, y) dv dz,$$

for some u which is in $C^\infty((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d)$ as a function of (t, v, z). For our purposes it is necessary to have u a continuous function of all its arguments (t, v, z, u, y) on $(0, \infty) \times (\mathbb{R}^d)^4$. Following McKean ([9], p. 64) this can be done by combining the forward and the backward equation. u (as a function of all its arguments) is a distribution solution of $Ku = 0$ on $(0, \infty) \times (\mathbb{R}^d)^4$, where

$$K = -2 \frac{\partial}{\partial t} + G^*_{v,z} + G_{w,y}$$

One shows as above that K is hypoelliptic which gives the desired conclusion.

We need only a very weak consequence of the existence of a smooth density u. This is that for every $M, 0 < \eta \leqq T, \lambda$ there exists a $K = K(M, \eta, T, \lambda) < \infty$ such that

$$P\{|\tilde{Z}^M_{t+s} - z| \leqq \rho | V_s = w, Z_s = y\} \leqq K\rho^d,$$
$$0 \leqq s \leqq T, \quad \eta \leqq t \leqq T, \quad |w|, |y|, |z| \leqq \lambda, \rho \leqq 1, \tag{4.18}$$

Now let $\varepsilon > 0$, $T < \infty$, and choose $M = M(\varepsilon, T)$ such that

$$P^{v_0}\left\{ |V_t| \leqq \frac{1}{2M} \text{ or } |V_t| \geqq \frac{M}{2} \text{ for some } t \leqq t \right\} \leqq \varepsilon,$$

For such an M we have

$$P^{v_0}\left\{\left|\int_s^t V_\sigma d\sigma\right| < \delta \text{ for some } 0 \leqq s, t \leqq T \text{ with } |t-s| \geqq \eta\right\}$$
$$\leqq \varepsilon + P\left\{\int_s^t \tilde{V}_\sigma^M d\sigma < \delta \text{ for some } 0 \leqq s, t \leqq T\right.$$
$$\left.\text{with } |t-s| > \eta, \text{ while } \frac{2}{M} \leqq |\tilde{V}_\sigma| \leqq \frac{M}{2} \text{ for all } \sigma \leqq T\right\}. \tag{4.19}$$

This is so because if V_σ is killed at the first time $|V_\sigma|$ leaves $(1/M, M)$ and similarly for $\tilde{V}_\sigma^M$, then the killed processes have the same distribution ([11], Chap. 5.24). Clearly

$$\left|\int_s^t \tilde{V}_\sigma^M d\sigma\right| \leqq \frac{\delta}{4} \text{ when } |t-s| \leqq \frac{\delta}{2M},$$

and

$$\left|\int_0^s \tilde{V}_\sigma^M d\sigma\right| \leqq TM, \qquad s, t \leqq T,$$

on the set $\{|\tilde{V}_\sigma^M| \leqq M/2 \text{ for } \sigma \leqq T\}$. We can use this to estimate the probability on the right-hand side of (4.19) by looking only at times which are multiples of $\rho = T/N$ for some integer N with $\delta/4M \leqq T/N \leqq \delta/2M$. The probability in question is bounded by

$$\sum_{\substack{0 \leqq i < j \leqq N \\ j-i \geqq \eta\rho^{-1}}} P\left\{\left|\int_{i\rho}^{j\rho} \tilde{V}_\sigma^M d\sigma\right| \leqq \frac{\delta}{2} \text{ but } |\tilde{V}_{i\rho}^M| \leqq \frac{M}{2}, \right.$$
$$\left. \text{and } |\tilde{Z}_{i\rho}^M| \leqq TM\right\}$$
$$\leqq N^2 \max_{\substack{|t-s| \geqq \eta \\ 0 \leqq s,t \leqq T \\ |w| \leqq M/2, |y| \leqq TM}} P\left\{\left|\tilde{Z}_t^M - y\right| \leqq \frac{\delta}{2} \middle| \tilde{V}_s^M = w, \tilde{Z}_s^M = y\right\}$$
$$\leqq N^2 K\left(\frac{\delta}{2}\right)^d \text{(by (4.18))} \leqq K\left(\frac{4MT}{\delta}\right)^2 \left(\frac{\delta}{2}\right)^d.$$

For $d \geqq 3$ the last expression tends to zero with δ so that (4.2) follows from (4,19).

Proof of Lemma 7. Apart from going over to the diffusion $W_t \equiv \Gamma V_t$ and letting the origin play the role of ∞, the proof of this lemma is in [9], Chap. 4.5. It is based on the observation that if u is the positive function on $[0, 1]$ which solves

$$u\left(\frac{r^2}{2}\right) = 1 + 2\int_r^1 \frac{t}{C_-(t)} dt \int_t^1 \frac{C_-(s)}{A_+(s)} su\left(\frac{s^2}{2}\right) ds,$$

then

$$Y_t \equiv e^{-t\wedge\tau} u(|\Gamma V_{t\wedge\tau}|^2/2)$$

is a positive supermartingale, where $\tau = \inf\{t \geqq 0: |V_t| \geqq 1\}$, and on the fact that $u(r) \to \infty$ as $r \downarrow 0$ if (4.4) holds. (We have C_- in contrast to C_+ in [9] because our u is decreasing rather than increasing). □

Example a. We already remarked that if r_{ij} in (4.5) vanishes for $|y| \geqq L$, then (2.2) trivially holds. If we take in addition r_{ij} of the form (4.6) with $\rho(y, v) = \rho^*(|y|)$ then a_{ij} and b_i of (2.3) and (2.4) become

$$\begin{aligned} a_{ij}(v) &= \int_{-\infty}^{+\infty} dt\, r_{ij}(tv, v, v) \\ &= \sum_k \int_{-\infty}^{+\infty} dt \int_{\mathbb{R}^d} \rho^*_{ik}(|tv + z|)\overline{\rho^*_{jk}}(|z|)dz \\ &= \frac{1}{|v|}\sum_k \int_0^{\infty} dt \int_{\mathbb{R}^d} \rho^*_{ik}(|te + z|)\overline{\rho^*_{jk}}(|z|)dz, \\ b_i(v) &= \frac{1}{2}\sum_j \frac{\partial}{\partial v_j} a_{ij}(v). \end{aligned}$$

In this case $\mathscr{L}$ has the form (4.10). With this $\mathscr{L}$ (2.6) is valid by Remark 6. Condition (IV) can be verified in the same way as for the Gaussian examples of [6], Theorem 4, since (4.8) and (4.9) assure that

$$\left(\frac{\partial}{\partial y_k}\right)^2\left(\frac{\partial}{\partial y_l}\right)^2 r_{ii}(y) \text{ exists, and}$$

$$\left|\left(\frac{\partial}{\partial y_k}\right)^2\left(\frac{\partial}{\partial y_l}\right)^2 r_{ii}(0) - \left(\frac{\partial}{\partial y_k}\right)^2\left(\frac{\partial}{\partial y_l}\right)^2 r_{ii}(u)\right| = O\left(\log\frac{\partial}{|u|}\right)^{\alpha}. \quad \square$$

Example b. It is clear that any field F of the form (4.11), (4.12) is stationary in x, because of the translation invariance of the point process P_ρ. It is also clear that the randomness in $K(x)$ and hence in $F(x, v)$ depends only on the position of the points p_n within distance L of x and the corresponding $H^{(n)}$. This is so because all $H^{(n)}(x)$ vanish for $|x| \geqq L$. In particular $F(x_1, v_1)$ and $F(x_2, v_2)$ will be independent when $|x_1 - x_2| > 2L$. More generally $\mathscr{G}_{A_1}$ and $\mathscr{G}_{A_2}$ will be independent when $d(A_1, A_2) > 2L$ and hence $\beta(\rho) = 0$ for $\rho > 2L$. Formula (2.2) is again immediate.

Let us now specialize to (4.15) with the distribution of $H^{(0)}(x_1)$ and $H^{(0)}(x_2)$ invariant under the change $x_i \to Ox_i$, and (4.13), (4.14). Clearly, $E\{F(x)\} = 0$ by (4.13), so that (II) holds. (IV) is easily derived from (4.14), and (I) holds for F because the $H^{(n)}(x)$ are assumed to be $C^2(\mathbb{R}^d)$. As for (V), the coefficients $a_{ij}(v)$ and $b_i(i)$ of (2.3) and (2.4) now are

$$\begin{aligned} a_{ij}(v) &= \int_{-\infty}^{+\infty} dt \sum_{n,m} E\{H_i^{(n)}(p_n)H_j^{(m)}(tv + p_m)\} \\ &= \int_{-\infty}^{+\infty} dt \sum_n E\{H_i^{(n)}(p_n)H_j^{(n)}(tv + p_n)\} \quad \text{(by (4.13))} \\ &= \int_{-\infty}^{+\infty} dt \int_{\mathbb{R}^d} \rho dz E\{H_i^{(0)}(z)H_j^{(0)}(tv + z)\} \end{aligned}$$

$$= \frac{2\rho}{|v|} \int_0^\infty dt \int_{\mathbb{R}^d} E\{H_i^{(0)}(z) H_j^{(0)}(te+z)\} dz.$$

($v = |v| Oe$ for some orthogonal O), and

$$b_i(v) = \frac{1}{2} \sum_j \frac{\partial}{\partial v_j} a_{ij}(v).$$

This $\mathscr{L}$ is given by (4.6) and (2.6) is again guaranteed by Remark 6.

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