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A limit theorem for sums of i.i.d. random variables with slowly varying tail probability

By

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§1. Introduction.

Let $X_1, X_2, \cdots$ be nonnegative independent random variables with common distribution function $F(x)$ belonging to the domain of attraction of stable law with index α $(0<\alpha<1)$; i.e.,

$$1-F(x) \sim 1/(x^{\alpha}L(x)) \quad \text{as} \quad x \to \infty, \tag{1.1}$$

for some slowly varying $L(x)$ $(\geqq 0)$. Here $f(x) \sim g(x)$ means that $\lim f(x)/g(x)=1$. Let $S_n = X_1 + X_2 + \cdots + X_n$, $n=1, 2, \cdots$. It is well known as Skorohod's invariance principle that, for suitably chosen $v(t)$, $S_{[nt]}/v(n)$ converges in law to the one-sided stable process in the function space $D([0, \infty) \to \boldsymbol{R})$ endowed with J_1-topology (see [7] for the definition and [9] for the result).

In this paper we will consider the extreme case of (1.1) as $\alpha \downarrow 0$; we treat the case where

$$1-F(x) \sim 1/L(x) \quad \text{as} \quad x \to \infty. \tag{1.2}$$

As is well known, under condition (1.2), every linear normalization $a_n S_n + b_n$ leads to a degenerate limiting distribution. However, if we allow non-linear normalizations, we have

Darling's Theorem ([2]). *If* (1.2) *is satisfied, then*

$$\lim_{n \to \infty} P[\frac{1}{n} L(S_n) \leqq x] = e^{-1/x}, \quad x>0.$$

(In [2] some technical conditions are assumed but they may be removed.) The convergence of the semi-group of the process $t \leadsto \frac{1}{n} L(S_{[nt]})$ was obtained by S. Watanabe [10], and the purpose of this paper is to show the weak convergence of this process in J_1-topology. Our idea of the proof is quite different from that of Darling or Watanabe and so may be read independently.

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§2. Main theorem.

Let $\xi=(\xi(t))_{t\geq 0}$ be the symmetric Cauchy process such that $E[e^{i\theta\xi(t)}]=e^{-\pi t|\theta|}$, $\theta\in\boldsymbol{R}$. In other words, ξ is the Lévy process with Lévy measure $x^{-2}dx$. Define

$$m(t)=\max_{0<s\leqq t}\Delta\xi(s) \quad \text{where} \quad \Delta\xi(s)=\xi(s)-\xi(s-)\,. \tag{2.1}$$

Thus $m=(m(t))_{t\geq 0}$ ($m(0)=0$) is the maximum process of the Poisson point process with intensity measure $x^{-2}dx\,dt$, and therefore it is easy to compute the finite-dimensional marginal distributions; for $0\leqq t_1<\cdots<t_n$ and $0\leqq x_1<\cdots<x_n$,

$$P[m(t_1)\leqq x_1, \cdots, m(t_n)\leqq x_n] = F(x_1)^{t_1}F(x_2)^{t_2-t_1}\cdots F(x_n)^{t_n-t_{n-1}}\,, \tag{2.2}$$

where $F(x)=e^{-1/x}$, $x>0$. Processes with the property (2.2) are often referred to as *extremal processes* (see [3]). Our main result is

Theorem 2.1. *Let $L(x)\geqq 0$, $x\geqq 0$, be a nondecreasing function varying slowly at infinity. Then, under the assumption* (1.2),

$$\frac{1}{n}L(S_{[nt]})\xrightarrow{\mathcal{D}} m(t) \quad as \quad n\to\infty \quad in \quad D([0,\infty)\to\boldsymbol{R})$$

where $\xrightarrow{\mathcal{D}}$ denotes the weak convergence in J_1-topology.

Proof. Since the convergence of finite-dimensional marginal distributions are obtained by [10], it remains to prove the tightness of the processes. However, it does not seem easy to check the well-known conditions for tightness such as Chentsov's moment condition (see page 128 of [1]), so we will adopt a direct method: On the probability space where ξ (and hence m) is defined we will construct processes ζ_n, $n=1, 2, \cdots$ which are distributed like $(1/n)L(S_{[nt]})$, $n=1, 2, \cdots$ and which converge to m almost surely in J_1-topology.

Now let $\eta(t)=\sum_{s\leq t}(\Delta^+\xi(s))^2$ where $\Delta^+\xi(s)=\max\{\Delta\xi(s), 0\}$. In other words, $\eta(t)=\int_0^{t+}\int_{x>0}x^2\,N(du\,dx)$, $N(du\,dx)$ being the Poisson random measure defined by the jumps of the Cauchy process ξ. Clearly $\eta=(\eta(t))_{t\geq 0}$ is a one-sided stable process with index 1/2 and the Lévy measure is given by $\mu[x,\infty)=x^{-1/2}$, $x>0$. Observe that $m(t)$ may be expressed by using η instead of ξ;

$$m(t)=\{\max_{0\leqq s\leqq t}\Delta\eta(s)\}^{1/2}, \quad \Delta\eta(s)=\eta(s)-\eta(s-)\,. \tag{2.3}$$

Indeed this is immediate from (2.1) and the definition of η. Let $F_0(x)=P[\eta(1)\leqq x]$, $x\in\boldsymbol{R}$. It is well known that

$$1-F_0(x)\sim x^{-1/2} \quad \text{as} \quad x\to\infty\,. \tag{2.4}$$

This may be confirmed as follows. Since $\eta(n)/n^2$ clearly converges in law to $\eta(1)$, appealing to Theorem 4 in page 124 of [5] we have that for every $x>0$,

$$n\{1-F_0(n^2x)\} \to x^{-1/2} \quad \text{as} \quad n\to\infty,$$

which proves (2.4). We also define $\phi(x)=F^{-1}(F_0(x))$ $(x\in\boldsymbol{R})$. Throughout the inverse of a nondecreasing function is chosen so that it is right-continuous;

$$F^{-1}(x) = \inf\{u\colon F(u)>x\},\quad 0<x<1.$$

Let, for $n=1, 2, \cdots$,

$$\eta_{ni} = \eta\Big(\frac{i}{n}\Big)-\eta\Big(\frac{i-1}{n}\Big),\quad i = 1, 2, \cdots, \tag{2.5}$$

$$\zeta_n(t) = \frac{1}{n} L(\textstyle\sum_{i\leqq nt} \phi(n^2\eta_{ni})),\quad t\geqq 0, \tag{2.6}$$

$$m_n(t) = \begin{cases} (\max_{i\leqq nt} \eta_{ni})^{1/2}, & t\geqq 1/n \\ 0, & 0\leqq t<1/n. \end{cases} \tag{2.7}$$

We are now ready to explain the story of the proof. We will prove the following three.

(1) $\zeta_n = (\zeta_n(t))_{t\geqq 0}$ is identical in law to $\Big(\frac{1}{n} L(\sum_{i\leqq nt} X_i)\Big)_{t\geqq 0}$, $n = 1, 2, \cdots$.

(2) $m_n \to m$ $(in\ J_1)$ a.s.

(3) $\sup_{0\leqq t\leqq T} |\zeta_n(t)-m_n(t)| \to 0$, a.s., for every $T>0$.

Indeed, (2) and (3) imply that $\zeta_n\to m$ (in J_1) a.s., which together with (1) clearly proves our Theorem.

The proof of (1) and (2) are easy. To see (1) it suffices to show that $\phi(\eta(1))$ (and hence $\phi(n^2\eta_{ni})$) is distributed like X_1. However, this is obvious from the definition $\phi(x)=F^{-1}(F_0(x))$. In fact ϕ was defined so that (1) holds. To see (2) define $\eta_n(t)=\eta([nt]/n)$, $n=1, 2, \cdots$. Since $\eta_n(t)\to\eta(t)$ (in J_1), we have from the continuity theorem that

$$\max_{i\leqq nt} \eta_{ni} = \max_{0\leqq s\leqq t} \Delta\eta_n(t) \to \max_{0\leqq s\leqq s} \Delta\eta(t) \quad (\text{in } J_1).$$

Therefore, considering the square root of each side, we obtain from (2.3) and (2.7) that $m_n(t)\to m(t)$ (in J_1) a.s., which is the desired result.

For the proof of (3) we need the next two key lemmas.

Lemma 2.2. *For every $\lambda>0$ and $T>0$,*

$$\lim_{n\to\infty} \sup_{0\leqq x\leqq T} \Big|\frac{1}{n} L(\lambda\phi(n^2x^2))-x\Big| = 0.$$

Lemma 2.3. *For every $\varepsilon>0$ and $T>0$ ($\varepsilon<T$) and for almost sure ω, there exist K_1, $K_2>0$ and n_0 depending on $\{\varepsilon, T, \omega\}$ such that, for every $t\in[\varepsilon, T]$ and $n\geqq n_0$,*

$$K_1\,\phi(n^2\,m_n(t)^2)\leqq\sum_{i\leqq nt}\phi(n^2\,\eta_{ni})\leqq K_2\,\phi(n^2\,m_n(t)^2)\,. \tag{2.8}$$

We postpone the proof until next section and return to the proof of (3). Since $L(x)$ is nondecreasing, applying $L(\cdot)$ to each side of (2.8) and dividing by n, we obtain

$$\frac{1}{n}\,L(K_1\,\phi(n^2\,m_n(t)^2))\leqq\zeta_n(t)\leqq\frac{1}{n}\,L(K_2\,\phi(n^2\,m_n(t)^2))\,.$$

Therefore, we have

$$\begin{aligned}\sup_{\varepsilon\leqq t\leqq T}\,&|\zeta_n(t)-m_n(t)|\\ &\leqq\sup_{0\leqq t\leqq T}\max_{i=1,2}\,\Big|\frac{1}{n}\,L(K_i\,\phi(n^2\,m_n(t)^2)-m_n(t)\Big|\\ &\leqq\max_{i=1,2}\,\sup_{0\leqq x\leqq m_n(T)}\Big|\frac{1}{n}\,L(K_i\,\phi(n^2\,x^2))-x\Big|\end{aligned} \tag{2.9}$$

which converges to 0 a.s. by Lemma 2.2. and (2). Since $\zeta_n(t)$ and $m_n(t)$ are monotone in t, we also see

$$\begin{aligned}\lim_{\varepsilon\to0}\,&\limsup_{n\to\infty}\,\sup_{0\leqq t\leqq\varepsilon}|\zeta_n(t)-m_n(t)|\\ &\leqq\lim_{\varepsilon\to0}\,\limsup_{n\to\infty}\,(\zeta_n(\varepsilon)+m_n(\varepsilon))\\ &=\lim_{\varepsilon\to0}\,2m(\varepsilon)\,=\,0\ \text{ a.s.}\end{aligned}$$

This completes the proof of (3) and hence of Theorem 2.1.

Remark 2.4. We have a very simple proof for the special case where $L(x)=\log(x+1)$. Let $M_0=0$ and $M_n=\max(X_1,\cdots,X_n)$, $n\geqq1$. Then we have

$$\frac{1}{n}\,L(M_{[nt]})\leqq\frac{1}{n}\,L(S_{[nt]})\leqq\frac{1}{n}\,L(nM_{[nt]})\,. \tag{2.10}$$

Since $\log(nx)=\log n+\log x$, it is easy to see that the first and the third process (and hence the second) should have the same limiting processes. Thus our problem may be reduced to the study of M_n, which was treated by [4] and [6]. However, for general slowly varying $L(x)$, we cannot apply this argument. For example, let $L(x)=\exp\sqrt{\log(x+1)}\cdot$ $L(x)$ is a slowly varying function and we can show that the first and the second processes in (2.10) converge to $m(t)$ but the third to $\sqrt{e}\,m(t)$.

§3. Proofs of Lemmas.

Proof of Lemma 2.2. Since $\phi(x)=F^{-1}(F_0(x))$, we have $1-F(\phi(x))=1-F_0(x)$ provided that $F(x)$ is continuous. In general it holds that

$$1-F(\phi(x))\leqq 1-F_0(x)\leqq 1-F(\phi(x)/2)\,. \tag{3.1}$$

By (1.2) both $1-F(\phi(x))$ and $1-F(\phi(x)/2)$ are asymptotically equal to $1/L(\phi(x))$ as $x\to\infty$, while $1-F_0(x)\sim x^{-1/2}$ (see (2.4)). Therefore, (3.1) implies that $L(\phi(x))\sim x^{1/2}$, or equivalently, $L(\phi(x^2))\sim x$ as $x\to\infty$. This also proves that

$$\lim_{x\to\infty} L(\lambda\phi(x^2))/x=1,\ \lambda>0\,, \tag{3.2}$$

since $L(x)$ is a slowly varying function. Thus we have for every $\lambda>0$ and $x\geqq 0$,

$$\lim_{n\to\infty} \frac{1}{n} L(\lambda\phi(n^2 x^2)) = x\,. \tag{3.3}$$

Since $\frac{1}{n} L(\lambda\phi(n^2x^2))$ is monotone in x and the limiting function is continuous, the convergence in (3.3) is automatically uniform for x on every finite interval, which completes the proof.

Proof of Lemma 2.3. It is easy to find K_1; $K_1=1$ is the desired number. To find K_2, define $\tilde{L}(x)=F_0^{-1}(F(x))$ $(=\phi^{-1}(x))$. By (2.4) we see that $F_0^{-1}(x)\sim 1/(1-x)^2$ as $x\to 1-0$. Therefore, $\tilde{L}(x)\sim 1/(1-F(x))^2\sim L(x)^2$ as $x\to\infty$, from which it follows that $\tilde{L}(x)$ is also a slowly varying function, and so has the canonical representation $\tilde{L}(x)=c(x)\exp\int_1^x \frac{\varepsilon(t)}{t}\,dt$, where $c(x)\to c>0$ and $\varepsilon(x)\to 0$ as $x\to\infty$ (see [8] page 2). Therefore $\phi(x)=\tilde{L}^{-1}(x)$ may be expressed as

$$\phi(x) = \exp\int_1^{q(x)x} \frac{\tilde{\varepsilon}(t)}{t}\,dt \tag{3.4}$$

where $\tilde{\varepsilon}(x)\to\infty$ and $q(x)\to 1/c$ as $x\to\infty$. Using (3.4) we easily see that, for every $a>0$,

$$\lim_{n\to\infty} n^2\,\phi(n^2 a)/\phi(2n^2 a) = 0\,. \tag{3.5}$$

Now let $a=a(\omega)=m(\varepsilon)^2/3$. Note that $a>0$, *a.s.* Since $m_n(\varepsilon)\to m(\varepsilon)$ a.s., we may and do assume that there exists $n_1>0$ such that

$$m_n(t)^2\geqq m_n(\varepsilon)^2\geqq 2a,\quad t\geqq\varepsilon,\quad n\geqq n_1\,. \tag{3.6}$$

We divide $\sum_{i\leqq nt}\phi(n^2\eta_{ni})$ into three parts;

$$S_{1,n}(t) = \textstyle\sum_{i\leqq nt}\phi(n^2\eta_{ni})\,I(\eta_{ni}<n^{-2})\,,$$

$$S_{2,n}(t) = \textstyle\sum_{i\leqq nt}\phi(n^2\eta_{ni})\,I(n^{-2}\leqq\eta_{ni}\leqq a)\,,$$

$$S_{3,n}(t) = \textstyle\sum_{i\leqq nt}\phi(n^2\eta_{ni})\,I(\eta_{ni}>a)\,.$$

Here, $I(\cdot)$ denotes the indicator function. Since ϕ is monotone, we clearly have

(3.7) $$S_{1,n}(t)\leqq nT\phi(1), \quad t\leqq T.$$

On the other hand, using (3.4), we see that $\lim_{n\to\infty} n/\phi(2n^2a)=0$. Therefore, by (3.7) and (3.6) there exists $n_2>0$ such that

(3.8) $$S_{1,n}(t)\leqq\phi(2n^2 a)\leqq\phi(n^2 m_n(t)^2), \quad \varepsilon\leqq t\leqq T,\ n>n_2.$$

We next consider $S_{2,n}(t)$. If $n^{-2}\leqq x\leqq a$, then we have

(3.9) $$0\leqq\phi(n^2 x)\leqq\phi(n^2 a)\ n^2x.$$

Combining this with (3.5), we see that there exists $n_3>0$ such that

(3.10) $$\phi(n^2 x)\leqq\phi(2n^2 a)\ x, \quad n\geqq n_3.$$

Therefore, we obtain that for $\varepsilon\leqq t\leqq T$ and $n\geqq n_3$,

(3.11) $$\begin{aligned}0\leqq S_{2,n}(t)&\leqq\phi(2n^2 a)\textstyle\sum_{i\leqq nt}\eta_{ni}\\ &=\phi(2n^2 a)\ \eta([nt]/n)\leqq\phi(n^2 m_n(t)^2)\ \eta(T).\end{aligned}$$

In the last inequality we used (3.6). Finally let us consider $S_{3,n}(t)$. Notice that there are only finitely many $t\in[0, T]$ such that $\Delta\eta(t)>a$. Thus if we denote by $K_3=K_3(\omega)$ the number of such t's, then $\sum_{i\leqq nT} I(\eta_{ni}>a)=K_3$ for all sufficiently large n (*a.s.*). Thus we have

(3.12) $$\begin{aligned}0\leqq S_{3,n}(t)&\leqq K_3\max_{i\leqq nt}\phi(n^2\eta_{ni})\\ &=K_3\,\phi(n^2 m_n(t)^2), \quad \varepsilon\leqq t\leqq T,\ n\geqq n_4.\end{aligned}$$

See (2.7) for the last equality. Now combining (3.8), (3.11) and (3.12) we obtain

(3.13) $$\textstyle\sum_{i\leqq nt}\phi(n^2\eta_{ni})\leqq K_2\,\phi(n^2 m_n(t)^2), \quad \varepsilon\leqq t\leqq T,\ n\geqq n_0,$$

where $K_2=1+\eta(T)+K_3$ and $n_0=\max(n_2, n_3, n_4)$.
Thus Lemma 2.3 is proved.

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