A Limit Theorem on Subintervals of Interrenewal Times

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Consider a renewal process $\{X_n, n \geq 1\}$ for which there is defined an associated sequence of independent and identically distributed random variables $\{B_n, n \geq 1\}$ such that B_n is the length of a subinterval of X_n. We show that when attention is restricted only to B-intervals, the asymptotic joint distribution of the residual life and total life of a B-interval is that of a renewal process generated by $\{B_n, n \geq 1\}$.

IN THE STUDY of stochastic systems, successful analysis is often dependent upon the process being regenerative. As part of the analysis, it may be essential to focus attention on a particular event that occurs during the regeneration cycle. For example, in Oliver's [1964] derivation of the expected waiting time in the $M/G/1$ queue, it is necessary to calculate the expected remaining service time of the customer in service, if any, at an arrival epoch. By considering only those times when the server is working, Oliver implied that the service times generate a renewal process and so the remaining service time is the equilibrium excess random variable. (Terms are defined in Section 1.) Though this argument is not rigorous, it provides the correct expression for the expected remaining service times as confirmed by Wolff [1970] and Brumelle [1971]. Questions remain, however, as to what other characteristics of a renewal process are inherited by these service times and whether such characteristics are also inherited by other types of events within regeneration cycles.

Consider the general setting of a renewal process in which each renewal interval X_n contains a subinterval B_n such that $\{B_n, n \geq 1\}$ is a sequence of nonnegative independent and identically distributed (i.i.d.) random variables. We prove that the limiting distributions of excess (residual) life and total life (spread) of such subintervals are the equilibrium distributions for the corresponding quantities in a renewal process generated by $\{B_n, n \geq 1\}$. This is true even if B_n is dependent on another part of the regeneration cycle. Such a case arises in Kleinrock's ([1975],

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p. 222) busy period analysis of the $M/G/1$ queue. Therein he recursively defines, for each busy period, a sequence $\{Y_i, i \geq 0\}$ such that Y_0 is the service time of the customer who initiates the busy period and Y_i, $i \geq 1$ is the interval in which all customers who arrive during Y_{i-1} are served. Kleinrock asserts, without proof, that the joint density of excess and total life for Y_i is identical to the corresponding density in a renewal process with interrenewal times distributed as Y_i. This conclusion is confirmed by the theorem presented in this paper.

1. NOTATION

Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative i.i.d. random variables, representing the time between events in a renewal process. Let $S_0, S_1, \cdots$ be the times at which renewals occur, i.e., $X_n = S_n - S_{n-1}$,

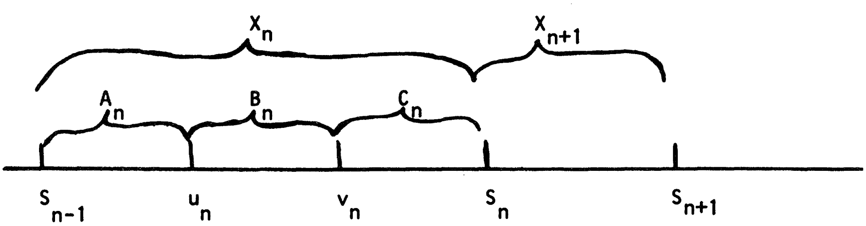

Figure 1

$n \geq 1$ for $S_0 < S_1 < S_2 \cdots$. Let $\{(A_n, B_n, C_n), n \geq 1\}$ be an associated sequence of i.i.d. nonnegative random elements of R^2 such that

$$A_n = u_n - S_{n-1}, \qquad B_n = v_n - u_n, \qquad C_n = S_n - v_n$$

where u_n and v_n are epochs within the nth renewal period, i.e.,

$$S_{n-1} \leq u_n \leq v_n \leq S_n.$$

Each renewal interval is divided into a beginning (A-interval), a middle (B-interval), and an end (C-interval) as shown in Figure 1. We allow A_i, B_i, and C_i to be dependent, but assume that the pairs (X_1, B_1), $(X_2, B_2) \cdots$ are independent.

Let $\{Y(t), t \geq 0\}$ be a continuous time stochastic process defined by

$$Y(t) = \begin{cases} 0 & \text{if} \quad t \in [S_{n+1}, u_n) \quad \text{for some } n; \\ 1 & \text{if} \quad t \in [u_n, v_n) \quad \text{for some } n, \\ 2 & \text{if} \quad t \in [v_n, S_n) \quad \text{for some } n, \end{cases} \tag{1}$$

$\{N(t), t \geq 0\}$ be the renewal process generated by $X_1, X_2, \cdots$, and

$$E'(t) = \begin{cases} S_{N(t)} + A_{N(t)+1} - t; & Y(t) = 0 \\ S_{N(t)} + A_{N(t)+1} + B_{N(t)+1} - t; & Y(t) = 1 \\ S_{N(t)+1} - t; & Y(t) = 2 \end{cases} \tag{2}$$

$$S'(t) = \begin{cases} A_{N(t)+1}; & Y(t) = 0 \\ B_{N(t)+1}; & Y(t) = 1 \\ C_{N(t)+1}; & Y(t) = 2. \end{cases} \tag{3}$$

Thus

$$\lim_{t\to\infty} P(E'(t) > a, S'(t) > b \mid Y(t) = 1)$$

is the asymptotic joint distribution function of the excess (residual) life and total life (spread) of a B-interval when attention is restricted only to B-intervals.

Consider a renewal process where the times between events are distributed as $\{B_n, n \geq 1\}$, and let $E(t)$ and $S(t)$ be the excess and spread at time t for this process. From Kleinrock (p. 172), the joint equilibrium density for these random variables is given by $dB(x)dy/E(B)$ where $B(x) = P(B_1 \leq x)$. Therefore,

$$\lim_{t\to\infty} P(E(t) > a, S(t) > b) = \int_b^\infty (x-a)dB(x)/E(B), \qquad 0 \leq a \leq b.$$

2. ASYMPTOTIC EXCESS AND SPREAD

THEOREM. *Let* $Y(t)$, $E'(t)$ *and* $S'(t)$ *be given by* (1), (2) *and* (3). *If* $F(x) = P\{X_i \leq x\}$ *is nonlattice and* $E(X) < \infty$, *then*

$$\lim_{t\to\infty} P\{E'(t) > a, S'(t) > b \mid Y(t) = 1\}$$

$$= \int_b^\infty (x-a)dB(x)/E(B), \qquad 0 \leq a \leq b.$$

Proof. Define the indicator function

$$I(t, a, b) = \begin{cases} 1 & \text{if } Y(t) = 1, E'(t) > a, S'(t) > b \\ 0 & \text{otherwise} \end{cases}$$

and the "rewards"

$$C_i = C_i(a, b) = \int_{S_{i-1}}^{S_i} I(t, a, b)dt.$$

Then

$$\lim_{t\to\infty}(E(\text{total reward by time } t)/t)$$

$$= \lim_{t\to\infty} \int_0^t P\{I(\tau, a, b) = 1\}d\tau/t = \lim_{t\to\infty} P[I(t, a, b) = 1]$$

since F is nonlattice and $I(t, a, b)$ lies in $D([0, \infty))$ (i.e., right-continuous and left-hand limits exist, see Miller [1972]). Therefore, we can apply the renewal-reward theorem (see, e.g., Ross [1970], p. 52) to get

$$\lim_{t\to\infty} P\{I(t, a, b) = 1\} = E\left[\int_0^{X_1} I(t, a, b)dt\right] \Big/ E(X).$$

It is clear that

$$\int_0^{X_1} I(t, a, b) = \begin{cases} B_1 - a & \text{if } B_1 \geq b \\ 0 & \text{otherwise} \end{cases}$$

so

$$\lim_{t\to\infty} P\{I(t, a, b) = 1\} = \int_b^{\infty} (x - a)dB(x)/E(X).$$

Now

$$\lim_{t\to\infty} P\{E'(t) > a, S'(t) > b \mid Y(t) = 1\}$$
$$= \lim_{t\to\infty} P\{I(t, a, b) = 1\}/\lim_{t\to\infty} P\{Y(t) = 1\}$$

and

$$\lim_{t\to\infty} P\{Y(t) = 1\} = E(B)/E(X),$$

hence

$$\lim_{t\to\infty} P\{E'(t) > a, S'(t) > b \mid Y(t) = 1\} = \int_b^{\infty} (x - a)dB(x)/E(B).$$

3. APPLICATIONS

Server Vacation Models

Levy and Yechiali [1975] study two $M/G/1$ systems in which the server leaves for a "vacation" whenever a busy period ends. In the first model, the server returns at the end of a single vacation and either begins serving customers who arrived during the vacation, or waits for the first customer to arrive. In the second model, if the server returns from a vacation to find the system empty, the server immediately takes another vacation and continues in this manner until there is at least one customer upon return. For each model, Levy and Yechiali use an extended Markov-chain representation to obtain the generating function of the number of customers in the system. This generating function is then used to derive the Laplace-Stieltjes transform of the waiting time and ultimately the expected waiting time in system.

Using the main result of this paper, these waiting time transforms (and the expected waits) can be obtained directly from probabilistic arguments. For example, consider Model 2 and let λ be the arrival rate, V the service time, and U the length of a single vacation. For any random variable Y, we denote its expectation by $E(Y)$.

In this system, the server takes successive vacations until returning to find at least one customer waiting. Therefore, whenever the server is not busy, the server is on vacation. Since the system is work-conserving, the steady-state probability that the server is busy is $\lambda E(V)$ as in the ordinary $M/G/1$ system. Let W_q be the steady-state waiting time in queue and N the number of customers seen by an arrival. Since Poisson arrivals see time averages,

$$E(W_q \mid N = n) = nE(V) + E(V_E)\cdot\lambda E(V) + E(U_E)(1 - \lambda E(V))$$

where V_E is the remaining service time of the customer in service, if any, and U_E is the remaining vacation time if any, at the arrival epoch. Unconditioning and using Little's formula,

$$E(W_q) = E(V_E)\lambda E(V)/(1 - \lambda E(V)) + E(U_E).$$

From the theorem, V_E and U_E are the equilibrium excess random variables for V and U. So

$$E(W_q) = E(V^2)\lambda E(V)/(2E(V)(1 - \lambda E(V))) + E(U^2)/(2E(U)),$$

and the mean wait in system is given by

$$E(W) = E(V) + \lambda E(V^2)/(2(1 - \lambda E(V))) + E(U^2)/(2E(U))$$

which corresponds to Equation 35 in Levy and Yechiali. Results for the other model can be similarly derived.

Approximation of $M/G/c$ Queues

One of the major difficulties in analyzing the $M/G/c$ queueing system is dealing with the joint distribution of the remaining service times of busy servers. This is apparent in two recent papers in which analysis is based on an approximation assumption designed to handle this difficulty.

Let the service time distribution be denoted by G and the equilibrium excess distribution of G be denoted by Ge. Nozaki and Ross [1978] obtain an approximation for average delay by assuming that at epochs when a customer enters service, the remaining service times of the services in progress, if any, are i.i.d. random variables with common distribution Ge. Tijms et al. [1980] use a recursive scheme at departure epochs to obtain approximations for the limiting probabilities of queue length. Their assumption is that at service completion epochs at which j, $i \leq j \leq c - 1$, customers are left behind, the remaining service times of the j customers being served are i.i.d. random variables with distribution Ge. For service completion epochs at which $j \geq c$ customers are left behind, they assume that the time until the next service completion has distribution $G^*(t) = G(ct)$. Though each paper provides motivation for its assumption, neither addresses the issue of what in the assumption is approximate and what is exact. Using the major result of this paper, the ambiguities are easily resolved.

Each of the assumptions involves three issues: the marginal distribution of each remaining service time, the times at which the process is observed, and the independence among the remaining service times. The first of these issues is resolved as a direct consequence of our theorem. By defining, for example, a sequence $\{B_n^{(k)}, n \geq 1\}$ for any $k \geq 1$ where $B_n^{(k)}$ is the kth service of the nth busy period, we can apply the theorem to obtain Ge as the limiting marginal distribution of the remaining service time of each server. Therefore this element of the assumptions is exact. The approximations result from using this equilibrium distribution at departure epochs which do not, in general, give rise to general-time probabilities, and from the independence assumption. This helps to clarify why the results can be expected to be more accurate in light traffic as noted in Nozaki and Ross, and when the number of servers increases to ∞, as noted in Timjs et al. In both of these cases, the probability of a queue forming decreases. One effect of this with respect to the assumption in Nozaki and Ross is that a greater proportion of the times at which customers enter service will be arrival epochs which, because of the Poisson assumption, give rise to equilibrium probabilities. The assumption in Timjs et al. also becomes more accurate in this respect since as the proportion of customers who have zero delay in queue increases, the system behaves more like an $M/G/\infty$ queue. This is significant since the departure process of the $M/G/\infty$ system is Poisson (see Gross and Harris [1974], p. 274) and therefore also results in general-time probabilities. In both assumptions as the probability of a queue decreases, the independence among the remaining service times increases. For the $M/G/\infty$ queue, where there is no queueing at all, the assumptions are equivalent and exact (see Takacs [1962], p. 161).

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Probability of Success in the Search for a Moving Target

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The probability of detection in the search for a randomly moving target is calculated for the case of a target whose motion is a diffusion process and known searcher path. The probability of detection can be calculated by solving a backward diffusion equation. Corwin [1980] gives a solution of the backward equation for a special case. In general, exact solutions do not exist and other methods are needed. In this paper, the backward equation is solved approximately by using a formal asymptotic method, valid when the intensity of the random motion is small. The general solution is illustrated for the case of spatially homogeneous drift and diffusion coefficients. In this case, the asymptotic solution can be evaluated analytically.

ALTHOUGH PROBLEMS of search for moving targets have received considerable attention in recent years, some apparently simple problems remained unsolved. These problems are actually not simple, and it is the motion of the target which makes them hard to solve. One of these problems is the calculation of the probability of detection in a search when the search path is specified. In this case, one does not try to find an "optimal" path, but gives a search path and then calculates the probability of detection at the end of the search. We shall find this probability by solving a backward diffusion equation. This procedure is not as removed from optimal search as it seems. First, once the probability of detection is known, the optimal path can be obtained by a nonlinear programming procedure (e.g. Ciervo [1976]). Second, it turns out that when studying optimal search problems, in order to solve the conditions giving an optimal path, one needs to solve the equation treated in this

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