The Annals of Statistics
1997, Vol. 25, No. 1, 105–137

A LIMIT THEORY FOR LONG-RANGE DEPENDENCE AND STATISTICAL INFERENCE ON RELATED MODELS[1]

BY YUZO HOSOYA

Tohoku University

This paper provides limit theorems for multivariate, possibly non-Gaussian stationary processes whose spectral density matrices may have singularities not restricted at the origin, applying those limiting results to the asymptotic theory of parameter estimation and testing for statistical models of long-range dependent processes. The central limit theorems are proved based on the assumption that the innovations of the stationary processes satisfy certain mixing conditions for their conditional moments, and the usual assumptions of exact martingale difference or the (transformed) Gaussianity for the innovation process are dispensed with. For the proofs of convergence of the covariances of quadratic forms, the concept of the multiple Fejér kernel is introduced. For the derivation of the asymptotic properties of the quasi-likelihood estimate and the quasi-likelihood ratio, the bracketing function approach is used instead of conventional regularity conditions on the model spectral density.

0. Introduction. This paper investigates vector-valued stationary processes with long-range dependence which possess a variety of singularities not necessarily limited to zero frequency, giving a limit theory for quadratic forms of observations from those processes. Then it considers the statistical inference based on the quasi-likelihood function giving the asymptotic properties for the quasi-maximum likelihood (QML) estimate and the quasi-likelihood ratio (QLR) statistics based on the limit theory under very general conditions. The asymptotic results obtained reveal a particular feature of long-range dependence whose modeling produces different asymptotics for related statistics based on the quasi-likelihood function.

A general framework for the asymptotic theory for parameter estimation and testing for short-range dependent stationary time-series models was given by Whittle (1952). Dealing with a linear scalar-valued process $\{z_t,\ t \in J\}$ given by

$$z_t = \sum_{j=0}^{\infty} \alpha_j(\theta) e_{t-j},$$

where $\{e_t\}$ is an i.i.d. $(0, \sigma^2)$ process with finite fourth-order moment, he proposed the minimizing value $\hat{\theta}$ of $\int_{-\pi}^{\pi} I_n(z, \omega)/f(\omega; \theta)\, d\omega$ for the estimate of θ, where $f(\omega; \theta)$ is a spectral density of the process $\{z_t\}$ and $I_n(z, \omega)$ is

Received August 1994; revised March 1996.

[1]Research supported in part by the Ministry of Education of Japan, Grant C07630019.

AMS 1991 *subject classifications*. Primary 62M10, 62M15, 62E30; secondary 60G10, 60F05.

Key words and phrases. Asymptotic theory, central limit theorems, mixing conditions, martingale differences, serial covariances, quadratic forms, bracketing function, long-range dependence, maximum likelihood estimation, likelihood ratio test.

the periodogram for $z_1, \ldots, z_n$. Assuming that σ^2 does not depend on θ, he then proposed $\hat{\sigma}^2 = \int_{-\pi}^{\pi} I_n(z, \omega)/g(\omega; \hat{\theta})\, d\omega$ for the estimate of σ^2, where $g(\omega; \theta) = |\sum_{j=0}^{\infty} \alpha_j(\theta) e^{ij\omega}|^2$. Hannan (1973) gave the asymptotic properties of $\hat{\theta}$ and $\hat{\sigma}^2$ under explicit regularity conditions. As for the case where σ^2 depends on θ, Hosoya (1974) and Dzhaparidze (1974) proposed the estimate which minimizes $\int_{-\pi}^{\pi} \{\log f(\omega; \theta) + I_n(z, \omega)/f(\omega; \theta)\}\, d\omega$ and gave the asymptotic properties. Those minimized objective functions approximate the normal log-likelihood up to a constant-order term, and, if $\{e_t\}$ is Gaussian, the minimizing value $\hat{\theta}$ is first-order efficient and the $\sqrt{n}(\hat{\theta} - \theta)$ is asymptotically normal $N(0, J(\theta)^{-1})$, where $J(\theta) = \lim_{n\to\infty}(1/n)J_n(\theta)$ for the Fisher information matrix $J_n(\theta)$ based on $z_1, \ldots, z_n$. The Whittle result was extended for vector-valued processes by Dunsmuir and Hannan (1976) and Dunsmuir (1979). Furthermore, assuming milder mixing conditions on the innovation process $\{e_t\}$, Hosoya and Taniguchi (1982) gave the asymptotic theory for the estimate which minimizes $\int_{-\pi}^{\pi} \{\log \det f(\omega; \theta) + \operatorname{tr} f(\omega; \theta)^{-1} I_n(z, \omega)\}\, d\omega$, where $f(\omega; \theta)$ and $I_n(z, \omega)$ denote this time a spectral density matrix and the periodogram matrix, respectively. That estimate will henceforth be called the quasi-maximum likelihood (QML) estimate.

A number of authors noted that long-range dependent models are imperative for some empirical time series [see, e.g., Mandelbrot and Wallis (1969), Granger and Joyeux (1980) and Cox (1984, 1991)], and out of the necessity to develop statistical methods to deal with them, some authors have extended the asymptotic theory of the Whittle estimate to long-range dependent stationary processes; Yajima (1985), Fox and Taqqu (1986) and Dahlhaus (1989) investigated Gaussian processes, and Giraitis and Surgailis (1990) dealt with non-Gaussian processes. Their investigations, however, are limited to the case where $\{z_t\}$ is a scalar-valued process whose spectral density is given by

$$f(\omega; \theta) = |\omega|^{-\beta(\theta)} g(\omega; \theta), \qquad 0 < \beta(\theta) < 1, \tag{0.1}$$

for regular $g(\omega; \theta)$, and the e_t are i.i.d. Recently Heyde and Gay (1993) dealt with a vector-valued non-Gaussian case under milder conditions on the innovation process, and Robinson (1995) proposed a semiparametric approach based on the local quasi-likelihood integration.

The approach of this paper has the following distinctive features: (1) It has a framework general enough to deal with a variety of multiple singularities of the spectral density not restricted to the type given in (0.1). (2) The innovation process is assumed only to satisfy a set of mixing conditions with respect to conditional moments (see Assumption A below) and does not require the assumption of Gaussianity (or contemporaneously transformed Gaussianity) or of martingale differences, and the limit theorems for quadratic forms (in particular, the convergence of the covariances) are shown under mild conditions. (3) The asymptotic theory of the QML estimate is based on the bracketing function method, and weaker regularity conditions for the spectral density matrix $f(\omega; \theta)$ with respect to θ are employed in contrast to the conventional approaches. As for the bracketing function approach, Daniels (1961), Huber

(1967) and Pollard (1985) dealt with the maximum likelihood estimate for i.i.d. sequences of observations, and Hosoya (1989a) dealt with the QML for short-range dependent stationary processes. While investigating multivariate non-Gaussian stationary processes, Heyde and Gay (1993) dealt with a limited case where Assumption E or F of this paper holds and also conventional regularity conditions for spectral density hold. Consequently, their asymptotic theory of the QML estimate does not reveal the role of the fourth-order cumulants.

The paper proceeds as follows. Section 1 is about limit theorems of quadratic forms. Theorem 1.1 shows the convergence of their expectation. Fox and Taqqu (1987) showed that, in case the weight function of a quadratic form effectively annihilates the singularities of the spectral density at the origin in the frequency domain, the clt of quadratic forms holds for a broad class of long-range dependent Gaussian stationary processes. Theorem 1.2 extends their idea to a general setup where the stationary process need not be Gaussian and the singularities of the spectral density matrix are not limited to the origin. In order to prove that theorem, the concept of the multiple Fejér kernel is effectively used in Lemma 1.1, which deals with the convergence of the covariances of quadratic forms. Employing the bracketing condition approach in place of the usual stringent regularities of the spectral density, Section 2 investigates the asymptotic properties of the QML estimate and the quasi-likelihood ratio (QLR) statistics in the presence of long-range dependent fourth-order stationary processes. Theorem 2.1 pertains to consistency of the estimates and Theorem 2.2 applies to the asymptotic normality. In both theorems the true structure of the observed process need not belong to a fitted parametric model. A similar result was given already in Hosoya (1989a), which, however, dealt only with short-range dependent processes. Theorem 2.3 gives a version of Theorem 2.2 under specific conditions so that the asymptotic theory of the QML estimate is in conformity with the Whittle result. A point to be emphasized is that, for non-Gaussian long-range dependent processes, one of those conditions (Assumption E) is likely to be violated except for such specific processes as the fractional ARIMA. Based on that theorem, Theorem 2.4 derives a nested χ^2 distribution of a set of QLR statistics. Section 3 gives the proofs of the lemmas and theorems.

Throughout the paper J denotes the set of all integers and L denotes the backward shift operator, that is, $Lx(t) = x(t-1)$; $\delta(\cdot, \cdot)$ signifies the indicator function such that $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ otherwise. Here I_p is the identity matrix of order p. The conjugate transpose of a matrix A is denoted A^* and the notation is retained also for the transpose of real A. For a square matrix A, $\det A$ and $\operatorname{tr} A$ imply the determinant and trace of A, respectively. The L^p-norm of a complex-valued function g on $(-\pi, \pi]$ is denoted by $||g||_p$, namely, $||g||_p = [\int_{-\pi}^{\pi} |g(\omega)|^p \, d\omega]^{1/p}$, and $c_1, c_2, \ldots$ denote generic, positive constants pertaining to each context of the proofs. For subscripted symbols A_{ij}, B_i, the symbols A or B without subscripts imply either a matrix or a column vector with components A_{ij} or B_i, respectively, unless otherwise indicated.

1. Limit theorems of quadratic forms. Let $\{z(t);\, t \in J\}$ be the vector-valued linear process

$$z(t) = \sum_{j=0}^{\infty} G(j)e(t-j), \qquad t \in J, \tag{1.1}$$

where the $z(t)$'s are q-vectors and the $e(t)$'s are p-vectors such that $E\{e(m)e(n)^*\} = \delta(m,n)K$ for K a nonsingular $p \times p$ matrix; the matrices $G(j)$ are $q \times p$ and the components of z, e and G are all real. This representation accommodates a rectangular G, which is useful for a state-space-type model. Assume throughout the paper that

$$\sum_{j=0}^{\infty} \operatorname{tr} G(j)KG(j)^* < \infty,$$

so that the process $\{z(t)\}$ is a second-order stationary process and has a spectral density matrix $f(\omega)$ representable as

$$f(\omega) = \frac{1}{2\pi} k(\omega)Kk(\omega)^*, \qquad -\pi < \omega \le \pi,$$

where $k(\omega) = \sum_{j=0}^{\infty} G(j)e^{i\omega j}$.

This section investigates the limiting properties of the quadratic form of the form $\int_{-\pi}^{\pi} \operatorname{tr}\{g(\omega)I_n(z,\omega)\}\, d\omega$, where $g(\omega)$ is a $q \times q$ matrix-valued function with complex-valued components $g_{\alpha\beta}(\omega)$, $1 \le \alpha, \beta \le q$, such that $g_{\alpha\beta}(\omega) = \overline{g_{\alpha\beta}(-\omega)}$ and $g(\omega) = g^*(\omega)$, and $I_n(z,\omega)$ is the periodogram matrix defined by

$$I_n(z,\omega) = w_n(\omega)w_n(\omega)^*, \qquad -\pi < \omega \le \pi,$$

where $w_n(\omega)$ is the finite Fourier transform defined by

$$w_n(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{n} z(t)e^{it\omega}.$$

The following theorem gives a condition for asymptotic unbiasedness of $\int_{-\pi}^{\pi} \operatorname{tr}\{g(\omega)I_n(z,\omega)\}\, d\omega$.

THEOREM 1.1. *Suppose that the pair* (g, f) *satisfies* $\operatorname{tr}\{g \cdot f\} \in L^u$ *for some* u, $1 < u \le 2$, *and suppose that there exists* $c > 0$ *such that*

$$\sup_{|\lambda|<\varepsilon} \| \operatorname{tr}[g(\cdot)\{f(\cdot) - f(\cdot - \lambda)\}]\|_u = O(\varepsilon^c) \tag{1.2}$$

as $\varepsilon \to 0$. *Then*

$$\int_{-\pi}^{\pi} \operatorname{tr}[g(\omega)E\{I_n(z,\omega)\} - g(\omega)f(\omega)]\, d\omega = O(n^{-c}).$$

Assume in what follows that the process $\{z(t)\}$ in (1.1) is full rank and linearly regular; therefore, $\int_{-\pi}^{\pi} \log\det f(\omega)\, d\omega > -\infty$, so that the process $\{z(t)\}$ is nondeterministic. Moreover, assume that $\{e(t)\}$ is fourth-order stationary and that

$$\sum_{t_1, t_2, t_3=-\infty}^{\infty} | \tilde{Q}^e_{\beta_1,\dots,\beta_4}(t_1, t_2, t_3) | < \infty, \tag{1.3}$$

where $\tilde{Q}^e_{\beta_1,\dots,\beta_4}(t_1, t_2, t_3)$ is the joint fourth cumulant of $e_{\beta_1}(t), e_{\beta_2}(t+t_1), e_{\beta_3}(t+t_2), e_{\beta_4}(t+t_3)$, so that the process $\{e(t)\}$ has a fourth-order spectral density $Q^e_{\beta_1,\dots,\beta 4}(\omega_1, \omega_2, \omega_3)$ given by

$$\begin{aligned} &Q^e_{\beta_1,\dots,\beta 4}(\omega_1, \omega_2, \omega_3) \\ &\quad = \frac{1}{(2\pi)^3} \sum_{t_1, t_2, t_3=-\infty}^{\infty} \exp\{-i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)\} \tilde{Q}^e_{\beta_1,\dots,\beta_4}(t_1, t_2, t_3). \end{aligned}$$

In contrast to Heyde and Gay (1993), Assumptions A(i) and (ii) are imposed on $\{e(t)\}$ to ensure the central limit theorem of quadratic forms, where their assumption of exact martingale difference is replaced by weaker conditional mixing conditions and also their ergodicity assumption is replaced by the Lindeberg-type condition A(ii). Assumption A(iii) is used for the convergence of the covariances of quadratic forms.

ASSUMPTION A. (i) There exists $\varepsilon > 0$ such that, for any $t < t_1 \le t_2 \le t_3 \le t_4$ and for each β_1, β_2,

$$\operatorname{Var}[E\{e_{\beta_1}(t_1)e_{\beta_2}(t_2)|\mathscr{B}(t)\} - \delta(t_1 - t_2, 0)K_{\beta_1\beta_2}] = O\{(t_1 - t)^{-2-\varepsilon}\},$$

and also

$$\begin{aligned} &E|E\{e_{\beta_1}(t_1)e_{\beta_2}(t_2)e_{\beta_3}(t_3)e_{\beta_4}(t_4)|\mathscr{B}(t)\} - E\{e_{\beta_1}(t_1)e_{\beta_2}(t_2)e_{\beta_3}(t_3)e_{\beta_4}(t_4)\}| \\ &\quad = O\{(t_1 - t)^{-1-\varepsilon}\} \end{aligned}$$

uniformly in t, where $\mathscr{B}(t)$ is the σ-field generated by $\{e(s); s \le t\}$.

(ii) For any $\varepsilon > 0$ and for any integer $M \ge 0$, there exists $B_\varepsilon > 0$ such that

$$E[T(n, s)^2\{T(n, s) > B_\varepsilon\}] < \varepsilon$$

uniformly in n, s, where

$$T(n, s) = \left[\sum_{\alpha, \beta=1}^{p} \sum_{r=0}^{M} \left\{ \sum_{t=1}^{n} (e_\alpha(t + s)e_\beta(t + s + r) - K_{\alpha\beta}\delta(0, r))/n^{1/2} \right\}^2 \right]^{1/2},$$

and $\{T(n, s) > B_\varepsilon\}$ is the indicator, which is equal to 1 if $T(n, s) > B_\varepsilon$ and equal to 0 otherwise.

(iii) Each component $Q^e_{\beta_1,\dots,\beta_4}(\omega_1, \omega_2, \omega_3)$ is uniformly γ-Lipschitz for some $\gamma > 0$; namely,

$$|Q^e_{\beta_1,\dots,\beta_4}(\omega_1 + \varepsilon_1, \omega_2 + \varepsilon_2, \omega_3 + \varepsilon_3) - Q^e_{\beta_1,\dots,\beta_4}(\omega_1, \omega_2, \omega_3)| < \left\{\max_i |\varepsilon_i|\right\}^\gamma$$

uniformly in $\omega_1, \omega_2, \omega_3$.

Hosoya and Taniguchi (1982, 1993) and Hosoya (1989a) showed the asymptotic convergence of the covariances of quadratic forms

$$\int_{-\pi}^{\pi} \mathrm{tr}\{g(\omega) I_n(z,\omega)\}\, d\omega$$

and a central limit theorem in the case where the spectral density matrix $f(\omega)$ is square-integrable under an assumption weaker than Assumption A. In order to prove the corresponding results on the quadratic forms for long-range dependent $\{z(t)\}$ with possibly non-square-integrable $f(\omega)$, this paper imposes a condition on g which has the effect of annulling the singularities of the spectral density f, and, by exploiting that effect, modifies the proofs in the preceding papers so as to apply to long-range dependent processes. For that purpose, the following condition seems pertinent.

CONDITION B. The pair $\{g^{(1)}(\omega), k^{(1)}(\omega)\}$ of complex-valued functions satisfies the following:

(i) $g^{(1)}$ is uniformly γ-Lipschitz for some $\gamma > 0$, that is, $|g^{(1)}(\omega) - g^{(1)}(\omega + \varepsilon)| < |\varepsilon|^{\gamma}$ uniformly in ω;

(ii) there exists $u > 1$ such that $\int_{-\pi}^{\pi} |k^{(1)}(\omega)|^{2u}\, d\omega < \infty$;

(iii) there exists $\gamma_1 > 0$ such that

$$\sup_{|\varepsilon|\le\varepsilon_1} ||g^{(1)}(\cdot)|k^{(1)}(\cdot+\varepsilon) - k^{(1)}(\cdot)|^2||_2 = O(|\varepsilon_1|^{\gamma_1});$$

(iv) $||g^{(1)}|k^{(1)}|^2||_2 < \infty$.

The following example illustrates how Condition B is justified for long-range dependent processes.

EXAMPLE 1.1. Suppose $\{z(t)\}$ is a scalar-valued fractionally integrated process:

$$(1-L)^d z(t) = e(t), \qquad 0 < d < 1/2,$$

where $\{e(t)\}$ is a white-noise process with mean 0 and with unit variance. In this case the frequency-response function $k(\omega; d)$ and the spectral density $f(\omega, d)$ are respectively given by

$$k(\omega; d) = (1 - e^{i\omega})^{-d}, \qquad f(\omega; d) = \frac{1}{2\pi}|k(\omega; d)|^2.$$

Set $h(\omega; d) = \partial f(\omega; d)^{-1}/\partial d = \{4\sin^2(\omega/2)\}^d \log\{4\sin^2(\omega/2)\}$ for $\omega \neq 0$ and set $h(\omega; d) = 0$ for $\omega = 0$. Now consider the pair $\{h(\omega; d), k(\omega; d)\}$ for a fixed d, $0 < d < 1/2$. It is evident that the pair satisfies Condition B(ii). Set $s(\omega) = 4\sin^2(\omega/2)$ so that $h(\omega; d) = s(\omega)^d \log s(\omega)$. Partition the torus $(-\pi, \pi]$ into two subsets $\Omega_1 = \{\omega \in [-2\varepsilon, 2\varepsilon]\}$ and $\Omega_2 = \{\omega \notin [-2\varepsilon, 2\varepsilon]\}$. If $\omega \in \Omega_1$, $|h(\omega; d) - h(\omega + \varepsilon; d)| \le |h(\omega; d)| + |h(\omega + \varepsilon; d)| \le \varepsilon^{c_1}$, due to the boundedness of $|s(\omega)|^{c_2} \log\{s(\omega)\}$ for any small $c_2 > 0$. On the other hand, if $\omega \in \Omega_2$, since $s(\omega)^{c_3}$ is uniformly Lipschitz for any $c_3 > 0$, so is the product in view of the boundedness of $|s(\omega)|^{c_2} \log\{s(\omega)\}$. Thus Condition B(i) is satisfied.

In order to show Condition B(iii), $|\varepsilon|$ is assumed, without loss of generality, to be sufficiently small so that $\varepsilon + a > 0$ for some a such that

$$\int_{-\pi}^{\pi} |\log s(\omega)|^4 |s(\omega)|^{-4a}\, d\omega < \infty,$$

which is possible because both $\int_{-\pi}^{\pi} |\log s(\omega)|^8\, d\omega$ and $\int_{-\pi}^{\pi} |s(\omega)|^{-8a}\, d\omega$ are finite if a is small enough. Then Condition B(iii) is satisfied, since

$$\begin{aligned}
&\int_{-\pi}^{\pi} \{|h(\omega;\, d)|\, |k(\omega;\, d+\varepsilon) - k(\omega;\, d)|^2\}^2\, d\omega \\
&\quad\le \int_{-\pi}^{\pi} \{|s(\omega)^d \log s(\omega)|\, |s(\omega)^{-d+\varepsilon} - s(\omega)^{-d}|\}^2\, d\omega \\
&\quad= \int_{-\pi}^{\pi} |\log s(\omega) s(\omega)^{-a}|^2 |s(\omega)^{a+\varepsilon} - s(\omega)^{a}|^2\, d\omega \\
&\quad\le c_3 \varepsilon^2.
\end{aligned}$$

We have the last inequality in view of the Schwarz inequality and the Taylor expansion of $s(\omega)^{a+\varepsilon}$ around $\varepsilon = 0$.

LEMMA 1.1. *Suppose that each pair of* $\{g^{(1)}_{\alpha_2\alpha_1}, k_{\alpha\beta}\}$, $\alpha = \alpha_1$ *or* α_2, *and* $\{g^{(2)}_{\alpha_4\alpha_3}, k_{\alpha\beta}\}$, $\alpha = \alpha_3$ *or* α_4, $\beta = 1, \dots, p$, *satisfies Condition* B *and suppose that Assumption* A(iii) *holds. Then we have*

$$\begin{aligned}
&\lim_{n\to\infty} n \operatorname{Cov}\left\{\int_{-\pi}^{\pi} g^{(1)}_{\alpha_2\alpha_1}(\omega) I_{\alpha_1\alpha_2}(z,\omega)\, d\omega,\ \int_{-\pi}^{\pi} g^{(2)}_{\alpha_4\alpha_3}(\omega) I_{\alpha_3\alpha_4}(z,\omega)\, d\omega\right\} \\
&\quad= 2\pi \int_{-\pi}^{\pi} g^{(1)}_{\alpha_2\alpha_1}(\omega)\overline{g^{(2)}_{\alpha_4\alpha_3}(\omega)} f_{\alpha_1\alpha_3}(\omega)\overline{f_{\alpha_2\alpha_4}(\omega)}\, d\omega \\
&\qquad+ 2\pi \int_{-\pi}^{\pi} g^{(1)}_{\alpha_2\alpha_1}(\omega)\overline{g^{(2)}_{\alpha_4\alpha_3}(-\omega)} f_{\alpha_1\alpha_4}(\omega)\overline{f_{\alpha_2\alpha_3}(\omega)}\, d\omega \\
&\qquad+ 2\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g^{(1)}(\omega_1) g^{(2)}(-\omega_2) Q^{z}_{\alpha_1,\dots,\alpha_4}(\omega_1, \omega_2, -\omega_2)\, d\omega_1\, d\omega_2.
\end{aligned} \tag{1.4}$$

REMARK 1.1. Fox and Taqqu (1987) gave a convergence result (1.4) in case g and f are scalar-valued and the product gf is bounded. However, since they exploit the Gaussian moment properties in such an essential way, their approach does not seem to allow straightforward extension to non-Gaussian cases.

REMARK 1.2. Heyde and Gay (1993) gave a result similar to Lemma 1.1, but without an explicit proof necessary for that. Although they claim the relationship

$$\lim_{T\to\infty} TEG_{T,X}(\theta) G'_{T,X}(\theta) = 4\pi \int_{-\pi}^{\pi} f_X^2(\omega,\theta) g(\omega,\theta) g'(\omega,\theta)\, d\omega,$$

to use their notation on pages 177–178, that does not seem to follow from their Lemma 1.

Since the process $\{z(t)\}$ is assumed full rank and linearly regular, the spectral density f has a factorization

$$f(\omega) = \Gamma(e^{-i\omega})\Gamma(e^{-i\omega})^*,$$

where $\Gamma(e^{-i\omega})$ is the boundary value of a $q \times q$ matrix-valued analytic function $\Gamma(z)$ in the unit disk [see Rozanov (1967)]. Using this Γ, set $k'(\omega) = \Gamma^{-1}(e^{-i\omega})k(\omega)$ and $g'(\omega) = \Gamma(e^{-i\omega})^* g(\omega)\Gamma(e^{-i\omega})$. Let M be any integer such that $M \geq 1$.

THEOREM 1.2. *Suppose that the process* $\{e(t)\}$ *satisfies Assumption* A *and suppose that all the pairs* $\{g^{(j)}_{\alpha_1\alpha_2}, k_{\alpha\beta}\}$ *and* $\{g'^{(j)}_{\alpha_1\alpha_2}, k'_{\alpha\beta}\}$, $\alpha_1, \alpha_2 = 1, \dots, q$, $\alpha = \alpha_1$ *or* α_2, $\beta = 1, \dots, p$, $j = 1, \dots, M$, *satisfy Condition* B. *Moreover, assume the pair* $(g^{(j)}, f)$ *satisfies* (1.2) *for some* $c > 1/2$. *Then, as* $n \to \infty$,

$$\text{(i)} \qquad \int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)I_n(z,\omega)\}\,d\omega \to \int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)f(\omega)\}\,d\omega$$

in probability.

(ii) *The quantities*

$$\sqrt{n}\int_{-\pi}^{\pi} \operatorname{tr}[g^{(j)}(\omega)\{I_n(z,\omega) - f(\omega)\}]\,d\omega, \qquad j = 1, \dots, M,$$

have, asymptotically, a jointly normal distribution with zero mean vector and covariance matrix whose (j, l) *element is*

$$\begin{aligned}
&4\pi\int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)f(\omega)g^{(l)}(\omega)f(\omega)\}\,d\omega\\
&\quad + 2\pi\sum_{\alpha_1,\dots,\alpha_4=1}^{q}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g^{(j)}_{\alpha_1\alpha_2}(\omega_1)g^{(l)}_{\alpha_3\alpha_4}(\omega_2)Q^z_{\alpha_1,\dots,\alpha_4}(-\omega_1,\omega_2,-\omega_2)\,d\omega_1\,d\omega_2,
\end{aligned}$$

where the second member above is equal to

$$\begin{aligned}
2\pi\sum_{\beta_1,\dots,\beta_4=1}^{p}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}&[k^*(\omega_1)g^{(j)}(\omega_1)k(\omega_1)]_{\beta_1\beta_2}[k^*(\omega_2)g^{(l)}(\omega_2)k(\omega_2)]_{\beta_3\beta_4}\\
&\times Q^e_{\beta_1,\dots,\beta_4}(\omega_1,\omega_2,-\omega_2)\,d\omega_1\,d\omega_2.
\end{aligned}$$

REMARK 1.3. Giraitis and Surgailis (1990) gave the clt for the quadratic forms for the case where $\{z(t)\}$ is a scalar-valued process and the innovations $\{e(t)\}$ are i.i.d. without the assumption of Gaussianity. However, they deal only with the singularity of the type (0.1) with respect to long-range dependency and besides they assume the convergence in (1.4) instead of establishing it.

REMARK 1.4. There are cases where, even if the spectral density f is not square-integrable or the pair (g, f) does not satisfy Condition B, the clt of serial covariances or quadratic forms, respectively, is still valid. See Hosoya (1993).

2. The quasi-likelihood approach to statistical inference. This section applies Theorems 1.1 and 1.2 for the derivation of the asymptotic properties of the QML estimate and the QLR statistics in long-range dependent situations. The results are given for a somewhat wider class of statistics than required. The asymptotic theory is based on the bracketing function method.

2.1. *The quasi-likelihood function.* For the purpose of statistical inference on a q-dimensional stationary process $\{z(t), t \in J\}$ based on its finite realization $z(1), \dots, z(n)$ when the generating mechanism is unknown, suppose that a parametric model which is structured by

$$z(t) = \sum_{j=0}^{\infty} G(j, \psi)e(t-j), \qquad t \in J,$$

is fitted where the matrices $G(j, \psi)$ and the vectors $e(t)$ are of the same size as in (1.1), $E(e(t)) = 0$ and $E(e(t)e(s)^*) = \delta(t, s)K(\psi)$. Suppose that $\psi \in \Psi$, where Ψ is a compact subset of R^s with nonempty interior. The coefficient matrices $G(j, \psi)$ are assumed to satisfy $\sum_{j=0}^{\infty} \operatorname{tr} G(j, \psi)K(\psi)G(j, \psi)^* < \infty$ and thus the fitted model process has a spectral density

$$f(\omega; \psi) = \frac{1}{2\pi} k(\omega; \psi)K(\psi)k(\omega; \psi)^*, \qquad -\pi < \omega \le \pi,$$

where $k(\omega; \psi) = \sum_{j=0}^{\infty} G(j, \psi)e^{i\omega j}$. For the sole purpose of deriving the quasi-likelihood function, assume that the process $\{z(t)\}$ is Gaussian. Choose the frequencies ω_j, $j = 1, \dots, n$, equispaced in the torus $(-\pi, \pi]$ in such a way that $f(\omega)$ is continuous at $\omega = \omega_j$. Then the finite Fourier transforms $w_n(\omega_j)$, $j = 1, \dots, n$, have a complex-valued multivariate normal distribution and for large n they are approximately independent, each with probability density function

$$\pi^{-2}\{\det f(\omega_j; \psi)\}^{-1/2} \exp\left[-\tfrac{1}{2} \operatorname{tr}\{f^{-1}(\omega_j; \psi)w_n(\omega_j)w_n(\omega_j)^*\}\right], \qquad j = 1, \dots, n$$

[see Hannan (1970), pages 224 and 225]. Since $w_n(\omega_j)$, $j = 1, \dots, n$, constitute a sufficient statistic for ψ, an approximate log-likelihood function of ψ based on $z(1), \dots, z(n)$ is given, up to constant multiplication, by

$$-\sum_{j=1}^{n} [\log \det f(\omega_j; \psi) + \operatorname{tr} f^{-1}(\omega_j; \psi)I_n(z, \omega_j)]. \tag{2.1}$$

In integral form, (2.1) has the expression

$$\bar{L}_n(\psi) = -n\left[\int_{-\pi}^{\pi} \log \det f(\omega; \psi)\, d\omega + \int_{-\pi}^{\pi} \operatorname{tr}\{f^{-1}(\omega; \psi)I_n(z; \omega)\}\, d\omega\right]. \tag{2.2}$$

The function $\bar{L}_n(\psi)$ is called the quasi-log-likelihood function [the approximation was originally proposed by Whittle (1952) for scalar-valued stationary processes; see also Dunsmuir and Hannan (1976) and Hosoya and Taniguchi (1982)].

EXAMPLE 2.1. Consider a bivariate fractional ARIMA process $\{z_1(t), z_2(t)\}$ which is generated by

$$\begin{bmatrix} \bar{A}_{11}(L) & \bar{A}_{12}(L) \\ \bar{A}_{21}(L) & \bar{A}_{22}(L) \end{bmatrix} \begin{bmatrix} (1-L)^{d_1} & \\ & (1-L)^{d_2} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \bar{B}_{11}(L) & \bar{B}_{12}(L) \\ \bar{B}_{21}(L) & \bar{B}_{22}(L) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

where $\{e_1(t),\ t \in J\}$ and $\{e_2(t),\ t \in J\}$ are white-noise processes with mean 0 and $\mathrm{Cov}\{e_i(t), e_j(s)\} = \delta(t,s)K_{ij},\ i, j = 1, 2$; the covariance matrix $K = \{K_{ij}\}$ is assumed positive definite; moreover, $0 < d_j < 1/2,\ j = 1, 2$, and L is the backward shift operator. The polynomials $\bar{A}_{ij}(L)$ and $\bar{B}_{ij}(L)$ of L are of order a and b, respectively, where $\bar{A}_{ij}(L) = \sum_{k=0}^{a} A_{ij}(k)L^k$ such that $\bar{A}(0) = I_2$ and $\det \bar{A}(z)$ and $\det \bar{B}(z)$ have all 0's outside the unit circle. Let θ be the vector whose components consist of d_1, d_2, the coefficients $A_{ij}(k)$ and $B_{ij}(k)$, and let μ be the vector $(K_{11}, K_{12}, K_{22})^*$. Define the parameter ψ by $\psi = (\theta^*, \mu^*)^*$. The process $\{z(t)\}$ then has the infinite-order MA representation:

$$\begin{aligned} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} &= \begin{bmatrix} (1-L)^{-d_1} & \\ & (1-L)^{-d_2} \end{bmatrix} \begin{bmatrix} \bar{A}_{11}(L) & \bar{A}_{12}(L) \\ \bar{A}_{21}(L) & \bar{A}_{22}(L) \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_{11}(L) & \bar{B}_{12}(L) \\ \bar{B}_{21}(L) & \bar{B}_{22}(L) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \\ &\equiv \begin{bmatrix} \bar{G}_{11}(L,\theta) & \bar{G}_{12}(L,\theta) \\ \bar{G}_{21}(L,\theta) & \bar{G}_{22}(L,\theta) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}, \end{aligned}$$

where the infinite-order polynomials $\bar{G}_{ij}(L) = \sum_{k=0}^{\infty} G_{ij}(k,\theta)L^k$ are determined by the last equation in (1.2) in view of the relationship

$$(1-L)^{-d_i} = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(d_i + k)}{k!\Gamma(d_i)} L^k.$$

It is obvious that $\sum_{k=0}^{\infty} \mathrm{tr}\, G(k,\theta)K(\mu)G(k,\theta)^* < \infty$.

EXAMPLE 2.2. Let $C_d(z)$ be the analytic function, defined in the unit circle of the complex plane, which is given by

$$C_d(z) = \sqrt{2\pi} \exp\left[\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\left\{|\omega|^{-d} \frac{e^{-i\omega} + z}{e^{-i\omega} - z}\right\} d\omega\right], \qquad 0 < d < 1.$$

Then the boundary value $C_d(e^{i\omega})$ of $C_d(z)$ satisfies $|C_d(e^{i\omega})|^2 = |\omega|^{-d}$. Suppose a bivariate process $\{z_1(t), z_2(t)\}$ is generated by

$$\begin{bmatrix} \bar{A}_{11}(L) & \bar{A}_{12}(L) \\ \bar{A}_{21}(L) & \bar{A}_{22}(L) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_{d_1}(L) & \\ & \bar{C}_{d_2}(L) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

where $\bar{A}(L)$ is defined as in Example 2.1 and $\bar{C}_d(L)$ is the lag polynomial generated by $C_d(z)$. Let $K(\mu)$ be the covariance matrix of $e(t)$. Then the spectral density matrix has the representation

$$f(\omega;\psi) = \frac{1}{2\pi} A(e^{i\omega})^{-1} C(e^{i\omega}) K(\mu) C^*(e^{i\omega}) A^*(e^{i\omega})^{-1},$$

where

$$C(e^{i\omega}) = \begin{bmatrix} C_{d_1}(e^{i\omega}) & 0 \\ 0 & C_{d_2}(e^{i\omega}) \end{bmatrix}.$$

In this model the spectral densities of $\{z_1(t)\}$ and $\{z_2(t)\}$ have singularities of the types $|\omega|^{-d_1}$ and $|\omega|^{-d_2}$ at zero frequency, respectively.

As will be seen in the following theorems, the functional dependency of the component $\int_{-\pi}^{\pi} \log\det f(\omega;\,\psi)\,d\omega$ in the quasi-log-likelihood function $\tilde{L}(\psi)$ on the parameter θ determining the coefficients $G(j)$ plays an important role in the determination of the asymptotic properties of the maximum-likelihood estimate and the likelihood ratio statistic if the observation process is non-Gaussian. To see this, consider the fractional ARIMA model given in Example 2.1. Since, for that process, we have

$$\int_{-\pi}^{\pi} \log\det A(e^{i\omega})\,d\omega = \int_{-\pi}^{\pi} \log\det B(e^{i\omega})\,d\omega = \int_{-\pi}^{\pi} \log|1 - e^{i\omega}|\,d\omega = 0,$$

it follows that

$$\int_{-\pi}^{\pi} \log\det f(\omega;\,\psi)\,d\omega = 2\pi K(\mu). \tag{2.3}$$

In view of (2.3) the quasi-log-likelihood function is expressed, up to constant multiplication, by

$$\tilde{L}_n(\psi) \equiv \tilde{L}_n(\theta,\mu) = -n\left[2\pi\det K(\mu) + \int_{-\pi}^{\pi} \operatorname{tr}\{f^{-1}(\omega;\,\psi)I_n(\omega,z)\}\,d\omega\right]; \tag{2.4}$$

whence the member on the left-hand side of (2.3) is independent of θ for fractional ARIMA models. In case the quasi-log-likelihood is given by (2.4), the likelihood ratio based on it has the usual χ^2 asymptotic distribution if the innovation process $\{e(t)\}$ satisfies a suitable fourth-order moment condition (Theorem 2.4). Otherwise, this asymptotic result is no longer valid. Consider the process of Example 2.2, for which

$$\int_{-\pi}^{\pi} \log\det f(\omega;\,\psi)\,d\omega = \text{const.} - (d_1 + d_2)\int_{-\pi}^{\pi} \log|\omega|\,d\omega + 2\pi\log\det K(\mu).$$

In this case the left-hand-side member is a function of d_1 and d_2. Thus (2.3) is violated.

Example 2.3. Suppose that a bivariate process $\{z(t)\}$ is generated by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} (1-B)^{-d_1} & a(1-B)^{-d_2} \\ b(1-B)^{-d_2} & (1-B)^{d_1} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}, \qquad d_1 \neq d_2.$$

Namely, each of $z_1(t)$ and $z_2(t)$ is a sum of two types of fractional difference processes. In general, the relation (2.3) does not hold for this kind of process, either.

The preceding examples illustrate that the quasi-likelihood function $\bar{L}_n(\psi)$ is preferred to $\tilde{L}(\psi)$ as a general expression, since the former function better approximates Gaussian likelihood functions while $\tilde{L}_n(\psi)$ does so only for a limited class of long-memory time series. Statistical estimation theories so far developed in the literature do not seem to make this distinction, giving consequently the same asymptotic result originally given by Whittle [see Yajima (1985), Fox and Taqqu (1986), Giraitis and Surgailis (1990) and Heyde and Gay (1993)]. As Theorem 2.2 shows, if (2.3) is violated, the large-sample theory based on the quasi-likelihood $\bar{L}_n(\psi)$ produces distinctly different results according to whether observations are Gaussian or not, and this has a consequence in the validity of the χ^2 asymptotics of the likelihood ratio test.

2.2. *The asymptotics of the quasi-maximum-likelihood estimate.* Assume henceforth that $\int_{-\pi}^{\pi} \log\det f(\omega;\psi)\,d\omega$ is differentiable with respect to ψ and, for each ω, $f(\omega;\psi)^{-1}$ is differentiable with respect to ψ almost everywhere. The derivatives are denoted, respectively, by $H_j(\psi) = \partial\int_{-\pi}^{\pi}\log\det f(\omega;\psi)/\partial\psi_j$ and $h_j(\omega;\psi) = \partial f^{-1}(\omega;\psi)/\partial\psi_j$ and h_j is assumed to be measurable with respect to ψ a.e. ω. Here $H(\psi)$ and $\mathrm{tr}\{h(\omega;\psi)f(\omega)\}$ represent, respectively, the s-vectors whose jth elements are $H_j(\psi)$ and $\mathrm{tr}\{h_j(\omega;\psi)f(\omega)\}$. The $h_j(\omega;\psi)$ are assumed separable throughout.

Let $S_{nj}(\psi)$ be defined as

$$S_{nj}(\psi) = H_j(\psi) + \int_{-\pi}^{\pi} \mathrm{tr}\{h_j(\omega;\psi)I_n(\omega)\}\,d\omega, \qquad j = 1,\dots,s,$$

and let $S_n(\psi)$ be the vector $\{S_{nj}(\psi)\}$. A value $\hat{\psi}_n$ such that $S_n(\hat{\psi}_n) = 0$ is said to be a quasi-maximum-likelihood (QML) estimate of ψ. Consistency of the QML estimate is established in Theorem 2.1 under the following assumption.

Assumption C. (i) The true process $\{z(t)\}$ has a spectral density $f(\omega) = (2\pi)^{-1}k(\omega)Kk(\omega)^*$ that satisfies:

(1) $\int_{-\pi}^{\pi} |k_{\alpha\beta}(\omega)|^{2u}\,d\omega < \infty$ for some u such that $1 < u \le 2$;
(2) there exists $c > 0$ such that

$$\sup_{|\lambda|<\varepsilon} \max_{\alpha,\beta} \; \|[f^{-1}(\cdot)\{f(\cdot) - f(\cdot-\lambda)\}]_{\alpha\beta}\|_u = O(\varepsilon^c). \tag{2.5}$$

(ii) For any $\varepsilon > 0$, there exist $a > 0$ and Hermitian matrix-valued bounded functions $\tilde{h}_j$ and $\bar{h}_j$ such that, if $|\psi_1 - \psi| < a$,

$$\tilde{h}_j(\omega) \le h_j(\omega,\psi_1) \le \bar{h}_j(\omega)$$

and

$$\max_{\alpha,\beta} \|[\{\bar{h}_j(\cdot) - \tilde{h}_j(\cdot)\}f(\cdot)]_{\alpha\beta}\|_v < \varepsilon, \tag{2.6}$$

where the inequality $A \le B$ implies that $B - A$ is nonnegative definite and $v = (u-1)/u$ for u given in (i)(2) above.

(iii) $R_j(\psi) \equiv H_j(\psi) + \int_{-\pi}^{\pi} \operatorname{tr}\{h_j(\omega, \psi)f(\omega)\}\,d\omega$ has a unique zero for all j at $\psi = \psi_0$ where ψ_0 is an interior point of Ψ.

(iv) $H_j(\psi)$ is continuous on Ψ.

REMARK 2.1. The assumption of Hermitian matrix-valued bracketing functions is not essential, but it is assumed only for the sake of expositional simplicity. Componentwise bracketing with respect to each real and imaginary part of h_j also works, but makes the exposition and proofs very tedious. The same remark applies also to Assumption D.

THEOREM 2.1. *Suppose $\tilde{\psi}_n$ is a sequence of measurable functions of $(z(1), \dots, z(n))$ taking values in Ψ such that $S_n(\tilde{\psi}_n) \to 0$ in probability as $n \to \infty$. Then, under Assumption* C, *$\tilde{\psi}_n$ tends to ψ_0 in probability.*

In order to deal with the clt of $\hat{\psi}_n$, the bracketing function approach is employed in the following way, where u and v are set as in Assumption C.

ASSUMPTION D. (i) For some $c > 1/2$, the relationship (2.5) holds.

(ii)
$$\lim_{r\to 0} \sup_{|\psi-\psi_0|\le r} \|[\{h_j(\cdot, \psi) - h_j(\cdot, \psi_0)\}f(\cdot)]_{\alpha\beta}\|_v < c_1$$

for some $c_1 > 0$, $j = 1, \dots, s$ and $\alpha, \beta = 1, \dots, q$.

(iii) Given $\varepsilon > 0$, there exist an integer $m(\varepsilon)$, a partition $U^1(r), \dots, U^{m(\varepsilon)}(r)$ of the ball in Ψ with center ψ_0 and radius r and square-integrable Hermitian matrix-valued functions $\bar{h}^i_j(\omega)$, $\tilde{h}^i_j(\omega)$ such that, for all sufficiently small r and for all j, $\tilde{h}^i_j(\omega) \le h_j(\omega; \psi) \le \bar{h}^i_j(\omega)$ if ψ is in $U^i(r)$. Also

$$\begin{aligned} &\|[k^*(\bar{h}^i_j - h^0_j)k]_{\beta_1\beta_2}\|_v \le \varepsilon r,\\ &\|[k^*(\tilde{h}^i_j - h^0_j)k]_{\beta_1\beta_2}\|_v \le \varepsilon r, \end{aligned} \tag{2.7}$$

where $h^0_j = h_j(\cdot, \psi_0)$.

(iv) $|R(\psi)| \ge a_1|\psi - \psi_0|$ for some $a_1 > 0$ in a neighborhood of ψ_0. Moreover, Condition B holds for all the pairs $\{\bar{h}_{j\alpha_2\alpha_1}, k_{\alpha\beta}\}$, $\{\tilde{h}_{j\alpha_2\alpha_1}, k_{\alpha\beta}\}$ and $\{h^0_{j\alpha_2\alpha_1}, k_{\alpha\beta}\}$, where $\alpha = \alpha_1$ or α_2 and $1 \le \beta \le p$.

THEOREM 2.2. *Suppose $\sqrt{n}S_n(\tilde{\psi}_n) \to 0$ and $\tilde{\psi}_n - \psi_0 \to 0$ in probability as $n \to \infty$, and Assumptions* A, C *and* D *hold. If R is differentiable at $\psi = \psi_0$ and the matrix of the derivatives $W_{ij} = \partial R_i/\partial\psi_j$ is denoted by W, $\sqrt{n}(\tilde{\psi}_n - \psi_0)$ has the asymptotic normal distribution with mean* 0 *and covariance matrix $W^{-1}U(W^*)^{-1}$, where U is the matrix whose (j, ℓ)th element is represented as*

$$\begin{aligned} U_{j\ell} = 4\pi &\int_{-\pi}^{\pi} \operatorname{tr}[h_j(\omega; \psi_0)f(\omega)h_\ell(\omega; \psi_0)f(\omega)]\,d\omega \\ &+ 2\pi \sum_{\beta_1,\dots,\beta_4=1}^{p} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} [k^*(\omega_1)h_j(\omega_1; \psi_0)k(\omega_1)]_{\beta_1\beta_2} \\ &\qquad \times [k^*(\omega_2)h_\ell(\omega_2; \psi_0)k(\omega_2)]_{\beta_3\beta_4} Q^e_{\beta_1,\dots,\beta_4}(\omega_1, \omega_2, -\omega_2)\,d\omega_1\,d\omega_2. \end{aligned}$$

EXAMPLE 2.4. Suppose that $\{z(t)\}$ is a scalar-valued process with spectral density $f(\omega;\, d) = |\omega|^{-d}$, $0 < d < 1$, and let d_0 and $\hat{d}_n$ be the true value and the QML estimate, respectively. Let $C_d(e^{i\omega})$ be the function defined in Example 2.2 so that $|C_d(e^{i\omega})|^2 = |\omega|^{-d}$. For this model the quasi-likelihood is given by

$$\bar{L}_n(d) = -\left[-d\int_{-\pi}^{\pi} \log|\omega|\, d\omega + \int_{-\pi}^{\pi} |\omega|^d I_n(z;\omega)\, d\omega\right],$$

whence $R(\psi)$ in Theorem 2.2 has the expression

$$R(d) = -\int_{-\pi}^{\pi} \log|\omega|\, d\omega + \int_{-\pi}^{\pi} \{\log|\omega|\}|\omega|^{d-d_0}\, d\omega,$$

which has a unique zero $d = d_0$. Moreover, $W(\psi)$ is given by

$$W(d) = \partial V(d)/\partial d = \int_{-\pi}^{\pi} \{\log|\omega|\}^2 |\omega|^{d-d_0}\, d\omega,$$

so that $W(d_0) = \int_{-\pi}^{\pi} (\log|\omega|)^2\, d\omega$. Accordingly, if the innovation $\{e(t)\}$ of this process satisfies Assumption A, Theorem 2.2 implies that $\sqrt{n}(\hat{d}_n - d_0)$ is asymptotically normally distributed with mean 0 and with variance

$$\left\{\int_{-\pi}^{\pi} (\log|\omega|)^2\, d\omega\right\}^{-2}\left[4\pi\int_{-\pi}^{\pi} (\log|\omega|)^2\, d\omega + 2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|\omega_1| \log|\omega_2| Q^e(\omega_1, \omega_2, -\omega_2)\, d\omega_1\, d\omega_2\right],$$

whence if the fourth-order spectrum Q^e vanishes as in the Gaussian case, the variance reduces to $4\pi[\int_{-\pi}^{\pi} (\log|\omega|)^2\, d\omega]^{-1}$; otherwise, the reduction is not justified.

REMARK 2.2. Note that Theorem 2.2 is not far-reaching enough to cover such nonregular statistical models which involve as an unknown parameter the location of an unbounded peak of the spectral density. Suppose $\{z(t)\}$ is a scalar-valued process where the spectral density is given by $f(\omega) = c|\omega - \theta|^{-\alpha}$, where θ is the unknown parameter and c and α are positive constants, $0 < \alpha < 1$. For such a model

$$h_\theta(\omega;\, \theta) \equiv df^{-1}(\omega;\, \theta)/d\theta = \begin{cases} c\alpha(\omega-\theta)^{\alpha-1}, & \text{if } \omega > \theta, \\ -c\alpha(\theta-\omega)^{\alpha-1}, & \text{if } \omega < \theta, \end{cases}$$

whence there do not exist bounded bracketing functions $\tilde{h}$ and $\bar{h}$ such that $\tilde{h}(\omega) \le h_\theta(\omega, \theta) \le \bar{h}(\omega)$ for any nonempty interval $\{\theta\colon |\theta - \theta_0| < r\}$. However, Theorem 2.2 applies to such cases where the locations of unbounded peaks are known or, even if the locations of singularities are unknown, the singularities produce only bounded peaks.

REMARK 2.3. All of the theorems of this paper are based on the assumption that the fourth-order spectral density of the process $\{e(t)\}$ is bounded. Long-range dependence in fourth-order serial moments of $\{e(t)\}$ and hence the existence of singularities in the fourth-order spectra would produce consequences different from the results of this paper. This problem remains open.

2.3. *The quasi-likelihood-ratio test.* For testing purposes, it is desirable to have a more specific characterization of the asymptotic distribution in Theorem 2.2. Suppose now that the fitted model in the last subsection has the following parametric structure: let Θ and M be compact subsets of R^l and R^m with nonempty interior, respectively, and $\Psi = \Theta \times M$, $s = l + m$; $\psi = (\theta^*, \mu^*)^*$, $G(j, \psi) = G(j, \theta)$ and $K(\psi) = K(\mu)$. Namely, the coefficients of the linear process $\{z(t)\}$ and the covariance of $\{e(t)\}$ are separately parametrized. In contrast to Theorem 2.2, the true process generating the observations is also assumed to belong to this parametric model. Denote by (θ_0, μ_0) the true value of (θ, μ). Let Φ be the covariance matrix of the innovation of the model process $\{z(\omega)\}$. Then, due to the known prediction theory of stationary processes, it holds that

$$\log\det\Phi = \frac{1}{(2\pi)^q}\int_{-\pi}^{\pi}\log\det f(\omega, \psi)\, d\omega.$$

ASSUMPTION E. The determinant $\det\Phi$ is functionally independent of θ, namely, $\det\Phi = S(\mu)$ for some function S of μ.

REMARK 2.4. The relation (2.3) holds for a general q-dimensional fractional ARIMA process with stationary ARMA part if each component process has the same fractional-difference order. Thus Assumption E is satisfied for that process, but not for the process exhibited in Example 2.2.

ASSUMPTION F. The joint fourth cumulant of $e_a(t_1), e_b(t_2), e_c(t_3), e_d(t_4)$ is equal to κ_{abcd} if $t_1 = t_2 = t_3 = t_4$ and is equal to 0 otherwise.

ASSUMPTION G. (i) The $h_j(\omega; \psi)$ are jointly Borel measurable with respect to (ω, ψ);

(ii) there exists a neighborhood N of ψ_0 such that $\int_{-\pi}^{\pi}|\operatorname{tr}\{h_j(\omega; \psi_1)\cdot f(\omega; \psi_2)\}|\, d\omega$ is bounded for $\psi_1, \psi_2 \in N$ and $j = 1, \dots, l+m$;

(iii) $V_{jk}(\psi) \equiv \int_{-\pi}^{\pi}\operatorname{tr}[f(\omega; \psi)h_j(\omega; \psi)f(\omega; \psi)h_k(\omega; \psi)]\, d\omega$ is continuous at $\psi = \psi_0$ and the matrix $V(\psi_0) = \{V_{jk}(\psi_0);\ 1 \le j, k \le l+m\}$ is invertible;

(iv) $\lim_{\psi_1 \to \psi}\int_{-\pi}^{\pi}|h_{j,\alpha\beta}(\omega; \psi_1) - h_{j,\alpha\beta}(\omega; \psi)|\, d\omega = 0$, $1 \le \alpha, \beta \le q$.

Under these additional assumptions, the asymptotic covariance matrices in Theorem 2.2 have more specific expressions. In particular, the matrix U in

Theorem 2.2 has the following expression. If $1 \le j, k \le l$, $U_{jk} = 4\pi V_{jk}$, if $l+1 \le j, k \le l+m$,

$$U_{jk} = 4\pi V_{jk} + \sum_{\beta_1,\dots,\beta_4} \frac{1}{(2\pi)^2}\left[\int_{-\pi}^{\pi} k^*(\omega) h_j(\omega;\, \psi_0)\, d\omega\right]_{\beta_1\beta_2} \times \left[\int_{-\pi}^{\pi} k^*(\omega) h_k(\omega;\, \psi_0)\, d\omega\right]_{\beta_3\beta_4}$$

and $U_{jk} = 0$ otherwise.

THEOREM 2.3. *Suppose that Assumptions* A, C, D, E, F *and* G *hold. Then* $\sqrt{n}(\tilde{\theta}_n - \theta_0)$ *and* $\sqrt{n}(\tilde{\mu}_n - \mu_0)$ *are asymptotically independently normally distributed with mean* 0 *and the covariance matrices which are given, respectively, by* $4\pi V_{(1)}^{-1}$ *and* $V_{(2)}^{-1} U_{(2)} V_{(2)}^{-1}$, *where* $\tilde{\psi}_n = (\tilde{\theta}_n, \tilde{\mu}_n)$ *and* $V_{(1)}, V_{(2)}, U_{(2)}$ *are submatrices of* V *and* U *such that* $V_{(1)} = \{V_{ij};\ 1 \le i, j \le l\}$, $V_{(2)} = \{V_{ij};\ l+1 \le i, j \le l+m\}$ *and* $U_{(2)} = \{U_{jk};\ l+1 \le j, k \le l+m\}$.

Theorem 2.3 enables the derivation of the asymptotic χ^2 distribution of the quasi-likelihood-ratio statistic based on the quasi-likelihood function (2.2). As Hosoya (1989b) pointed out, model selection problems often require testing a null hypothesis in the presence of a hierarchy of alternative hypotheses. In view of their application to such situations, a set of quasi-log-likelihood ratio statistics is considered below instead of the conventional LR statistic. Let $\Theta_0 \subset \Theta_1 \subset \cdots \subset \Theta_r \subset \Theta$ be nested sebsets of Θ such that, for $0 \le l_0 < l_1 < \cdots < l_r \le l$,

$$\Theta_i = \{\theta \in \Theta,\ \theta_j = 0 \text{ for } l_i + 1 \le j \le l\}, \qquad i = 0, \dots, r.$$

This requirement can be imposed without loss of much generality since a suitable transformation of Θ could reduce a hierarchical structure into the above type. Now suppose that a null hypothesis is given by $\Theta_0 \times M$ and the hierarchy of alternative hypotheses is given by $\Theta_i \times M$, $i = 1, \dots, r$. Assume that, for each $i = 0, 1, \dots, r$, there exists a sequence of statistics $\tilde{\psi}_n(i) = (\tilde{\theta}_n(i)^*, \tilde{\mu}_n(i)^*)^* \in \Theta_i \times M$ such that:

1. $\sqrt{n} S_n(\tilde{\psi}_n(i)) \to 0$ in probability as $n \to \infty$;
2. $\tilde{\psi}_n(i) \to \psi_0 = (\theta_0^*, \mu_0^*)^* \in \Theta_0 \times M$ in probability as $n \to \infty$.

Define the quasi-log-likelihood ratios $\bar{L}_{n,ij}$ by $\bar{L}_{n,ij} = \bar{L}_n(\tilde{\theta}_n(i), \tilde{\mu}_n(i)) - \bar{L}_n(\tilde{\theta}_n(j), \tilde{\mu}_n(j))$, where $\bar{L}_n$ is defined in (2.2).

THEOREM 2.4. *Suppose that Assumptions* A, D, E, F *and* G *hold. Then* $\bar{L}_{n,01}, \dots, \bar{L}_{n,0r}$ *are asymptotically jointly distributed as* $-2\pi \sum_{j=1}^{i} \chi_j^2$, $i = 1, \dots, r$, *where the* χ_j^2*'s are independent* χ^2 *random variables with* $l_j - l_{j-1}$ *degrees of freedom.*

3. Proofs.

3.1. *Proofs for Section* 1.

PROOF OF THEOREM 1.1. Let $K_n(\omega)$ be the Fejér kernel which is defined by $K_n(\omega) = [\sin(n\omega/2)/\sin(\omega/2)]^2/n$ and let $v = u/(u-1)$. For any $h \in L^v$ such that $\| h \|_v < 1$, we have

$$\left|\int_{-\pi}^{\pi} h(\omega)\,\mathrm{tr}\left[\frac{1}{2\pi} g(\omega)\left\{\int_{-\pi}^{\pi} f(\lambda)K_n(\omega-\lambda)\,d\lambda - f(\omega)\right\}\right] d\omega\right|$$
$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\omega)\|\,\mathrm{tr}\{g(f-f_{-\omega})\}\|_u\,d\omega,$$

where $f_{-\omega}$ is defined by $f_{-\omega}(\lambda) = f(\lambda-\omega)$. On the other hand, it follows from the relationship

$$K_n(\omega) \le \min[(n+1), c_1/\{(n+1)\omega^2\}]$$

that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\omega)\,\|\,\mathrm{tr}\{g\cdot(f-f_{-\omega})\}\,\|_u\; d\omega \\
&\quad\le \frac{n+1}{2\pi}\int_{-1/n}^{1/n} \|\,\mathrm{tr}\{g\cdot(f-f_{-\omega})\}\,\|_u\; d\omega \\
&\qquad + \frac{c_2}{2\pi}\int_{1/n\le|\omega|\le\pi} \frac{\|\,\mathrm{tr}\{g\cdot(f-f_{-\omega})\}\,\|_u}{(n+1)\omega^2}\,d\omega.
\end{aligned} \tag{3.1}$$

For sufficiently large n, (1.2) implies that

$$\frac{n+1}{2\pi}\int_{-1/n}^{1/n} \|\,\mathrm{tr}\{g\cdot(f-f_{-\omega})\}\,\|_u\; d\omega \le \left(\frac{1}{n}\right)^c \frac{n+1}{\pi n}. \tag{3.2}$$

On the other hand, for the second member on the right-hand side of (3.1), we have

$$\frac{1}{n+1}\int_{1/n\le|\omega|\le\pi} \|\,\mathrm{tr}\{g\cdot(f-f_{-\omega})\}\,\|_u\,/\omega^2\,d\omega \le c_3 n^{-c} \tag{3.3}$$

[see Hosoya (1989a), page 408]. Then it follows from (3.2) and (3.3) that

$$\|\,\mathrm{tr}[g(\cdot)\{(2\pi)^{-1}\int_{-\pi}^{\pi} f(\omega)K_n(\cdot-\omega)\,d\omega - f(\cdot)\}]\|_u = O(n^{-c}).$$

It follows from the Hölder inequality that

$$\left|\int_{-\pi}^{\pi} \mathrm{tr}[g(\omega)E\{I_n(z,\omega)\} - g(\omega)f(\omega)]\,d\omega\right|$$
$$\le c_1\left[\int_{-\pi}^{\pi} |\,\mathrm{tr}[g(\omega)\{E(I_n(z,\omega)) - f(\omega)\}]|^u\,d\omega\right]^{1/u}.$$

Since

$$E(I_n(z,\omega)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega-\lambda)K_n(\lambda)\,d\lambda,$$

the desired result follows. □

The next theorem is necessary in order to establish the clt of quadratic forms in Theorem 1.2; it is given without proof [for the proof see Hosoya and Taniguchi (1982, 1993), and for a related approach see Findley and Wei (1993)]. Denote by $C^z_{\alpha\beta}(r)$, $1 \le \alpha, \beta \le q$, the sample serial covariance constructed from the partial realization $\{z(1),\dots,z(n)\}$; namely, $C^z_{\alpha\beta}(r) = (1/n)\sum_{t=1}^{n-r} z_\alpha(t) \times z_\beta(t+r)$ for $0 \le r \le n-1$ and $C^z_{\alpha\beta}(r) = C^z_{\alpha\beta}(-r)$ for $-n+1 \le r \le 0$. Denote the population serial covariances by $\gamma^z_{\alpha\beta}(r) = E(z_\alpha(t)z_\beta(t+r))$.

THEOREM 3.1. *If Assumption* A *holds and if the spectral densities* $f_{\beta\beta}$, $\beta = 1,\dots,q$ *are square-integrable, then* $\sqrt{n}\{C^z_{\alpha_1\alpha_2}(r) - \gamma^z_{\alpha_1\alpha_2}(r)\}$, $\alpha_1,\alpha_2 = 1,\dots,q$, $0 \le r \le M$, *for any* $M \ge 0$ *have the joint asymptotic normal distribution whose mean is* 0 *and with asymptotic covariance between* $\sqrt{n}\{C^z_{\alpha_1\alpha_2}(r_1) - \gamma^z_{\alpha_1\alpha_2}(r_1)\}$ *and* $\sqrt{n}\{C^z_{\alpha_3\alpha_4}(r_2) - \gamma^z_{\alpha_3\alpha_4}(r_2)\}$ *given as*

$$\begin{aligned}
2\pi\int_{-\pi}^{\pi}&\big[f_{\alpha_1\alpha_3}(\omega)\overline{f_{\alpha_2\alpha_4}(\omega)}\exp\{-i(r_2-r_1)\omega\} \\
&+ f_{\alpha_1\alpha_4}(\omega)\overline{f_{\alpha_2\alpha_3}(\omega)}\exp\{i(r_1+r_2)\omega\}\big]\,d\omega \\
&+ 2\pi\sum_{\beta_1,\dots,\beta_4=1}^{p}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp\{ir_1\omega_1+ir_2\omega_2\}k_{\alpha_1\beta_1}(\omega_1)k_{\alpha_2\beta_2}(-\omega_1) \\
&\qquad\times k_{\alpha_3\beta_3}(\omega_2)k_{\alpha_4\beta_4}(-\omega_2)Q^e_{\beta_1,\dots,\beta_4}(\omega_1,-\omega_2,\omega_2)\,d\omega_1\,d\omega_2.
\end{aligned}$$

PROOF OF LEMMA 1.1. Since the proof is very involved, it is divided into two separate lemmas (Lemmas 3.1 and 3.2) in what follows. For simplicity, write $g^{(1)}_{\alpha_2\alpha_1} = g^{(1)}$ and $g^{(2)}_{\alpha_4\alpha_3} = g^{(2)}$ and also set

$$\tilde g^{(1)}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g^{(1)}(\omega)e^{it\omega}\,d\omega,$$
$$\tilde g^{(2)}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g^{(2)}(\omega)e^{it\omega}\,d\omega.$$

Then we have

$$\begin{aligned}
n\,\mathrm{Cov}&\left\{\int_{-\pi}^{\pi} g^{(1)}(\omega)I_{n\alpha_1\alpha_2}(z,\omega)\,d\omega, \int_{-\pi}^{\pi} g^{(2)}(\omega)I_{n\alpha_3\alpha_4}(z,\omega)\,d\omega\right\} \\
&= \frac{1}{n}\sum_{t_1,\dots,t_4=1}^{n} \tilde g^{(1)}(t_1-t_2)\tilde g^{(2)}(t_3-t_4)\{\gamma^z_{\alpha_1\alpha_3}(t_3-t_1)\gamma^z_{\alpha_2\alpha_4}(t_4-t_2) \\
&\quad + \gamma^z_{\alpha_1\alpha_4}(t_4-t_1)\gamma^z_{\alpha_2\alpha_3}(t_3-t_2) + \tilde Q^z_{\alpha_1,\dots,\alpha_4}(t_2-t_1, t_3-t_1, t_4-t_1)\}.
\end{aligned}$$

Now define φ_n and $K_n^{(4)}$, respectively, by

$$\varphi_n(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{n} \exp\{it\omega\}$$

and

$$K_n^{(4)}(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{1}{n} \varphi_n(\omega_1)\varphi_n(\omega_2)\varphi_n(\omega_3)\varphi_n(\omega_4)$$

[$K_n^{(4)}$ is regarded as the quadruple Fejér kernel in view of the properties below]. In the following arguments, we frequently use the bound $|\varphi_n(\omega)| \le \min[n/(2\pi)^{1/2}, (\pi/2)^{1/2}\omega^{-1}]$. First, it is shown that, as $n \to \infty$,

$$\begin{aligned} D_n &\equiv \frac{1}{n} \sum \tilde{g}^{(1)}(t_1 - t_2)\tilde{g}^{(2)}(t_3 - t_4)\gamma^z(t_3 - t_1)\gamma^z(t_4 - t_2) \\ &\to 2\pi \int_{-\pi}^{\pi} g^{(1)}(\omega)\overline{g^{(2)}(\omega)} f_{\alpha_1\alpha_3}(\omega)\overline{f_{\alpha_2\alpha_4}(\omega)}\, d\omega. \end{aligned} \tag{3.4}$$

The convergence of

$$\frac{1}{n} \sum \tilde{g}^{(1)}(t_1 - t_2)\tilde{g}^{(2)}(t_3 - t_4)\gamma^z_{\alpha_1\alpha_4}(t_4 - t_1)\gamma^z_{\alpha_2\alpha_3}(t_3 - t_2)$$

can be shown similarly and the proof is omitted. Now D_n is expressed as follows:

$$\begin{aligned} D_n &= \sum \frac{1}{n(2\pi)^2} \int \cdots \int \exp\{i(t_1 - t_2)\omega_1\} \exp\{-i(t_3 - t_4)\omega_2\} \\ &\qquad \times \exp\{i(t_3 - t_1)\omega_3\} \exp\{-i(t_4 - t_2)\omega_4\} \\ &\qquad \times g^{(1)}(\omega_1)\overline{g^{(2)}(\omega_2)} f_{\alpha_1\alpha_3}(\omega_3)\overline{f_{\alpha_2\alpha_4}(\omega_4)}\, d\omega_1 \cdots d\omega_4 \\ &= \int \cdots \int K_n^{(4)}(\omega_1 - \omega_3, -\omega_1 + \omega_4, -\omega_2 + \omega_3, \omega_2 - \omega_4) \\ &\qquad \times g^{(1)}(\omega_1)\overline{g^{(2)}(\omega_2)} f_{\alpha_1\alpha_3}(\omega_3)\overline{f_{\alpha_2\alpha_4}(\omega_4)}\, d\omega_1 \cdots d\omega_4. \end{aligned} \tag{3.5}$$

Since for any ω_1 it holds that

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} K_n^{(4)}(\omega_1 - \omega_3, -\omega_1 + \omega_4, \omega_2 - \omega_3, \omega_2 - \omega_4)\, d\omega_2\, d\omega_3\, d\omega_4 = 2\pi,$$

we have the equality

$$\begin{aligned} &\int_{-\pi}^{\pi} g^{(1)}(\omega_1)\overline{g^{(2)}(\omega_1)} f_{\alpha_1\alpha_3}(\omega_1)\overline{f_{\alpha_2\alpha_4}(\omega_1)}\, d\omega_1 \\ &\quad = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{2\pi} K_n^{(4)}(\omega_1 - \omega_3, -\omega_1 + \omega_4, -\omega_2 + \omega_3, \omega_2 - \omega_4) \\ &\qquad \times g^{(1)}(\omega_1)\overline{g^{(2)}(\omega_1)} f_{\alpha_1\alpha_3}(\omega_1)\overline{f_{\alpha_2\alpha_4}(\omega_1)}\, d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4. \end{aligned} \tag{3.6}$$

Set $\lambda_2 = \omega_2 - \omega_1$, $\lambda_3 = \omega_3 - \omega_1$, $\lambda_4 = \omega_4 - \omega_1$, and set

$$\begin{aligned}
& g^{(1)}(\omega)\overline{g^{(2)}(\lambda_2+\omega)} f_{\alpha_1\alpha_3}(\lambda_3+\omega) f_{\alpha_2\alpha_4}(\lambda_4+\omega) \\
& \quad - g^{(1)}(\omega)\overline{g^{(2)}(\omega)} f_{\alpha_1\alpha_3}(\omega)\overline{f_{\alpha_2\alpha_4}(\omega)} \\
& \quad = g^{(1)}(\omega)\{\overline{g^{(2)}(\lambda_2+\omega)} - \overline{g^{(2)}(\omega)}\} f_{\alpha_1\alpha_3}(\lambda_3+\omega) f_{\alpha_2\alpha_4}(\lambda_4+\omega) \\
& \qquad + g^{(1)}(\omega)\overline{g^{(2)}(\omega)}\{f_{\alpha_1\alpha_3}(\lambda_3+\omega) - f_{\alpha_1\alpha_3}(\omega)\} f_{\alpha_2\alpha_4}(\lambda_4+\omega) \\
& \qquad + g^{(1)}(\omega)\overline{g^{(2)}(\omega)} f_{\alpha_1\alpha_3}(\omega)\{f_{\alpha_2\alpha_4}(\lambda_4+\omega) - f_{\alpha_2\alpha_4}(\omega)\} \\
& \quad \equiv C_1 + C_2 + C_3.
\end{aligned}$$

Then, in view of (3.5) and (3.6), what is needed to prove (3.4) is the three propositions given in the following lemma. Let D be the domain of integration for $(\lambda_2, \lambda_3, \lambda_4)$.

LEMMA 3.1.

$$\lim_{n\to\infty} \frac{1}{n}\int_{-\pi}^{\pi}\left[\int_D K_n^{(4)}(-\lambda_3, -\lambda_4, -\lambda_2+\lambda_3, \lambda_2-\lambda_4) C_j \, d\lambda_2\, d\lambda_3 \lambda_4\right] d\omega = 0, \quad j = 1, 2, 3. \tag{3.7.j}$$

PROOF. In order to prove (3.7.1), set β so that $1/(2+\gamma) < \beta < 1/2$ and define D^1, D^2, D^3, D^4, the subsets of D, as follows:

$$\begin{aligned}
& D^1 = \{(\lambda_2, \lambda_3, \lambda_4) \in D;\ |\lambda_3 \pm \lambda_4| \le n^{-\beta};\ |\lambda_2| \le 2n^{-\beta}\}, \\
& D^2 = \text{the complement of } D^1 \text{ in } D,\ D^3 = \{|\lambda_3 \pm \lambda_4| \le n^{-\beta}, |\lambda_2| > 2n^{-\beta}\}, \\
& D^4 = \{|\lambda_3 + \lambda_4| > n^{-\beta} \text{ or } |\lambda_3 - \lambda_4| > n^{-\beta}\}.
\end{aligned}$$

Then set

$$\begin{aligned}
A_1 = \int_{-\pi}^{\pi} d\omega \int_{D^1} & \frac{1}{2\pi n} K_n^4(-\lambda_3, -\lambda_4, -\lambda_2+\lambda_3, \lambda_2-\lambda_4) \\
& \times g^{(1)}(\omega)\left\{\overline{g^{(2)}(\lambda_2+\omega)} - \overline{g^{(2)}(\omega)}\right\} \\
& \times f_{\alpha_1\alpha_3}(\lambda_3+\omega)\overline{f_{\alpha_2\alpha_4}(\lambda_4+\omega)}\, d\lambda_2\, d\lambda_3\, d\lambda_4
\end{aligned}$$

and let A_2, A_3 be the integral for which the domain of integration D^1 is replaced by D^2 and D^3, respectively. It follows from Condition B(i) that

$$\begin{aligned}
|A_1| \le c_1 n^{-(1+\gamma\beta)} \int_{|\lambda_2 \pm \lambda_3| < n^{-\beta}} & |f_{\alpha_1\alpha_3}(\lambda_3) f_{\alpha_2\alpha_4}(\lambda_4)| \\
& \times \left\{\int_{-\pi}^{\pi} |\varphi_n(-\lambda_3+\omega)\varphi_n(-\lambda_4+\omega)|\, d\omega\right\} \\
& \times \left\{\int_{-\pi}^{\pi} |\varphi_n(-\lambda_2+\lambda_3)\varphi_n(\lambda_2-\lambda_4)|\, d\lambda_2\right\} d\lambda_3\, d\lambda_4
\end{aligned}$$

$$\leq c_2 n^{1-\gamma\beta} \int_{|\lambda_3 \pm \lambda_4| < n^{-\beta}} |f_{\alpha_1\alpha_3}(\lambda_3) f_{\alpha_2\alpha_4}(\lambda_4)| \, d\lambda_3 \, d\lambda_4$$
$$\leq c_2 n^{1-(\gamma\beta+2\beta)},$$

where the last inequality is due to Condition B(ii). Since $\beta > 1/(2+\gamma)$, $|A_1| \to 0$ as $n \to \infty$. For $(\lambda_2, \lambda_3, \lambda_4) \in D^2$, we have the inequality

$$\int_{|\lambda_2| > 2n^{-\beta}} |\varphi_n(-\lambda_2 + \lambda_3)\varphi_n(\lambda_2 - \lambda_4)| \, d\lambda_2$$
$$\leq \int_{|\lambda_2| > 2n^{-\beta}} |\varphi_n(-\lambda_2)\varphi_n\{\lambda_2 + (\lambda_3 - \lambda_4)\}| \, d\lambda_2 = O(n^{\beta});$$

hence

$$|A_2| \leq c_4 n^{\beta} \int_{|\lambda_3 \pm \lambda_4| < n^{-\beta}} |f_{\alpha_1\alpha_3}(\lambda_3) \overline{f_{\alpha_2\alpha_4}(\lambda_4)}| \, d\lambda_3 \, d\lambda_4 \leq c_5 n^{\beta} n^{-2\beta}.$$

Thus it is concluded that $\lim_{n\to\infty} |A_2| = 0$. For $|\lambda_3 - \lambda_4| > n^{-\beta}$, we have

$$\int_{-\pi}^{\pi} |\varphi_n(-\lambda_3 + \omega)\varphi_n(-\lambda_4 + \omega)| \, d\omega \int_{-\pi}^{\pi} |\varphi_n(-\lambda_2 + \lambda_3)\varphi_n(\lambda_2 - \lambda_4)| \, d\lambda_2$$
$$= O(n^{-2\beta}),$$

and the same relationship holds for $|\lambda_3 + \lambda_4| > n^{-\beta}$ as is seen by changing the variable λ_3 to $-\lambda_3$ in the above integral. Consequently, $|A_3| \leq c_6 n^{2\beta}/n$, and then since $\beta < 1/2$, $\lim_{n\to\infty} |A_3| = 0$. Similarly, we have $|A_4| \to 0$. This proves (3.7.1).

In order to deal with (3.7.2), note first that we have the relationship

$$\int_{-\pi}^{\pi} K^4(-\lambda_3, -\lambda_4, -\lambda_2 + \lambda_3, \lambda_2 - \lambda_4) \, d\lambda_2 = \sqrt{2\pi}\varphi_n(-\lambda_3)\varphi_n(-\lambda_4)\varphi_n(\lambda_3 - \lambda_4),$$

because

$$\int_{-\pi}^{\pi} \varphi_n(-\lambda_2 + \lambda_3)\varphi_n(\lambda_2 - \lambda_4) \, d\lambda_2$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{t=1}^{n} \sum_{t_1=1}^{n} \exp\{i(t - t_1)\lambda_2\} \exp\{it\lambda_3\} \exp\{-it_1\lambda_4\} \, d\lambda_2$$
$$= \sum_{t=1}^{n} \exp\{it(\lambda_3 - \lambda_4)\}.$$

Now set

$$F_1(\lambda_3) = \int_{-\pi}^{\pi} |g^{(1)}(\omega)| \, |f_{\alpha_1\alpha_3}(\lambda_3 + \omega) - f_{\alpha_1\alpha_3}(\omega)| \, d\omega$$

and

$$F_2(\omega, \lambda_3) = \int_{-\pi}^{\pi} |f_{\alpha_2\alpha_4}(\lambda_4 + \omega)| \, |\varphi_n(-\lambda_4)\varphi_n(\lambda_3 - \lambda_4)| \, d\lambda_4.$$

If $|\lambda_3| \ge n^{-\beta}$, $F_1(\lambda_3)$ is bounded and

$$
(3.8)\quad \begin{aligned}
|F_2(\omega,\lambda_3)| \le & \left[\int_{-\pi}^{\pi} |f_{\alpha_2\alpha_4}(\lambda_4+\omega)|^u \, d\lambda_4\right]^{1/u} \\
& \times \left[\int_{-\pi}^{\pi} |\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4)|^v d\lambda_4\right]^{1/v},
\end{aligned}
$$

where $v = u/(u-1)$. Since the first factor above is bounded and the second factor is $O(n^{\beta+(v-1)/v})$, it follows that, if D' and D'' are the subdomains of integration with respect to $(\lambda_2, \lambda_3, \lambda_4)$ in which $|\lambda_3| \ge n^{-\beta}$ and $|\lambda_3| < n^{-\beta}$, respectively,

$$
(3.9)\quad \begin{aligned}
& \left|\frac{1}{n}\int_{-\pi}^{\pi}\int_{D'} K_n(-\lambda_3, \lambda_4, -\lambda_2+\lambda_3, \lambda_2-\lambda_4) C_2 \, d\lambda_2 \, d\lambda_3 \, d\lambda_4 \, d\omega\right| \\
&\quad = \left|\frac{2\pi}{n}\int_{-\pi}^{\pi}\int_{D'} g^{(1)}(\omega) g^{(2)}(\omega)\{f_{\alpha_1\alpha_3}(\lambda_3+\omega) - f_{\alpha_1\alpha_3}(\omega)\} f_{\alpha_2\alpha_3}(\lambda_4+\omega)\right. \\
&\qquad\qquad \left. \times \varphi_n(-\lambda_3)\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4) \, d\omega \, d\lambda_3 \, d\lambda_4\right| \\
&\quad \le \frac{c_8}{n}\int_{|\lambda_3|\ge n^{-\beta}} d\lambda_3 \int_{-\pi}^{\pi} |\varphi_n(-\lambda_3) F_1(\lambda_3) F_2(\omega,\lambda_3)| \, d\omega \\
&\quad \le c_9 n^{\beta+\{(v-1)/v\}-1}\int_{|\lambda_3|\ge n^{-\beta}} |\varphi_n(-\lambda_3)| d\lambda_3 \\
&\quad \le c_{10} n^{2\beta+\{(v-1)/v\}-1}\log n.
\end{aligned}
$$

Hence, if β is set this time so as to satisfy $0 < \beta < (2v)^{-1}$, the left-hand-side member of (3.9) can be made arbitrarily small for sufficiently large n. Consider then the case $|\lambda_3| < n^{-\beta}$. The domain of integration D'' is divided into two parts $D''_{(1)} = \{(\lambda_3, \lambda_4) \in D'';\ |\lambda_4| \ge 2n^{-\beta}\}$ and $D''_{(2)} = \{D''; |\lambda_4| \le 2n^{-\beta}\}$. The integration on $D''_{(1)}$ can be dealt with quite similarly as above in view of the inequalities:

$$
\begin{aligned}
& \int_{n^{-\beta}\le|\lambda_4|\le\pi} |f_{\alpha_2\alpha_4}(\lambda_4+\omega)| \, |\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4)| \, d\lambda_4 \\
&\quad \le \left[\int_{-\pi}^{\pi} |f_{\alpha_2\alpha_4}(\lambda_4+\omega)|^u \, d\lambda_4\right]^{1/u}\left[\int_{|\lambda_3-\lambda_4|\ge n^{-\beta}} |\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4)|^v \, d\lambda_4\right]^{1/v} \\
&\quad = O(n^{\beta+(v-1)/v}).
\end{aligned}
$$

In order to consider the integration on $D''_{(2)}$, set

$$
\begin{aligned}
& \frac{1}{n}\int_{-\pi}^{\pi}\int_{D''_{(2)}} g^{(1)}(\omega) g^{(2)}(\omega)\{f_{\alpha_1\alpha_3}(\lambda_3+\omega) - f_{\alpha_1\alpha_3}(\omega)\} f_{\alpha_2\alpha_4}(\lambda_4+\omega) \\
&\qquad \times \varphi_n(-\lambda_3)\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4) \, d\omega \, d\lambda_3 \, d\lambda_4
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n}\int_{-\pi}^{\pi}\int_{D''_{(2)}} g^{(1)}(\omega)\{g^{(2)}(\omega)\}\{f_{\alpha_1\alpha_3}(\lambda_3+\omega)-f_{\alpha_1\alpha_3}(\omega)\}\\
&\qquad\times\{f_{\alpha_2\alpha_4}(\lambda_4+\omega)-f_{\alpha_2\alpha_4}(\omega)\}\\
&\qquad\times\varphi_n(-\lambda_3)\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4)\,d\omega\,d\lambda_3\,d\lambda_4\\
&\quad+\frac{1}{n}\int_{-\pi}^{\pi}\int_{D''_{(2)}} g^{(1)}(\omega)g^{(2)}(\omega)\{f_{\alpha_1\alpha_3}(\lambda_3+\omega)-f_{\alpha_1\alpha_3}(\omega)\}f_{\alpha_2\alpha_4}(\omega)\\
&\qquad\times\varphi_n(-\lambda_3)\varphi_n(-\lambda_4)\varphi_n(\lambda_3-\lambda_4)\,d\omega\,d\lambda_3\,d\lambda_4\\
&\equiv E_1+E_2.
\end{aligned}
$$

By the Schwarz inequality and Condition B(iii), there exists $c_{11}>0$ such that

$$
\begin{aligned}
&\int_{-\pi}^{\pi}|g^{(1)}(\omega)\{f_{\alpha_1\alpha_3}(\lambda_3+\omega)-f_{\alpha_1\alpha_3}(\omega)\}|\\
&\quad\times|g^{(2)}(\omega)\{f_{\alpha_2\alpha_4}(\lambda_4+\omega)-f_{\alpha_2\alpha_4}(\omega)\}|\,d\omega\\
&\quad\le c_{11}|\lambda_3|^{\gamma_1}|\lambda_4|^{\gamma_2}.
\end{aligned}
\tag{3.10}
$$

Moreover, we have

$$
\int_{|\lambda_3|\le n^{-\beta}}|\lambda_3|^{\gamma_1}|\varphi_n(-\lambda_3)\varphi_n(\lambda_3-\lambda_4)|\,d\lambda_3=O(n^{1-\beta\gamma_1})
\tag{3.11}
$$

and

$$
\int_{|\lambda_4|\le 2n^{-\beta}}|\lambda_4|^{\gamma_1}|\varphi_n(-\lambda_4)|\,d\lambda_4=O(n^{-\beta\gamma_1}).
\tag{3.12}
$$

Therefore, it follows that $|E_1|=O(n^{-2\beta\gamma_1})$. As for E_2, in view of the Schwarz inequality and Conditions B(iii) and (iv), we have

$$
\left|\int_{-\pi}^{\pi}g^{(1)}(\omega)g^{(2)}(\omega)\{f_{\alpha_1\alpha_3}(\lambda_3+\omega)-f_{\alpha_1\alpha_3}(\omega)\}f_{\alpha_2\alpha_4}(\omega)\,d\omega\right|=O(|\lambda_3|^{-\gamma_1}).
$$

Hence, since $\int_{-\pi}^{\pi}|\varphi_n(\lambda_4)|d\lambda_4=O(\log n)$, it follows from (3.11) that $|E_2|\le c_{13}n^{-1}(\log n)n^{1-\gamma_1\beta}=o(1)$. This completes the proof of (3.7.2). Finally, since $g^{(1)}(\omega)$ is essentially bounded, and $\|g^{(1)}f_{\alpha_1\alpha_3}\|_v<\infty$, and also since

$$
\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}K_n(-\lambda_3,-\lambda_4,-\lambda_2+\lambda_3,\lambda_2-\lambda_4)\,d\lambda_2\,d\lambda_3=(2\pi)^2|\varphi_n(-\lambda_4)|^2,
$$

it follows that

$$
\begin{aligned}
&\Bigg|\int\cdots\int g^{(1)}(\omega)g^{(2)}(\omega)f_{\alpha_1\alpha_3}(\omega)\{f_{\alpha_2\alpha_4}(\lambda_4+\omega)-f_{\alpha_2\alpha_4}(\omega)\}\\
&\qquad\times K_n(-\lambda_3,\lambda_3,-\lambda_2+\lambda_3,\lambda_2-\lambda_4)\,d\omega\,d\lambda_2\,d\lambda_3\,d\lambda_4\Bigg|
\end{aligned}
$$

$$\leq \frac{c_{14}}{n} \int_{-\pi}^{\pi} \int_{G_1} g^{(1)}(\omega) g^{(2)}(\omega) f_{\alpha_1\alpha_3}(\omega) \{ f_{\alpha_2\alpha_4}(\lambda_4 + \omega) - f_{\alpha_2\alpha_4}(\omega) \}$$
$$\times |\varphi_n(-\lambda_4)|^2 \, d\lambda_4 \, d\omega$$
$$+ \frac{c_{15}}{n} \int_{G_2} \| g^{(2)}(\cdot) \{ f_{\alpha_2\alpha_4}(\cdot + \lambda_4) - f(\cdot) \} \|_2 |\varphi_n(-\lambda_4)|^2 \, d\lambda_4$$
$$\equiv F_1 + F_2,$$

where $G_1 = \{ n^{-(1-\varepsilon)/2} < |\lambda_4| \leq \pi \}$ and $G_2 = \{ |\lambda_4| \leq n^{-(1-\varepsilon)/2} \}$ for some sufficiently small positive ε. Then it evidently follows that $F_1 = O(n^{-\varepsilon})$. As for F_2, we have

$$F_2 \leq \frac{c_{15}}{n} \int_{G_2} |\lambda_4|^{\gamma_1} |\varphi_n(-\lambda_4)|^2 \, d\lambda_4 \leq \frac{c_{16}}{n} n^{1-\gamma_1} = O(n^{-\gamma_1}).$$

This proves (3.7.3). □

LEMMA 3.2. *If $Q^e(\omega_1, \omega_2, \omega_3)$ is uniformly γ_2-Lipschitz, $\gamma_2 > 0$, $g^{(1)}(\omega)$ and $g^{(2)}(\omega)$ are essentially bounded, and if Conditions* B(i) *and* (iii) *hold*,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t_1,\ldots,t_4=1}^{n} \tilde{g}^{(1)}(t_1 - t_2) \tilde{g}^{(2)}(t_3 - t_4) \tilde{Q}^z_{\alpha_1,\ldots,\alpha_4}(t_2 - t_1, t_3 - t_1, t_4 - t_1)$$
$$= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g^{(1)}(\omega_1) g^{(2)}(\omega_2) Q^z_{\alpha_1,\ldots,\alpha_4}(\omega_1, \omega_2, -\omega_2) \, d\omega_1 \, d\omega_2.$$

PROOF. What is required to show is that

$$\int_{-\pi}^{\pi} \int_{\pi}^{\pi} |g^{(1)}(\omega_1) g^{(2)}(\omega_2)| \, | Q^z_{\alpha_1,\ldots,\alpha_4}(\omega_1 + \lambda_3, -\omega_2 + \lambda_4, \omega_2 + \lambda_5)$$
$$- Q^z_{\alpha_1,\ldots,\alpha_4}(\omega_1, -\omega_2, \omega_2) | \, d\omega_1 \, d\omega_2$$

is bounded in $(\lambda_3, \lambda_4, \lambda_5)$ and its supremum for $|\lambda_j| < \varepsilon$ is of the order $O(\varepsilon^{\gamma_2})$ for some $\gamma_2 > 0$. Since

$$Q^z_{\alpha_1,\ldots,\alpha_4}(\omega_1, \omega_2, \omega_3) = \sum_{\beta_1,\ldots,\beta_4=1}^{p} k_{\alpha_1\beta_1}(\omega_1 + \omega_2 + \omega_3) k_{\alpha_2\beta_2}(-\omega_1) k_{\alpha_3\beta_3}(-\omega_2)$$
$$\times k_{\alpha_4\beta_4}(-\omega_3) Q^e_{\beta_1,\ldots,\beta_4}(\omega_1 + \omega_2 + \omega_3, \omega_2, \omega_3),$$

the difference $Q^z_{\alpha_1,\ldots,\alpha_4}(\omega_1 + \lambda_3, -\omega_2 + \lambda_4, \omega_2 + \lambda_5) - Q^z_{\alpha_1,\ldots,\alpha_4}(\omega_1, -\omega_2, \omega_2)$ is expressed as the sum of the product terms, for example, such as

$$\{ k_{\alpha_1\beta_1}(\omega_1 + \lambda_3 + \lambda_4 + \lambda_5) - k_{\alpha_1\beta_1}(\omega_1 + \lambda_4 + \lambda_5) \} k_{\alpha_2\beta_2}(-\omega_1 - \lambda_3) k_{\alpha_3\beta_3}(\omega_2 - \lambda_4)$$
$$\times k_{\alpha_4\beta_4}(-\omega_2 - \lambda_5) Q^e_{\beta_1,\ldots,\beta_4}(\omega_1 + \lambda_3, -\omega_2 + \lambda_4, \omega_2 + \lambda_5)$$

and

$$k_{\alpha_1\beta_1}(\omega_1)k_{\alpha_2\beta_2}(-\omega_1)k_{\alpha_3\beta_3}(\omega_2)k_{\alpha_4\beta_4}(-\omega_2) \times \{Q^e_{\beta_1,\dots,\beta_4}(\omega_1, -\omega_2, \omega_2+\lambda_5) - Q^e_{\beta_1,\dots,\beta_4}(\omega_1, -\omega_2, \omega_2)\}.$$

The former product, however, is seen to be in modulus less than or equal to

$$c_{11}\int_{-\pi}^{\pi} |k_{\alpha_1\beta_1}(\omega_1+\lambda_3+\lambda_4+\lambda_5) - k_{\alpha_1\beta_1}(\omega_1+\lambda_4+\lambda_5)|^2 g^{(1)}(\omega_1)\,d\omega_1$$

in view of the Schwarz inequality. Then Conditions B(i) and (iii) imply the required result. For the latter, the Schwarz inequality and the uniform Lipschitz condition imply that it is of the order $O(\varepsilon^{\gamma_2})$ in modulus for $|\lambda_5| < \varepsilon$. The boundedness can be proved quite similarly by means of the Schwarz inequality and the boundedness of Q^e. $\square$

Proof of Theorem 1.2. Set $f'(\omega) = \Gamma^{-1}(e^{-i\omega})f(\omega)\Gamma^{-1}(e^{-i\omega})^*$ where, in view of the construction, $f'(\omega)$ is equal a.e. to the identity matrix. Also define the coefficients $G'(j)$ by $k'(\omega) \equiv \sum G'(j)e^{i\omega j}$, and define a new process $\{z'(t),\ t \in J\}$ by $z'(t) = \sum_{j=0}^{\infty} G'(j)e(t-j)$. It is evident in view of the assumptions of this theorem and Theorem 1.1 that

$$\begin{aligned}(3.13)\qquad &\lim_{n\to\infty}\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}[g^{(j)}(\omega)E\{I_n(z,\omega)\} - g^{(j)}(\omega)f(\omega)]\,d\omega \\ &\quad = \lim_{n\to\infty}\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}[g^{(j)}(\omega)E\{I_n(z',\omega)\} - g'^{(j)}(\omega)f'(\omega)]\,d\omega = 0.\end{aligned}$$

It is shown below that $\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}\{g^{(j)}(\omega)I_n(z,\omega) - g^{(j)}(\omega)f(\omega)\}\,d\omega$ has the same asymptotic distribution as $\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}\{g'^{(j)}(\omega)I_n(z',\omega) - g'^{(j)}(\omega)f'(\omega)\}\,d\omega$. In order to see this, let $\{\tilde{z}(t)\}$ be the joint process $\{z(t), z'(t)\}$; then, in view of the construction of $\{z'(t)\}$, the process $\{\tilde{z}(t)\}$ has a joint spectral density matrix

$$\tilde{f}(\omega) = \begin{bmatrix} f(\omega) & h(\omega)^* \\ h(\omega) & f'(\omega)\end{bmatrix} \quad \text{where } h(\omega) = \Gamma(e^{-i\omega})^{-1}f(\omega).$$

Let $\tilde{g}^{(j)}(\omega)$ be the matrix of the form either

$$\tilde{g}^{(j)}(\omega) = \begin{bmatrix} g^{(j)}(\omega) & 0 \\ 0 & 0\end{bmatrix} \quad \text{or} \quad \tilde{g}^{(j)}(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & g'^{(j)}(\omega)\end{bmatrix},$$

and apply Lemma 1.1 to the quadratic form

$$\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}\{\tilde{g}^{(j)}(\omega)I_n(\tilde{z},\omega) - \tilde{g}^{(j)}(\omega)\tilde{f}(\omega)\}\,d\omega, \qquad j = 1,\dots,s.$$

For example,

$$\lim_{n\to\infty} n \operatorname{Cov}\left[\int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)I_n(z, x)\}\, d\omega, \int_{-\pi}^{\pi} \operatorname{tr}\{g'^{(j)}(\omega)I_n(z', x)\}\, d\omega\right]$$
$$= 4\pi \int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)h(\omega)g^{(j)}(\omega)h(\omega)\}\, d\omega$$
$$+ 2\pi \sum_{\beta_1,\dots,\beta_4=1}^{p} \left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \{k(\omega_1)^* g^{(j)}(\omega_1)k(\omega_1)\}_{\beta_1\beta_2} \times \{k(\omega_2)^* g^{(j)}(\omega_2)k(\omega_2)\}_{\beta_3\beta_4} \times Q^e_{\beta_1,\dots,\beta_4}(-\omega_1, \omega_2, -\omega_2)\, d\omega_1\, d\omega_2\right].$$

Now it follows from the equalities

$$\operatorname{tr}\{g^{(j)}(\omega)f(\omega)g^{(j)}(\omega)f(\omega)\} = \operatorname{tr}\{g'^{(j)}(\omega)f'(\omega)g'^{(j)}(\omega)f'(\omega)\} = \operatorname{tr}\{g^{(j)}(\omega)h(\omega)^* g^{(j)}(\omega)h(\omega)\}$$

and

$$k(\omega)^* g^{(j)}(\omega)k(\omega) = k'(\omega)^* g'^{(j)}(\omega)k'(\omega)$$

that

$$\lim_{n\to\infty} n \operatorname{Var}\left[\int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)I_n(z, \omega)\}\, d\omega\right]$$
$$= \lim_{n\to\infty} n \operatorname{Var}\left[\int_{-\pi}^{\pi} \operatorname{tr}\{g'^{(j)}(\omega)I_n(z', \omega)\}\, d\omega\right]$$
$$= \lim_{n\to\infty} n \operatorname{Cov}\left[\int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)I_n(z, \omega)\}\, d\omega, \int_{-\pi}^{\pi} \operatorname{tr}\{g'^{(j)}(\omega)I_n(z', \omega)\}\, d\omega\right]$$
$$(3.14)\qquad = 4\pi \int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)f(\omega)g^{(j)}(\omega)f(\omega)\}\, d\omega$$
$$+ 2\pi \sum_{\beta_1,\dots,\beta_4=1}^{p} \left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \{k(\omega_1)^* g^{(j)}(\omega_1)k(\omega_1)\}_{\beta_1\beta_2} \times \{k(\omega_2)^* g^{(j)}(\omega_2)k(\omega_2)\}_{\beta_3\beta_4} \times Q^e_{\beta_1,\dots,\beta_4}(\omega_1, \omega_2, -\omega_2)\, d\omega_1\, d\omega_2\right].$$

The equalities (3.13) and (3.14) imply that the variance of the difference between

$$\sqrt{n} \int_{-\pi}^{\pi} \operatorname{tr}\{g^{(j)}(\omega)I_n(z, \omega) - g^{(j)}(\omega)f(\omega)\}\, d\omega$$

and

$$\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}\{g'^{(j)}(\omega)I_n(z',\omega)-g'^{(j)}(\omega)f'(\omega)\}\,d\omega$$

tends to 0 as $n\to\infty$ for each j so that the former quantities for $j=1,\dots,s$ have the same joint asymptotic distribution as the latter. On the other hand, the process $\{z'(t)\}$ satisfies the conditions of Theorem 3.1, so that the quantities

$$\sqrt{n}\int_{-\pi}^{\pi}\operatorname{tr}\{g'^{(j)}(\omega)I_n(z',\omega)-g'^{(j)}(\omega)f'(\omega)\}\,d\omega,\qquad j=1,\dots,s,$$

are seen to have the limit normal distribution with mean 0 and covariance matrix determined by the formula (1.4) in view of Lemma 1.1 and the Berstein lemma [see Hannan (1970), page 242]. This completes the proof. □

3.2. *Proofs for Section* 2.

LEMMA 3.3. *If $g_{\alpha_2\alpha_1}(\omega)$ is essentially bounded and $f_{\alpha_1\alpha_2}(\omega)$ is in L^u, $u>1$, then*

$$\lim_{n\to\infty}\operatorname{Var}\left\{\int_{-\pi}^{\pi}g_{\alpha_2\alpha_1}(\omega)I_{\alpha_1\alpha_2}(z,\omega)\,d\omega\right\}=0.$$

PROOF. It follows from (3.5) that

$$\begin{aligned}
&\left|n^{-2}\sum\tilde{g}_{\alpha_1\alpha_2}(t_1-t_2)\tilde{g}_{\alpha_1\alpha_2}(t_3-t_4)\gamma^z_{\alpha_1\alpha_1}(t_3-t_1)\gamma^z_{\alpha_2\alpha_2}(t_4-t_2)\right|\\
&\qquad\le n^{-1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|K^4(\omega_1-\omega_3,-\omega_1+\omega_4,-\omega_2+\omega_3,\omega_2-\omega_4)|\\
&\qquad\qquad\times|f_{\alpha_1\alpha_1}(\omega_3)f_{\alpha_2\alpha_2}(\omega_4)|\,d\omega_1\cdots d\omega_4\\
&\qquad\le n^{-1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\varphi_n(-\omega_2+\omega_3)|\,|\varphi_n(\omega_2-\omega_4)|\\
&\qquad\qquad\times|f_{\alpha_1\alpha_1}(\omega_3)f_{\alpha_2\alpha_2}(\omega_4)|\,d\omega_2\,d\omega_3\,d\omega_4,
\end{aligned}$$

where, by means of the decomposition of the domain of integration into $\{|\omega_3-\omega_4|\le n^{-1/2}\}$ and $\{|\omega_2-\omega_4|>n^{-1/2}\}$ and by an argument similar to the proof of Lemma 3.1, the last member in the above inequalities is seen to be of order $o(1)$. In the same way, we have

$$\left|n^{-2}\sum\tilde{g}_{\alpha_1\alpha_2}(t_1-t_2)\tilde{g}_{\alpha_1\alpha_2}(t_3-t_4)\gamma^z_{\alpha_1\alpha_2}(t_4-t_1)\gamma^z_{\alpha_2\alpha_1}(t_3-t_2)\right|=o(1).$$

As for the member involving the fourth-order cumulants,

$$\begin{aligned}
&\left|n^{-2}\sum_{t_1,\dots,t_4=1}^{n}\tilde{g}_{\alpha_1\alpha_2}(t_1-t_2)\tilde{g}_{\alpha_1\alpha_2}(t_3-t_4)\tilde{Q}^z_{\alpha_1\alpha_2\alpha_1\alpha_2}(t_2-t_1,t_3-t_1,t_4-t_1)\right|\\
&\qquad\le n^{-1}\sum_{\beta_1,\dots,\beta_4}\int\cdots\int|K^{(4)}(\omega_1-\omega_3-\omega_4-\omega_5,\\
&\qquad\qquad-\omega_1+\omega_3,\omega_2+\omega_4,-\omega_2+\omega_5)|
\end{aligned}$$

$$\times |k_{\alpha_1\beta_2}(\omega_3+\omega_4+\omega_5)k_{\alpha_2\beta_2}(-\omega_3)k_{\alpha_1\beta_3}(-\omega_4)k_{\alpha_2\beta_4}(-\omega_5)|\,d\omega_1\cdots d\omega_5$$

$$\le c_1 n^{-1}\sum_{\beta_1,\dots,\beta_4}\int\cdots\int|\varphi(\omega_2+\omega_4)\varphi(-\omega_2+\omega_5)|$$

$$\times |k_{\alpha_1\beta_1}(-\omega_4)k_{\alpha_2\beta_4}(-\omega_5)|\,d\omega_2\,d\omega_4\,d\omega_5.$$

Again, by means of the decomposition of the domain of integration into $\{|\omega_4 - \omega_5| \le n^{-1/2}\}$ and $\{|\omega_4-\omega_5| > n^{-1/2}\}$, the last member is seen to be of order $o(1)$, since $k_{\alpha_1\beta_3}, k_{\alpha_2\beta_4} \in L^{2u}$ for some $u > 1$. □

PROOF OF THEOREM 2.1. Given $\varepsilon_1 > 0$, let $B(a(\psi))$ be the open ball $\{\psi_1\colon |\psi_1-\psi| < a(\psi)\}$ and let $\hat{h}_j(\omega)$ and $\tilde{h}_j(\omega)$ be the bracketing functions which satisfy (2.2) for ε_1 and $a = a(\psi)$. Then we have

$$\begin{aligned}\sup_{B(a(\psi))}|S_{nj}(\psi_1)-S_{nj}(\psi)| &\le \sup_{B(a(\psi))}|H_j(\psi_1)-H_j(\psi)| \\ &\quad+\int_{-\pi}^{\pi}\operatorname{tr}[(\bar{h}_j(\omega)-\tilde{h}_j(\omega))f(\omega)]\,d\omega \\ &\quad+\int_{-\pi}^{\pi}\operatorname{tr}\{\bar{h}_j(\omega)-\tilde{h}_j(\omega)\}\{I_n-E(I_n)\}\,d\omega \\ &\le c_1\varepsilon_1+o_p(1),\end{aligned}$$

so that

$$\sup_{B(a(\psi))}|S_n(\psi_1)-S_n(\psi)| \le c_1 s\varepsilon_1 + o_p(1).$$

Since $R(\psi)$ is continuous due to Assumption C, given an open neighborhood N of ψ_0, there is $\varepsilon_2 > 0$ such that $\inf_{\Psi/N}|R(\psi)| > \varepsilon_2$. Suppose that $B_j = B(r(\psi_j))$, $j=1,\dots,k$, is an open finite subcover of Ψ/N. Then

$$\begin{aligned}\inf_{\Psi/N}|S_n(\psi)| &\ge \inf_j|R(\psi_j)| - \sup_{B_j}|S_n(\psi)| + \sup_j|S_n(\psi_j)-R(\psi_j)| \\ &\ge \varepsilon_2 - c_1 s\varepsilon_1 + o_p(1),\end{aligned}$$

since $\sup_j|S_n(\psi_j)-R(\psi_j)| \to 0$ in probability in view of Lemma 3.3. Now choose ε_1 so that $\varepsilon_2 - c_1 s\varepsilon_1 > 0$ and set $\varepsilon = \varepsilon_2 - c_1 s\varepsilon_1$. Then the proof is complete. □

The next lemma constitutes the main part of Theorem 2.2.

LEMMA 3.4. *$\sqrt{n}\{S_n(\psi_0) + R(\tilde{\psi}_n)\} \to 0$ in probability as $n\to\infty$ if $\sqrt{n}S_n(\tilde{\psi}_n) \to 0$ in probability and $\Pr\{|\tilde{\psi}_n - \psi_0| \le d_0\} \to 0$ as $n \to \infty$ given any sufficiently small $d_0 > 0$.*

PROOF. It is shown below that in probability

$$\sup_{|\psi-\psi_0|\le d_0}|S_n(\psi)-S_n(\psi_0)-R(\psi)|/\{n^{-1/2}+|R(\psi)|\}\to 0 \tag{3.15}$$

as $n \to \infty$. Then the lemma follows from it by an argument quite similar to that given by Huber (1967), page 230. For the sake of notational simplicity, denote $h_j(\omega, \psi) - h_j(\omega, \psi_0)$ by $h - h^0$ and set $T_n(g) = \int_{-\pi}^{\pi} \operatorname{tr}\{(g - h^0)(I_n - E(I_n))\}\, d\omega$. Also set $d_0 = 1$ without loss of generality. Notice in the inequality

$$\sup_{|\psi| \le 1} \left| \int_{-\pi}^{\pi} \operatorname{tr}\{(h - h^0)(I_n - f)\}\, d\omega \right|$$
$$\le \sup_{|\psi| \le 1} |T_n(h)| + \sup_{|\psi| \le 1} \left| \int_{-\pi}^{\pi} \operatorname{tr}\{(h - h^0)(EI_n - f)\}\, d\omega \right|,$$

the second member on the right-hand side is bounded by

$$c_1 \sum_{\alpha, \beta = 1}^{q} \|\{(h - h^0) f\}_{\alpha\beta}\|_v \|\{f^{-1}(EI_n - f)\}_{\beta\alpha}\|_u,$$

where, by Assumption C(ii), $\|\{(h - h^0) f\}_{\alpha\beta}\|_u < c_2$ for some $c_2 > 0$ and $\|\{f^{-1}(EI_n - f)\}_{\beta\alpha}\|_u^u = O(n^{-c})$ for some $c > 1/2$ in view of Assumption D(i) and Theorem 1.1. Therefore, it suffices to show that, as $n \to \infty$,

$$\sup_{|\psi| \le 1} |T_n(h)| / (n^{-1/2} + |R(\psi)|) \to 0.$$

Choose l_0 such that $n/2 < 4^{l_0 + 1} < n$ and let $B(l)$ be the ball with center ψ_0 and radius 2^{-l}, $l = 0, 1, \dots, l_0$, and let $A(l)$ denote the difference $B(l-1) \backslash B(l)$. Given $\varepsilon > 0$, let $U^1, \dots, U^m$ be a partition of $A(l)$ which satisfies (2.7) for ε' determined below. Set $Q(2^{-l}) = \max_{\beta_1 \beta_2} \{\|[k^*(\bar{h}^i - h^0)k]_{\beta_1\beta_2}\|_v + \|[k^*(\tilde{h}^i - h^0)k]_{\beta_1\beta_2}\|_v\}$. Now in view of the property that the product of nonnegative definite Hermitian matrices is nonnegative definite, we have, for a pair of Hermitian bracketing functions $\bar{h}^i_j$ and $\tilde{h}^i_j$,

$$T_n(h) \le T_n(\bar{h}^i) + \int_{-\pi}^{\pi} \operatorname{tr}\{(\bar{h}^i - \tilde{h}^i) E(I_n)\}\, d\omega,$$

whereas, for $\psi \in U^i$, it follows from Assumption D(iii) that

$$\left| \int_{-\pi}^{\pi} \operatorname{tr}\{(\bar{h}^i - \tilde{h}^i) E(I_n)\}\, d\omega \right|$$
$$\le \|\operatorname{tr}\{(\bar{h}^i - \tilde{h}^i) f\}\|_v \|\operatorname{tr}\{f^{-1}(E(I_n) - f)\}\|_u + \|\operatorname{tr}\{(\bar{h}^i - \tilde{h}) f\}\|_1$$
$$\le \left(\frac{c_1 \varepsilon}{4}\right) 2^{-l},$$

since

$$\|\operatorname{tr}\{(\bar{h}^i - \tilde{h}) f\}\|_1 \le \|\operatorname{tr}\{(\bar{h}^i - \tilde{h}) f\}\|_v$$
$$\le c_1 \max_{\beta_1 \beta_2} \{\|[k^*(\bar{h}^i - h^0)k]_{\beta_1\beta_2}\|_v + \|[k^*(\tilde{h}^i - h^0)k]_{\beta_1\beta_2}\|_v\}.$$

Therefore, in view of Assumption D(iii), we have

$$\begin{aligned}\text{(3.16)}\qquad &\Pr\Big[\sup_{A(l)} T_n(h)/\{n^{-1/2}+|R(\psi)|\} > \varepsilon\Big] \\ &\quad \le m(\varepsilon')\max_i \Pr\{\sqrt{n}T_n(\bar{h}^i) > \varepsilon a_1\sqrt{n}2^{-(l+1)}\}.\end{aligned}$$

It follows from the asymptotic covariance formula of Theorem 3.1 that, for sufficiently large n,

$$\begin{aligned}\operatorname{Var}\{T_n(\bar{h}^i)\} &\le c_2\Big[\max_{\alpha_1\alpha_2}\|[(\bar{h}^i-h^0)f]_{\alpha_1\alpha_2}\|_2^2 \\ &\qquad + \max_{\beta_1\beta_2}\|[k^*(\bar{h}^i-h^0)k]_{\beta_1\beta_2}\|_2^2\Big] \\ &\le c_3 Q(2^{-l}).\end{aligned}$$

Thus, in view of the Schwarz inequality, the right-hand-side member in (3.16) is not greater than

$$m(\varepsilon')c_3Q(2^{-l})/(\varepsilon a_1\sqrt{n}2^{-(l+1)})^2 = m(\varepsilon')c_3Q(2^{-l})4^{l+1}/(n\varepsilon^2).$$

A similar bound can be given to $\Pr\{\inf_{A(l)} T_n(h) > -\varepsilon\}$ by the bracketing method and consequently we have

$$\Pr\Big[\sup_{A(l)}|T_n(h)|/\{n^{-1/2}+|R(\psi)|\} > \varepsilon\Big] \le 8m(\varepsilon')c_3Q(2^{-l})4^l/(n\varepsilon^2).$$

Furthermore, it is shown in a similar way that

$$\Pr\Big[\sup_{B(l_0)}|T_n(h)|/\{n^{-1/2}+|R(\psi)|\} > \varepsilon\Big] \le c_3Q(2^{-l_0})/\varepsilon^2.$$

Set l' and ε' such that, for $l \ge l'$, $8m(\varepsilon')c_3Q(2^{-l})/\varepsilon^2 < \varepsilon$. Then

$$\begin{aligned}&\Pr\Big[\sup_{B_0}|T_n(h)|/\{n^{-1/2}+|R(\psi)|\} > \varepsilon\Big] \\ &\quad \le \left(\sum_{l=0}^{l'-1}+\sum_{l'}^{l_0}\right)\Pr\Big[\sup_{A(l)}|T_n(h)|/\{n^{-1/2}+|R(\psi)|\} > \varepsilon\Big] \\ &\qquad + \Pr\Big[\sup_{B(l_0)}|T_n(h)|/\{n^{-1/2}+|R(\psi)|\} > \varepsilon\Big] \\ &\quad \le 8m(\varepsilon')c_3Q(1)(4^{l'}-1)/(3n\varepsilon^2) + \varepsilon(4^{l_0+1}-1)/(3n) + c_3Q(2^{-l_0}/\varepsilon^2).\end{aligned}$$

Since l' is independent of n, the first and the third terms above tend to 0 as $n \to \infty$ and the second term is less than ε, whence the lemma follows. □

PROOF OF THEOREM 2.2. Theorem 1.2 implies that, under the assumptions of Theorem 2.2, $\sqrt{n}S_n(\psi_0)$ tends to a multivariate normal distribution with mean 0 and with covariance matrix U. Then the theorem is the immediate consequence of Lemma 3.4. □

PROOF OF THEOREM 2.3. The theorem is a specific case of Theorem 2.2, and the proof proceeds similarly to Hosoya (1989a), pages 414 and 415. The latter result is extended straightforwardly to non-square-integrable $f(\omega, \psi)$ under Assumption D. □

PROOF OF THEOREM 2.4. Assumption D(iv) implies that, in a neighborhood of ψ_0,

$$\partial \bar{L}_n/\partial\psi_j = -n\left[H_j(\psi) + \int_{-\pi}^{\pi} \operatorname{tr}\{h_j(\omega;\psi)I_n(z;\omega)\}\,d\omega\right].$$

Set $\psi_t(l) = t\psi_0 + (1-t)\tilde{\psi}_n(l)$. Then, for $i = 0, \dots, r$,

$$\begin{aligned}\bar{L}(\tilde{\psi}_n(i)) - \bar{L}(\psi_0) &= \sum_{j=1}^{l+m}\int_0^1 \sqrt{n}\{S_{nj}(\psi_t) - S_{nj}(\tilde{\psi}_n(i))\}\,dt\sqrt{n}(\tilde{\psi}_n - \psi_0)_j + o_p(1)\\ &= \tfrac{1}{2}\sum_j\sum_k V_{jk}\sqrt{n}(\tilde{\psi}_n(i) - \psi_0)_j\sqrt{n}(\tilde{\psi}_n(i) - \psi_0)_k + o_p(1),\end{aligned}$$

where the last equality holds in view of Lemma 3.4. Therefore, Theorem 2.3 implies that

$$\bar{L}_{n,0i} = \{\bar{L}_n(\tilde{\psi}_n(0)) - \bar{L}_n(\psi_0)\} - \{\bar{L}_n(\tilde{\psi}_n(i)) - \bar{L}_n(\psi_0)\}$$

has the same asymptotic distribution as

$$-\tfrac{1}{2}\left\{\sum_{j,k=1}^{l_i} V^{jk}(i)\sqrt{n}S_{nj}(\psi_0)\sqrt{n}S_{nk}(\psi_0) - \sum_{j,k=1}^{l_0} V^{jk}(0)\sqrt{n}S_{nj}(\psi_0)\sqrt{n}S_{nj}(\psi_0)\right\},$$

where $V(i) = \{V_{ij},\ i, j = 1, \dots, l_i\}$, and $V^{jk}(i)$ signifies the (j,k) element of $V(i)^{-1}$. Let C be a lower-triangular real matrix such that $C^*C = 4\pi V^{-1}$. Let $C^{(i)}$ be the triangular matrix $\{C_{j,k};\ j,k = 1,\dots,l_i\}$. Set $v(n) = C^{-1}\sqrt{n}\bar{S}_n(\psi_0)$, where $\bar{S}_n(\psi_0)$ is the vector $\{S_{nj}(\psi_0),\ j = 1,\dots,l_i\}$. Then Theorem 2.3 implies that the $v_j(n),\ j = 1,\dots,l_i$, are asymptotically independently standard normally distributed. Denote by $v^{(i)}(n)$ the column l_i-vector $\{v_j(n),\ j = 1,\dots,l_i\}$. Then, since

$$\begin{aligned}\bar{L}_{n,0i} &= -\tfrac{1}{2}\{v^{(i)}(n)C^{(i)*-1}V(i)^{-1}C^{(i)-1}v^{(i)}(n)\\ &\qquad - v^{(0)}(n)^*C^{(0)*-1}V(0)^{-1}C^{(0)-1}v^{(0)}(n)\}\\ &= -2\pi\sum_{j=l_0+1}^{l_i} v_j(n)^2,\end{aligned}$$

the theorem follows. □

Acknowledgments. The author is grateful to the two referees for their helpful comments.

REFERENCES

COX, D. R. (1984). Long-range dependence: a review. In *Statistics: An Appraisal* (H. A. David and H. T. David, eds.) 55–73. Iowa State Univ. Press.

COX, D. R. (1991). Long-range dependence, non-linearity and time irreversibility. *J. Time Ser. Anal.* **12** 329–335.

DAHLHAUS, R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Statist.* **17** 1749–1766.

DANIELS, H. E. (1961). The asymptotic efficiency of a maximum likelihood estimator. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **1** 151–163. Univ. California Press, Berkeley.

DUNSMUIR, W. (1979). A central limit theorem for parameter estimation in stationary vector time series and its application to models for a signal observed with noise. *Ann. Statist.* **7** 490–506.

DUNSMUIR, W. and HANNAN, E. J. (1976). Vector linear time series models. *Adv. in Appl. Probab.* **8** 339–364.

DZHAPARIDZE, K. O. (1974). A new method for estimating spectral parameters of a stationary regular time series. *Theory Probab. Appl.* **19** 122–132.

FINDLEY, D. F. and WEI, C. Z. (1993). Moment bound useful for deriving time series model selection procedures. *Statist. Sinica* **3** 453–480.

FOX, R. and TAQQU, M. S. (1986). Large sample properties of parameter estimates for strongly dependent stationary Gaussian time series. *Ann. Statist.* **14** 517–532.

FOX, R. and TAQQU, M. S. (1987). Central limit theorems for quadratic forms in random variables having long-range dependence. *Probab. Theory Related Fields* **74** 213–240.

GIRAITIS, L. and SURGAILIS, D. (1990). A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotical normality of Whittle's estimate. *Probab. Theory Related Fields* **86** 87–104.

GRANGER, C. W. J. and JOYEUX, R. (1980). An introduction to long-memory time series and fractional differencing. *J. Time Ser. Anal.* **1** 15–29.

HANNAN, E. J. (1970). *Multiple Time Series*. Wiley, New York.

HANNAN, E. J. (1973). The asymptotic theory of linear time-series models. *J. Appl. Probab.* **10** 130–145.

HEYDE, C. C. and GAY, G. (1993). Smoothed periodogram asymptotics and estimation for processes and fields with possible long-range dependence. *Stochastic Process. Appl.* **45** 169–182.

HOSOYA, Y. (1974). Estimation problems on stationary time series models. Ph.D. dissertation, Dept. Statistics, Yale Univ.

HOSOYA, Y. (1989a). The bracketing condition for limit theorems on stationary linear processes. *Ann. Statist.* **17** 401–418.

HOSOYA, Y. (1989b). Hierarchical statistical models and a generalized likelihood ratio test. *J. Roy. Statist. Soc. Ser. B* **51** 435–447.

HOSOYA, Y. (1993). Limit theorems for statistical inference on stationary processes with strong dependence. In *Statistical Sciences and Data Analysis. Proceedings of the Third Pacific Area Statistical Conference* (K. Matsusita, M. L. Puri and T. Hayakawa, eds.) 151–163. Interscience, Utrecht.

HOSOYA, Y. and TANIGUCHI, M. (1982). A central limit theorem for stationary processes and the parameter estimation of linear processes. *Ann. Statist.* **10** 132–153.

HOSOYA, Y. and TANIGUCHI, M. (1993). Correction to Y. Hosoya and M. Taniguchi. *Ann. Statist.* **21** 115–117.

HUBER, P. (1967). The behavior of maximum-likelihood estimates under nonstandard conditions. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **1** 221–231. Univ. California Press, Berkeley.

MANDELBROT, B. B. and WALLIS, J. R. (1969). Some long-run properties of geophysical records. *Water Resource Research* **5** 321–340.

POLLARD, D. (1985). New ways to prove central limit theorems. *Econometric Theory* **1** 295–314.

ROBINSON, P. M. (1995). Gaussian semiparametric estimation of long range dependence. *Ann. Statist.* **23** 1630–1661.

ROZANOV, Y. A. (1967). *Stationary Random Process.* Holden-Day, San Francisco.

YAJIMA, Y. (1985). On estimation of long-memory time series models. *Austral. J. Statist.* **27** 303–320.

WHITTLE, P. (1952). Some results in time series analysis. *Skand. Aktuarietidskr.* **35** 48–60.

FACULTY OF ECONOMICS
TOHOKU UNIVERSITY
KAWAUCHI, SENDAI 980
JAPAN
E-MAIL: hosoya@econ.tohoku.ac.jp