A LIMITATION THEOREM FOR CESÀRO SUMMABLE SERIES

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1. Introduction

We consider the Cesàro summability, for integral orders of the series $\sum_{\nu=0}^{\alpha} a_\nu d_\nu$. In this paper we establish a limitation theorem for this series. Results of this character, but not overlapping with those in this paper, were given by Hardy and Littlewood [7] and by Andersen [1]. Andersen's result was extended by Bosanquet and Chow [5], and further extended by Bosanquet [4].

Notation. We write

$$A_n^0 = A_n = a_0 + a_1 + \cdots + a_n, \quad A_n^k = A_0^{k-1} + A_1^{k-1} + \cdots + A_n^{k-1}$$

and we get the identities [6]

$$A_n^k = \sum_{\nu=0}^{n} \binom{n-\nu+k-1}{k-1} A_\nu, \quad A_n^k = \sum_{\nu=0}^{n} \binom{n-\nu+k}{k} a_\nu, \quad E_n^k = A_n^k$$

when $a_0 = 1$, $a_n = 0$, for $n > 0$ i.e. when $A_n = 1$, for all n. So

$$E_n^k = \binom{n+k}{k} \sim \frac{n^k}{k!}.$$

$\sum a_n$ is said to be summable (C, k) to A if $A_n^k/E_n^k \to A$ as $n \to \infty$, or equivalently if $k!\, A_n^k/n^k \to A$.

We write $\Delta d_n = d_n - d_{n-1}$, following L. S. Bosanquet [3]. We will use the following identity (see L. S. Bosanquet [3]):

$$\Delta^k (U_n V_n) = \sum_{\nu=0}^{k} \binom{k}{\nu} \Delta^\nu U_n \Delta^{k-\nu} V_{n-\nu}. \tag{1.1}$$

2. Statement of the theorem and two lemmas.

THEOREM 1. *Suppose that $d_n > 0$, for $n \geq 0$, and*

(i) $d_{n+1} = o(1)$ *as* $n \to \infty$,

(ii) $\displaystyle \frac{d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \binom{n-\nu+k}{k} \frac{1}{d_{\nu+k+1}} = O(1)$,

(iii) $|\Delta^j (1/d\nu + k + 1)| \leq K |\Delta^{j-1} (1/d_\nu + k + 1)|$,

$j = 1, 2, \cdots, k+1$; $k \geq 0$, *k an integer*; Δ *operating on* ν.

Then $A_n^k = o(n^k/d_{n+1})$ whenever $\sum_{\nu=0}^{\infty} a_\nu d_\nu$ is summable (C, k).

We require the following lemmas.

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LEMMA A. *In order that* $t_m = \sum c_{m,n} S_n \to 0$ $(m \to \infty)$ $(m = 1, 2, \cdots)$ *whenever* $S_n \to 0$ $(n \to \infty)$, *it is necessary and sufficient that*

(i) $\sum | C_{m,n} | < H$, *where* H *is independent of* m *and*

(ii) $C_{m,n} \to 0$ *for each* n, *when* $m \to \infty$.

Lemma A is given by Hardy [6, Theorem 4], which follows from a theorem given by Toeplitz [9]. Toeplitz considers only "triangular" transformations in which $C_{m,n} = 0$ for $n > m$. Steinhaus [8] made extension for general transformations.

LEMMA B. *If* d_n *satisfies conditions of Theorem* 1 *then*

$$\frac{d_{n+1}}{n^k} \sum_{\nu=0}^{\infty} \nu^k \left| \Delta^{k+1} \left\{ \frac{1}{d_{\nu+k+1}} \binom{n-\nu-1}{k} \right\} \right| = O(1).$$

We have

$$\begin{aligned} \frac{d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \left| \Delta^{k+1} \left\{ \frac{1}{d_{\nu+k+1}} \binom{n-\nu-1}{k} \right\} \right| & \\ \leq \alpha \frac{d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \left| \binom{n-\nu+k}{k} \right| \left| \Delta^{k+1} \left(\frac{1}{d_{\nu+k+1}} \right) \right| & \\ + \frac{\alpha d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \left| \Delta \binom{n-\nu+k-1}{k} \right| \left| \Delta^k \left(\frac{1}{d_{\nu+k+1}} \right) \right| & \\ + \cdots + \frac{\alpha d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \left| \Delta^k \binom{n-\nu}{k} \right| \left| \Delta \left(\frac{1}{d_{\nu+k+1}} \right) \right| & \\ + \frac{\alpha d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \left| \Delta^{k+1} \binom{n-\nu-1}{k} \frac{1}{d_{\nu+k+1}} \right| & \end{aligned} \tag{2.1}$$

using identity (1.1), where the α's are various constants.

By (2.1) it will be enough to prove that

$$\frac{d_{n+1}}{n^k} \sum_{\nu=0} \nu^k \left| \Delta^j \binom{n-\nu+k-j}{k} \right| \left| \Delta^{k+1-j} \left(\frac{1}{d_{\nu+k+1}} \right) \right| = O(1), j = 0, 1, \cdots, k+1. \tag{2.2}$$

But we have

$$\begin{aligned} \left| \Delta^j \binom{n-\nu+k-j}{k} \right| & \leq \beta \binom{n-\nu+k-j}{k} + \beta \binom{n-\nu+k-j+1}{k} + \cdots + \beta \binom{n-\nu+k}{k} \\ & \leq K \binom{n-\nu+k}{k} \end{aligned} \tag{2.3}$$

where $j = 0, 1, \cdots, k+1$ and the β's are various positive constants; and

$$\Delta^{k+1-j}(1/d_{\nu+k+1}) < K' \left| \Delta^{k-j}(1/d_{\nu+k+1}) \right| \qquad j = 0, \cdots, k, \tag{2.4}$$

by hypothesis (iii).

Then since

$$\frac{d_{n+1}}{n^k} \sum_{\nu=0}^{k} \nu^k \binom{n-\nu+k}{k} \frac{1}{d_{\nu+k+1}} = O(1) \tag{2.5}$$

by hypothesis (ii), (2.2) follows immediately from (2.3), (2.4) and (2.5).

3. Proof of the theorem

We can assume that $\sum_{\nu=0}^{\infty} a_\nu d_\nu$ is summable (C, k) to 0. Then $C_n^k/n^k \to 0$ as $n \to \infty$. Let $C_n = \sum_{\nu=0}^{n} a_\nu d_\nu$; then

$$\Delta C_n = \sum_{\nu=0}^{n} a_\nu d_\nu - \sum_{\nu=0}^{n-1} a_\nu d_\nu = a_n d_n. \tag{3.1}$$

Now

$$A_n^k = \sum_{\nu=0}^{n} \binom{n-\nu+k}{k} a_\nu = \sum_{\nu=0}^{n} \binom{n-\nu+k}{k} \frac{\Delta C_\nu}{d_\nu} \qquad \text{(by (3.1))}$$
$$= \sum_{\nu=0}^{n} (-1)^{k+1} C_\nu^k \Delta^{k+1} \left\{ \frac{1}{d_{\nu+k+1}} \binom{n-\nu-1}{k} \right\}.$$

So

$$\frac{d_{n+1} A_n^k}{n^k} = \frac{d_{n+1}}{n^k} \sum_{\nu=0}^{n} (-1)^{k+1} \nu^k \frac{C_\nu^k}{\nu^k} \Delta^{k+1} \left\{ \frac{1}{d_{\nu+k+1}} \binom{n-\nu-1}{k} \right\}$$
$$= \sum_{\nu=0}^{n} \frac{C_\nu^k}{\nu^k} \gamma_{n,\nu},$$

where

$$\gamma_{n,\nu} = \frac{(-1)^{k+1} d_{n+1}}{n^k} \nu^k \Delta^{k+1} \left\{ \frac{1}{d_{\nu+k+1}} \binom{n-\nu-1}{k} \right\}. \tag{3.3}$$

Then, by Lemma B,

$$\sum_{\nu=0}^{n} |\gamma_{n,\nu}| = \frac{d_{n+1}}{n^k} \sum_{\nu=0}^{n} \nu^k \left| \Delta^{k+1} \left\{ \frac{1}{d_{\nu+k+1}} \binom{n-\nu-1}{k} \right\} \right| < H \tag{3.4}$$

Next from (3.5),

$$\gamma_{n,\nu} = \frac{\alpha d_{n+1}}{n^k} \nu^k \binom{n-\nu+k}{k} \Delta^{k+1} \left(\frac{1}{d_{\nu+k+1}} \right)$$
$$+ \frac{\alpha d_{n+1}}{n^k} \nu^k \Delta \binom{n-\nu+k-1}{k} \Delta^k \left(\frac{1}{d_{\nu+k+1}} \right)$$
$$+ \cdots + \frac{\alpha d_{n+1}}{n^k} \nu^k \Delta^k \binom{n-\nu}{k} \Delta \left(\frac{1}{d_{\nu+k+1}} \right)$$
$$+ \frac{\alpha d_{n+1}}{n^k} \nu^k \Delta^{k+1} \binom{n-\nu-1}{k} \frac{1}{d_{\nu+k+1}}$$

(using identity (1.1) where the α's are various constants)

$$= \frac{d_{n+1}}{n^k} \nu^k O(n^k) = o(1) \qquad \text{(by hypothesis (i))}.$$

So $\gamma_{n,\nu} \to 0$ as $n \to \infty$, for each ν. It follows that conditions (i) and (ii) of Lemma A are satisfied and hence $d_{n+1} A_n^k / n^k = o(1)$.

Added in proof. Condition (ii) of the theorem could be replaced by (3.4), and this would then widen the class d_n covered by the theorem.

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References

1. A. F. Andersen, *Comparison theorems in the theory of Cesaro summability*, Proc. London Math. Soc. (2), vol. 27 (1928), pp. 39–71.
2. L. S. Bosanquet, *Note on the Bohr-Hardy theorem*, J. London Math. Soc., vol. 17 (1942), pp. 68–73.
3. ———, *Note on convergence and summability factors*, J. London Math. Soc., vol. 20 (1945), pp. 39–48.
4. ———, *An extension of a theorem of Andersen*, J. London Math. Soc., vol. 25 (1950), pp. 72–80.
5. L. S. Bosanquet and H. C. Chow, *Some analogues of a theorem of Andersen*, J. London Math. Soc., vol. 16 (1941), pp. 42–48.
6. G. H. Hardy, *Divergent series*, Oxford Univ. Press, Oxford, 1949, pp. 42–63.
7. G. H. Hardy and J. E. Littlewood, *A theorem in the theory of summable divergent series*, Proc. London Math. Soc. (2), vol. 27 (1928), pp. 327–348.
8. H. Steinhaus, Prace Mat., vol. 22 (1911), pp. 121–134.
9. O. Toeplitz, Prace Mat., vol. 22 (1911), pp. 113–119.

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