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A linear approximation to the solution of a one-dimensional Stefan problem and its geophysical implications

Jean Claude Mareschal *Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada*

Anthony F. Gangi *Geophysics Department, Texas A and M University, College Station, Texas 77843, USA*

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Summary. The motion of a phase boundary in the Earth caused by temperature and pressure excitations at the Earth's surface is determined under a linear approximation. The solution is found as a sum of convolutions of pressure and temperature Green's functions with the corresponding excitations. The Green's functions are given under the form of Laplace transforms that can be inverted either by numerical evaluation of a branch cut integral or by inversion of a series expansion. This solution is a generalization of a solution previously derived by Gjevik. This latter solution is the first term in the series expansion. The relaxation times associated with the phase boundary motion are of the order of 10^5–10^7yr for the olivine–spinel phase transition and of 10^6–10^7yr for the basalt–eclogite transition. The linear approximation remains valid for long times only if the phase boundary moves slowly.

1 Introduction

Geophysical and geological observations show that vertical crustal movements take place in different regions of the Earth with various characteristic times (Wellman 1972; Artyushkov & Mescherikov 1969). Among the movements of this type, the post-glacial uplift is one of the most studied and best understood. It is generally accepted that the post-glacial uplift can be explained as the response of a viscoelastic Earth to the removal of a load (i.e. the melting of the pleistocene glaciers), at the Earth's surface (Haskell 1935, 1936; McConnell 1965, 1968; Peltier 1974). In that case, the driving force of the vertical motion is gravity and a definite correlation should exist between gravity anomalies and vertical crustal motions. The more recent gravity data (Kaula 1972; Gaposhkin 1964) indicate that, as far as post-glacial rebound is concerned, there is a general correlation between the direction of the Earth's surface motion and the gravity anomaly. However, this is not true for other vertical crustal movements that are not directly related to the end of the pleistocene glaciation, such as those observed today in the Russian platform or in the Eastern United States: in this case, only two-thirds of the observed motions show the expected correlation with the gravity anomaly (Artyushkov & Mescherikov 1969).

Very large vertical motions of the Earth's crust are also evidenced by the thickness of sediment deposited in geosynclines. For instance, in the Gulf coast geosyncline (Ewing, Edgar & Antoine 1970), more than 20 000 m of sediment have been deposited at an almost constant shallow depth. The geological evidence indicates that similar depositions happened in other geosynclines (Rocky Mountains, Appalachian). The subsidence of low-density sediments into a higher density substratum cannot be explained by isostatic adjustments alone (Jeffreys 1970, p. 418). Moreover, the sediments that have been deposited below sea level have subsequently been uplifted by several hundreds of metres. Rapid uplifts have also been observed within continents (Colorado Plateau or Hoggar plateau in Africa); the driving mechanism for these uplifts is not yet well understood.

For these reasons, various authors have suggested that these vertical crustal movements are caused by the response of phase boundaries within the Earth to changes in pressure and temperature (Fermor 1914; Goldschmidt 1922; Holmes 1926; Lovering 1958; Kennedy 1959). If density increases with depth, the effect of an increase in pressure is the conversion into the high-density phase of low-density materials and a subsidence of the Earth's surface; an increase in temperature would have the opposite effect. If sediments are deposited at the Earth's surface, both the temperature and the pressure in the Earth will be increased; the effect of the pressure increase is felt instantaneously at the phase boundary and subsidence will occur; the temperature perturbation will not reach the phase boundary and produce an uplift before a very long time has elapsed. This theory will thus explain why the Earth's surface subsides upon deposition of sediments and why the sediments are subsequently uplifted. This effect combined with isostatic adjustments could explain the amount of sediment deposition in geosynclines (Wetherill 1961).

Among the phase transitions that have been suggested to be related to uplift and subsidence of the Earth's surface, the most important is the basalt–eclogite transition that would occur at the depth of the Mohorovicic discontinuity (Lovering 1958; Kennedy 1959; Ito & Kennedy 1971). This transition is accompanied by two sharp increases in density (from 3.0 to 3.2 g/cm^{3}and from 3.2 to 3.45 g/cm^3, Ito & Kennedy 1970) and could therefore explain significant amount of uplift and subsidence. Geophysicists do not agree on the interpretation of the experimental data on the basalt–eclogite transition; some (Ringwood & Green 1966) consider that these experimental data make unlikely the hypothesis of a phase change *M*-discontinuity while others (Kennedy & Ito 1972) claim that these data strongly support the hypothesis. The geophysical data on the structure of the *M*-discontinuity and the upper mantle does not provide an unambiguous answer to the questions about the nature of the Moho. The observation of seismic relfections from the Moho (Clowes, Kanasewich & Cumming 1968) would indicate that, at the Moho, there is a sharp seismic velocity and density change rather than the diffuse transition region that would be expected to accompany a phase change. On the other hand, the combined inversion of travel-time and free-oscillations data (Press 1971; Anderson & Hart 1976) indicate a high upper-mantle density that seems more compatible with an eclogite composition than with the peridotite composition advocated by the opponents of the phase change hypothesis. Many other arguments have been given in favour or against the phase-change hypothesis (see Wyllie 1971, chapter 5, for a review). A compromise that has been suggested by Wyllie (1963) is that the oceanic Moho is not associated with the basalt–eclogite transition, but that the transition is present in tectonically active continental regions. It is the authors opinion that there is no sufficient evidence today to give a definite answer to the question of the nature of the Mohorovicic discontinuity; however, they feel that it is worthwhile to continue the investigations that have been undertaken on some geophysical implications of the phase-change hypothesis even if the hypothesis remains controversial.

If it is assumed that the composition of the upper mantle is not eclogite, but peridotite or pyrolite, other shallow phase changes will occur: they are the transformation of pyrolite from plagioclase to garnet and from garnet to spinel (Green & Ringwood 1967; Wyllie 1971, p, 119–121). If, following Wyllie's (1963) compromise, the composition of the upper mantle is assumed to be peridotite under the oceans, these phase transitions would occur below the oceans, the former at a depth of 30 km and the latter of about 60 km. If, in contradiction to the Moho phase-change hypothesis, the upper mantle has a peridotite composition everywhere, the latter phase change would also occur below the continents. Although the density contrast (of the order of 3 per cent) associated with these phase changes is much smaller than in the basalt–eclogite transition, these transitions may have important effects on the topography of the Earth's surface and in particular they could be responsible for about 20 per cent of the midoceanic ridges elevation (Sclater 1972).

The determination of the phase-boundary motion is called the problem of Stefan. This is a difficult problem to solve analytically because it is non-linear and special solutions have to be determined which cannot be superposed (Carslaw & Jaeger 1959, chapter XI; Rubinstein 1971). Special solutions to the problem can be determined by numerical methods. This approach has been taken (restricting ourselves to the geophysical literature) by McDonald & Ness (1960), Van De Lindt (1967), Joyner (1967) and O'Connell & Wasserburg (1967). McDonald & Ness investigated the effect of a Mohorovicic discontinuity phase change on uplift and subsidence, but did not take into account isostatic adjustments. The effect of isostasy has been included in this model by Van De Lindt (1967) and Joyner (1967). These two models produced deposition of thick sediments in shallow water and in some cases a succession of cycles of erosion and sedimentation.

Although the behaviour of the phase change can very accurately be determined by these numerical approximations, analytical approximations even if they are less accurate, have definite advantages: they demonstrate more clearly the physical phenomena involved and they allow an easy determination of the important physical parameters.

Analytical approximations to the solution of Stefan-like problems have been used by O'Connell & Wasserburg (1967, 1972) in their study of the uplift and subsidence of sedimentary basins underlain by phase changes. A much simpler approach for obtaining analytical approximations has been taken by Gjevik (1972, 1973). In Gjevik's model, the Earth's surface is supposed to be at an infinite distance from the phase boundary which therefore is never affected by a near-surface temperature change. For deep phase changes (i.e. at a depth > 300 km), these thermal effects can indeed be neglected since they require a time of the order of the Earth's age to reach the phase boundary by conduction. But this is not the case for a shallower phase transition (i.e. in the lithosphere). The major purpose of this paper is thus to derive a generalization of Gjevik's solution that would include the thermal effects. In the author's opinion, this approximation is easier to derive and to handle than the three different approximations used by O'Connell & Wasserburg. The range of validity of the linear approximation used to derive the solution will be discussed more fully than in Gjevik's and limits to the use of the solution will be set. Some geophysical applications of these results and their implications will be sketched.

2 Linear approximation to the solution of a Stefan problem

2.1 STATEMENT OF THE PROBLEM

Let us consider a one-dimensional model of the Earth in which a phase transition occurs at a depth depending on the temperature and the pressure. Let ρ_1 and ρ_2 be the density, k_1 and

k_2 the thermal conductivity and c_1 and c_2 the heat capacity of the two phases. The phase boundary motion will be described in a system of coordinates fixed to the Earth's surface ($x = 0$ is the surface and $x = x_m(t)$ is the phase boundary). In that system of coordinates, the deepest phase moves with a velocity dy/dt determined by the mass conservation condition which gives

$$\frac{dy}{dt} = \frac{\rho_2 - \rho_1}{\rho_1} \frac{dx_m}{dt}. \tag{1}$$

The temperature T_m and the pressure P_m for which the two phases are in equilibrium are related by the integrated Clausius–Clapeyron equation which is assumed to be linear. Therefore, we have at the interface

$$T_m = T_c + \gamma P_m \tag{2}$$

(γ is the inverse slope of the Clausius–Clapeyron equation). If $P_0(t)$ is the pressure excitation applied on the Earth's surface, we have that

$$T_m = T_c + \gamma P_0(t) + \gamma g \rho_1 x_m(t). \tag{3}$$

When the phase boundary moves and material is transformed from one phase to another, heat is produced (or absorbed) proportionally to the amount of material undergoing the transformation. In order for heat conservation to hold, we must have that (Carslaw & Jaeger 1959, p. 284)

$$L\rho_1 \frac{dx_m}{dt} = k_2 \left.\frac{\partial T_2}{\partial x}\right|_{x_m} - k_1 \left.\frac{\partial T_1}{\partial x}\right|_{x_m}. \tag{4}$$

(L is the latent heat of transformation; T_1 and T_2 are the temperature in each phase; the condition (4) is referred to as the Stefan boundary condition.)

The motion of the phase boundary will depend on how this extra heat is carried away; this is determined by the solution of the heat equation

$$\frac{\partial T_1}{\partial t} = \mathscr{K}_1 \frac{\partial^2 T_1}{\partial x^2};\ x < x_m(t) \tag{5a}$$

$$\frac{\partial T_2}{\partial t} + \frac{\partial T_2}{\partial x}\frac{dy}{dt} = \mathscr{K}_2 \frac{\partial^2 T_2}{\partial x^2};\ x > x_m(t) \tag{5b}$$

with its boundary and initial conditions ($\mathscr{K}_1$ and $\mathscr{K}_2$ are the thermal diffusivity of the two phases).

It will be assumed that no heat sources are present and that the system is initially in thermal equilibrium. The initial temperature distribution is thus

$$T_1(x, t = 0) = \beta_1 x \tag{6a}$$

$$T_2(x, t = 0) = \beta_1 x_0 + \beta_2 (x - x_0) \tag{6b}$$

and the initial phase-boundary depth, x_0, is determined by

$$\beta_1 x_0 = T_c + \gamma g \rho_1 x_0. \tag{7}$$

The boundary conditions are:

(i) The temperature at the surface $x = 0$ is specified

$$T_1(x = 0, t) = T_0(t). \tag{8}$$

(ii) The temperature gradient at $x = \infty$ remains constant

$$\lim_{x\to\infty} \frac{\partial T_2}{\partial x}(x,t) = \beta_2. \tag{9}$$

(iii) At the interface between the two phases, the temperature is continuous and equal to the transition temperature T_m given by equation (3).
(iv) At the interface, the heat flow is discontinuous and obeys the Stefan boundary condition given by equation (4).

The major difference between this statement of the problem and Gjevik's (1972) is the boundary condition (8), which Gjevik replaced by the condition

$$\lim_{x\to-\infty} T_1(x,t) = \beta_1 x.$$

Gjevik also replaced the boundary condition (11) by the condition that the temperature (instead of the gradient) remains constant at a fixed point, but it can be verified easily that both conditions would give the same results.

2.2 METHOD OF SOLUTION

The solution of this problem can be derived by the method of Gjevik (1972; Mareschal 1975). As another derivation is given in the Appendix, we shall simply recall the major steps and the hypotheses that are necessary to obtain the solution. A new coordinate system, z, fixed to the phase boundary is introduced, as

$$z = x - x_m(t) \tag{10}$$

and the temperature $T_1(z,t)$ and $T_2(z,t)$ are replaced by the temperature perturbations defined by

$$\theta_1(z,t) = T_1(z,t) - \beta_1[z + x_m(t)]. \tag{11a}$$

$$\theta_2(z,t) = T_2(z,t) - \beta_2 z - \beta_1 x_m(t). \tag{11b}$$

Two hypotheses will be made in order to linearize the new set of partial differential equations and boundary conditions:

(i) Because the coordinate system (fixed to the phase boundary) is moving with respect to the material, convection terms of the form $(\partial\theta_1/\partial z)(dx_m/dt)$ and $(\partial\theta_2/\partial z)(dx_m/dt)$ are introduced in the heat equation. These quadratic terms will be assumed negligible by comparison to the conduction terms $\mathscr{K}_1(\partial^2\theta_1/\partial z^2)$ and $\mathscr{K}_2(\partial^2\theta_2/\partial z^2)$. By neglecting those terms, it is thus assumed that transport of heat away from the boundary is more rapid by conduction than by this 'convection'. The range of validity of this approximation will be discussed more fully later.
(ii) It will be assumed that the amplitude of the phase-boundary motion is smaller than the distance between the phase boundary and the free surface. $x_m(t) - x_0 \ll x_0$ and therefore the boundary condition at the surface $\theta_1(z = -x_m(t), t) = T_0(t)$ can be approximated by $\theta_1(z = -x_0, t) = T_0(t)$.

Since the system of equations and boundary conditions have been linearized, Laplace transform techniques (Doetsch 1963) can be used to determine its solution. The Laplace transform $f(s)$ of the function $f(t)$ is defined by

$$f(s) = \int_0^\infty \exp(-st) f(t)\, dt \tag{12}$$

The phase-boundary motion $z_m(t)$ defined by $x_m(t) - x_0$ has the following form in the transform variable

$$z_m(s) = \frac{[-\gamma P_0(s) + T_0(s)\exp(-\sqrt{(\tau_0 s)})]\,[k_2/\sqrt{\mathscr{H}_2} + k_1/\sqrt{\mathscr{H}_1}\coth\sqrt{(\tau_0 s)}]}{L\rho_1\sqrt{s} + (\gamma g\rho_2 - \beta_2)\,\rho_1 k_2/\rho_2\sqrt{\mathscr{H}_2} + (\gamma g\rho_1 - \beta_1)\,k_1/\sqrt{\mathscr{H}_1}\coth\sqrt{(\tau_0 s)}} \tag{13}$$

where we have defined $\tau_0 = x_0^2/\mathscr{H}_1$.

The form of the solution shows that the motion of the phase boundary is the convolution (in time) of the pressure or temperature excitation with a Green's function. Before determining the time dependent motion of the phase boundary, the transform can be simplified by assuming that:

(i) The thermal properties of the two phases are the same: $k_1 = k_2 = k$, $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$ and therefore $\beta_1 = \beta_2 = \beta$.

(ii) In equation (13), $\rho_1 \simeq \rho_2 = \rho$ and therefore, $\beta_2\rho_1 = \beta_1\rho_2 = \beta\rho$. This assumption may not be very good for phase changes at Moho depth where the transition temperature and the geothermal gradient are probably close: in that case, the inverse Laplace transforms will be slightly more complex, but the same type of analysis could be applied. The assumptions introduced at this stage are not as critical as the linearizing assumptions made earlier, since they do not alter the general form of the result. Under these assumptions $z_m(s)$ reduces to

$$z_m(s) = \frac{[-\gamma P_0(s) + T_0(s)\exp(-\sqrt{(\tau_0 s)})]/(\gamma g\rho - \beta)}{\sqrt{\tau s}\,[1 - \exp(-2\sqrt{(\tau_0 s)})] + 1} \tag{14}$$

where

$$\tau = \left\{\frac{L\rho\sqrt{\mathscr{H}}}{2k(\gamma g\rho - \beta)}\right\}^2 \tag{15}$$

This solution differs from Gjevik's solution in two aspects; it includes the effect of the surface temperature $T_0(s)\exp[-\sqrt{(\tau_0 s)}]$ and the effect of the image of the phase boundary with respect to the surface $z = -x_0$ (term $\sqrt{(\tau s)}\exp[-2\sqrt{(\tau_0 s)}]$ in the denominator). In the limit $\tau_0 \to \infty$, the equation (14) is identical to Gjevik's solution.

3 Inversion of the Laplace transform

3.1 DETERMINATION OF THE GREEN'S FUNCTION FOR A PRESSURE PERTURBATION

The Green's function (i.e. the phase boundary motion perturbation caused by a pressure $P_0(t) = P_0\delta(t/\tau)$ can be determined by two different methods.

The inversion integral for the Laplace transform is given by

$$z_m(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} z_m(s)\exp(st)\,ds. \tag{16}$$

As the function $z_m(s)$ has no poles on the branch $\mathrm{Re}(\sqrt{s}) > 0$ the contour of integration can be deformed and the integral (16) can be replaced by an integral along the branch cut of $\sqrt{s}$, as follows

$$z_m(t) = \frac{z_0}{\pi}\int_0^\infty \frac{\exp(-xt)\,(1 - \cos 2\sqrt{(\tau_0 x)})\,dx}{1 - 2\sqrt{(\tau x)}\sin 2\sqrt{(\tau_0 x)} + 2\tau x\,(1 - \cos 2\sqrt{\tau_0 x})} \tag{17}$$

where

$z_0 = -\gamma P_0/(\gamma g\rho - \beta)$.

The determination of the solution by the numerical integration of equation (17) will be particularly easy for large values of t. Furthermore, by application of Watson's lemma (Sirovich 1971, p. 65), the following asymptotic expansion of the solution can be determined

$$\frac{z_m(t)}{z_0} = \frac{3}{2}\sqrt{\left(\frac{\tau_0^2 \tau^3}{\pi t^5}\right)} + 0\,(t)^{-7/2} \tag{18}$$

where

$$\lim_{t\to\infty} 0\,(t)^{-n}\, t^n$$

is bounded.

The Green's function determined by numerical integration of equation (17) is plotted on Fig. 1. It can be observed on the figure that the closer the phase boundary is to the surface, the faster the equilibrium will be reached. This result, which can be inferred from

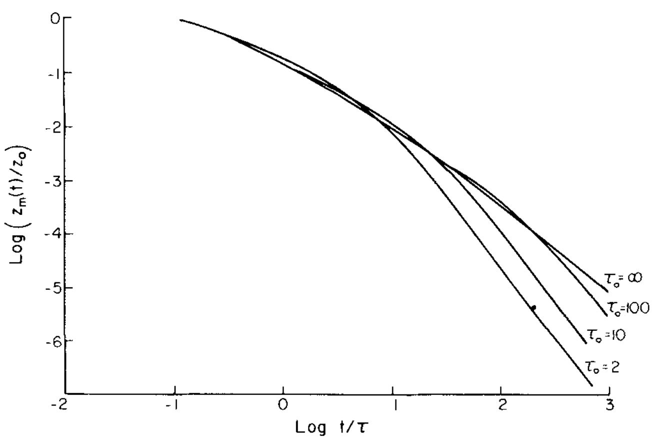

Figure 1. Pressure Green's function. Gjevik's solution corresponds to $\tau_0 = \infty$.

the asymptotic expansion (18), can be expected for intuitive reasons: the moving boundary is slowed down by the rising temperature; the closer the heat source (moving boundary) is to the isothermal surface, the smaller the increase in temperature and thus the effect on phase boundary motion.

Instead of integrating numerically equation (17), we can expand in series the Laplace transform of the Green's function and invert the series term by term (Doetsch 1963, p. 118). This expansion is formally equivalent to the method of images (Carslaw & Jaeger 1959, p. 273). The series expansion of the Green's function is

$$\frac{z_m(s)}{z_0} = \frac{1}{\sqrt{\tau s} + 1} \sum_{k=0}^{\infty} \left\{\frac{\exp\left(-2\sqrt{(\tau_0 s)}\right)}{\sqrt{\tau s} + 1}\right\}^k. \tag{19}$$

The series converges uniformly and absolutely on any closed region of the branch $\mathrm{Re}\,(\sqrt{s}) > 0$ (the convergence is more rapid for large values of s, i.e. small values of t). Inverting the two first terms of the series expansion, we obtain (Oberhettinger & Badii 1973, p. 229)

$$\frac{z_m(t)}{z_0} = \sqrt{\left(\frac{\tau}{\pi t}\right)} - \exp(t/\tau)\operatorname{erfc}\sqrt{(t/\tau)} + (\sqrt{(\tau/\pi t)} + 2\sqrt{(t/\pi\tau)})\exp(-\tau_0/t)$$
$$-(2(1 + t/\tau + \sqrt{(\tau_0/\tau)})\exp(2\sqrt{(\tau_0/\tau)} + t/\tau)\operatorname{erfc}(\sqrt{(\tau_0/\tau)} + \sqrt{(t/\tau)}) \quad (20)$$

where erfc is the complementary error function (see Gautschi 1965) defined by

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt \quad (21)$$

The first term of the series expansion of the Laplace transform is the solution of Gjevik (1972) and if τ_0/τ is sufficiently large, it is not very different from the exact solution. Higher order terms can be understood as the contribution of the images of the heat source. For small t/τ_0, the nth order term goes to zero as $\exp[-(2n)^2\tau_0/t]$ and can be neglected. The comparison between the solution obtained by numerical integration of equation (17) and the term by term inversion of the series expansion shows that, for all cases of practical interest, the series expansion limited to the first two terms provides accurate results (Mareschal 1975).

3.2 DETERMINATION OF THE GREEN'S FUNCTION FOR A TEMPERATURE PERTURBATION

The Green's function for a temperature perturbation $T_0(t) = T_0\delta(t/\tau)$ can be evaluated by the same techniques. The branch cut integral for $z_m(t)$

$$z_m(t) = \frac{z_0}{\pi}\int_0^\infty \frac{\exp(-xt)\sin\sqrt{(\tau_0 x)}\,dx}{1 - 2\sqrt{\tau x}\sin 2\sqrt{(\tau_0 x)} + 2\tau x(1 - \cos 2\sqrt{(\tau_0 x)})} \quad (22)$$

yields the following asymptotic development

$$\frac{z_m(t)}{z_0} = \sqrt{\left(\frac{\tau_0\tau^2}{4\pi t^3}\right)} + 0\,(t)^{-5/2}\ (t \to \infty) \quad (23)$$

(where $z_0 = T_0/(\gamma g\rho - \beta)$).

The Green's function, which was evaluated by numerical integration, has been plotted for different values of τ_0/τ on Fig. 2. This Green's function is no longer singular for $t = 0$; its maximum is located approximately at $t = 0.2\,\tau_0$ and its amplitude decreases exponentially with increasing τ_0. Note that this Green's function is the convolution of $\exp(-\tau_0/4t)$ and the Green's function for a pressure induced phase boundary motion.

The series expansion of the transform can also be inverted term by term leading to

$$\frac{z_m(t)}{z_0} = \sqrt{(\tau/\pi t)}\exp(\tau_0/4t) - \exp(\sqrt{(\tau_0/\tau)} + t/\tau)\operatorname{erfc}[\sqrt{(\tau_0/4t)} + \sqrt{(t/\tau)}]$$
$$-(2 + 2t/\tau + 3\sqrt{(\tau_0/\tau)})\exp(3\sqrt{(\tau_0/\tau)} + t/\tau)\operatorname{erfc}(\tfrac{3}{2}\sqrt{(\tau_0/t)} + \sqrt{(t/\tau)}) \quad (24)$$
$$+(\sqrt{(\tau/\pi t)} + 2\sqrt{(t/\pi\tau)})\exp(-9\tau_0/4t).$$

In all cases of practical interest, the results obtained by limiting the series expansion to its first two terms are very close to the exact solution obtained by numerical integration of the branch cut integral.

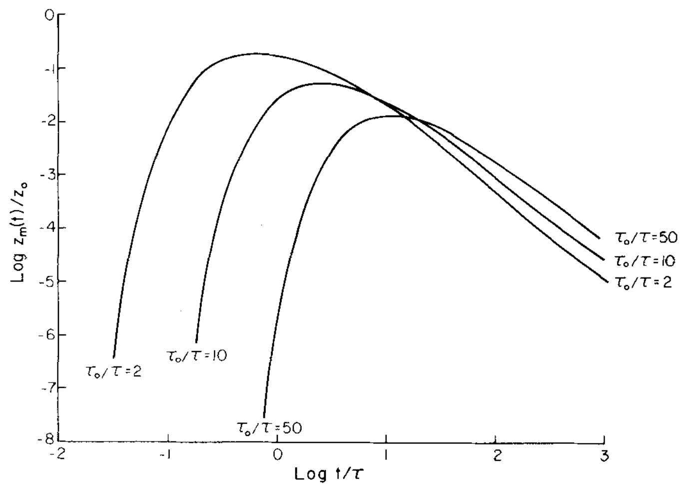

Figure 2. Temperature Green's function. No corresponding solution is obtained by Gjevik's method.

3.3 APPLICATIONS

The techniques described in the preceding Section can be used to determine the phase boundary motions for different type of pressure or temperature excitations. In particular, the motions of the phase boundary following the application of step or ramp function pressure and temperature excitations are plotted in Figs 3, 4, 5 and 6. Analytical expressions for the phase boundary motion can be obtained by inverting the first terms in the series expansion of the transform (Mareschal 1975).

The difference between this solution and Gjevik's can be seen directly on the figures where Gjevik's solution corresponds to the case $\tau_0 = \infty$. This difference appears non-negligible for values of $\tau_0/\tau < 10$. It can be seen that under the effect of the boundary condition at the surface, the boundary reaches its equilibrium position more rapidly (independently of any temperature perturbation at the surface). The closer the phase boundary is to the surface, the faster is equilibrium reached.

For a temperature perturbation, no motion would occur under Gjevik's hypothesis.. In the solution developed here, the motion of the phase boundary is significant after a time of the order of $\tau_0/5$.

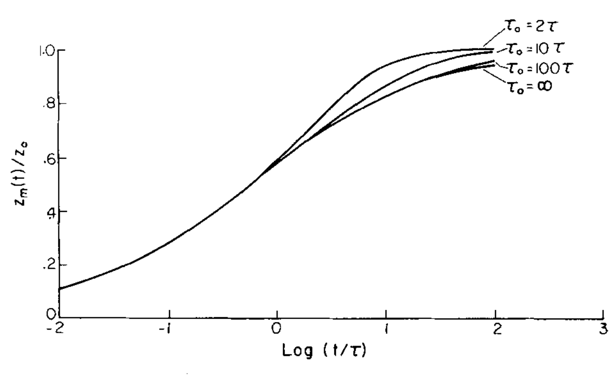

Figure 3. Phase boundary motion for a Heaviside function pressure excitation (normalized to the final equilibrium position z_0).

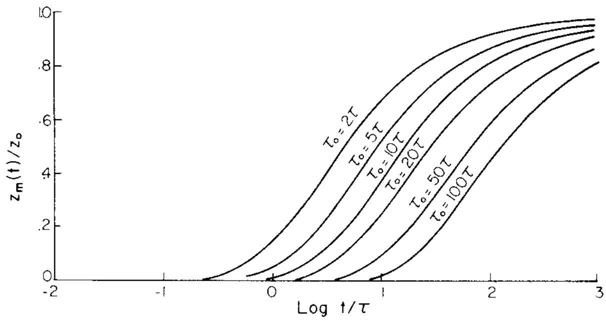

Figure 4. Phase boundary motion for a Heaviside function temperature excitation (normalized to the final equilibrium position z_0).

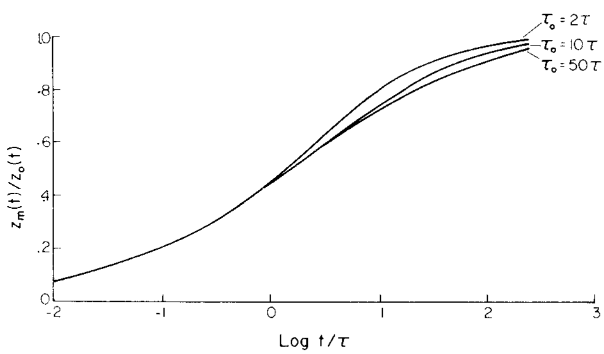

Figure 5. Phase boundary motion for a linear increase in pressure. The motion at time t is normalized to the change in equilibrium position of the phase boundary between time 0 and t.

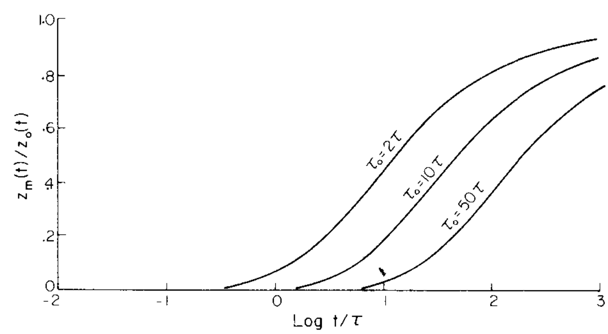

Figure 6. Phase boundary motion for a linear increase in temperature. The motion at time t is normalized to the change in equilibrium position between time 0 and t.

The amplitude of the phase boundary motion can be determined either by application of the Tauberian theorems (Hladik 1969, p. 104) to the equation (14) or by simpler considerations. For a step pressure excitation of amplitude P_0, the amplitude of phase boundary motion is given by $-\gamma P_0/(\gamma g\rho - \beta)$ for a step temperature excitation T_0 the amplitude of motion is $T_0/(\gamma g\rho - \beta)$. A similar result was obtained by O'Connell & Wasserburg (1967) and Gjevik (1972).

The time constant τ_0 is the characteristic heat propagation time from the surface to the phase boundary and vice versa. Its order of magnitude (10^8yr) for phase transitions at a depth of about 50 km is comparable to the duration of a geosyncline and to the time for formation of a sedimentary basin.

The time constant τ can be considered as the relaxation constant associated with the phase boundary motion. It is the time necessary for the latent heat produced (or absorbed) at the phase boundary to be removed (or carred in) by conduction. The value of that time constant depends on the parameters of the phase transition considered and on the geothermal conditions that exist in the region of the Earth where the phase transition occurs. Since the latent heat L is related to the slope of the Clapeyron curve $1/\gamma$ by the Clausius–Clapeyron equation (Landau & Lifchitz 1967, p. 318) in the following manner

$$L = (-T_m \Delta\rho/\gamma\rho_1\rho_2) \tag{25}$$

(where T_m is the transition temperature and $\Delta\rho$ the density contrast between two phases) τ can be expressed as

$$\tau = \left\{\frac{T_m\sqrt{\mathscr{H}}\,\Delta\rho}{2\gamma k\,\rho(\gamma g\rho - \beta)}\right\}^2. \tag{26}$$

This time constant is identical to Gjevik's (1972). Within a factor $4/\pi$ of difference between the definitions, it is also the same as O'Connell & Wasserburg's (1972) time constant when the physical properties of the two phases are assumed equal.

In Table 1(a), the relaxation time τ for the basalt–eclogite transition is given for different values of the thermal conductivity and of the difference between the geothermal and transition temperature gradient; the actual value of these parameters is very uncertain and may vary greatly from one region to another. However, it can be observed that if all the parameters are kept within the range allowed by the experimental data, it is not possible to obtain relaxation times smaller than 10^6 yr. For the phase changes in the peridotite, the time constant would be about 10 times smaller because the smaller density contrast in the equation (26). The order of magnitude of that relaxation constant is in accordance not only with the time of formation of sedimentary basins, but also with the period of tilts and up-

Table 1. (a) Value of the time constant τ (in 10^6 yr) for the gabbro–eclogite transformation for different values of the thermal conductivity (k in mW/m/K) and the difference between the transition temperature and the geothermal gradient $\gamma g\rho - \beta$ in K/km. The values assumed for the other parameters are: $\gamma = 60$ K/kbar, $\rho_1 = 3.0$ g/cm³, $\rho_2 = 3.5$ g/cm³, $\mathscr{H} = 0.01$ cm²/s, $T_m = 1000$ K.

$\gamma g\rho - \beta$	k 0.14	0.2	0.26	0.32
1	285	140	83	55
2	71	35	21	14
4	18	8.7	5.2	3.9
6	8	3.9	2.3	1.5
8	4.5	2.2	1.3	0.8
10	2.9	1.4	0.8	0.55

(b) Value of the time constant τ (in 10^6 yr) for the olivine–spinel phase transition for different values of the thermal conductivity (k in mW/m/K) and the Clausius–Clapeyron slope (γ in K/kbar). The values assumed for the other parameters are: $\beta = 0$, $\rho_1 = 4$ g/cm³, $\rho_2 = 4.4$ g/cm³, $\mathscr{H} = 0.01$ cm²/s, $T_m = 2000$ K.

γ	k 0.2	0.4	0.08
20	26	6.5	1.6
30	5	1.3	0.31
50	0.65	0.16	0.042
100	0.042	0.01	0.003

lifts directly related to mountain building (Wellman 1972). It is much larger than the time constant associated with post-glacial rebound.

Although our time constant is similar to Gjevik's and the inclusion of thermal effects does not change the response of the olivine–spinel transition to surface loading, it is worthwhile rediscussing Gjevik's conclusion that the olivine–spinel transition could produce the post-glacial rebound. In Table 1(b), the time constants corresponding to the olivine–spinel transition are given for different values of the slope of the Clausius–Clapeyron curve and the thermal conductivity. It appears that this time constant is very sensitive to the value of the Clausius–Clapeyron slope. A value as small as 6000 yr has been suggested by Gjevik who assumed a slope of the order of 100 K/kbar for the Clausius–Clapeyron curve.

However, the experimental data show that the olivine–spinel and all solid–solid phase transitions in the mantle have slopes of the order of 30 K/kbar (Ringwood 1972). In this case, it is impossible to obtain a relaxation time smaller than 500 000 yr. One implication is that none of these phase transitions would affect significantly the post-glacial rebound. A similar conclusion has been reached by O'Connell (1976). However, it is correct that, as pointed out by Gjevik, there is a range of values for the parameters of the olivine–spinel transition which can produce the observed relaxation time for the post-glacial rebound.

3.4 VALIDITY OF THE LINEAR APPROXIMATION USED IN THIS PAPER

The results presented so far have been obtained under several simplifying hypotheses. The neglect of the convection terms of the type $(\partial\theta/\partial z)(dx_\mathrm{m}/dt)$ in the partial differential equations in the most restrictive because the method of solution depends on it.

For this approximation to be true, the condition

$$\mathscr{K}_1 \frac{\partial^2 \theta_1}{\partial z^2} \gg \frac{\partial \theta_1}{\partial z} \frac{dx_\mathrm{m}}{dt} \tag{27}$$

must be satisfied. Each one of these terms can be evaluated from the solution for the temperature fields obtained under the linear approximation (Mareschal 1975).

For linearly increasing pressure $P_0(t) = \dot{P}_0 t H(t)$ it can be verified that the condition always holds for short time (i.e. $t < \tau$). For long time, the condition will hold only if

$$t < 4\mathscr{K}(\gamma g\rho - \beta)^2/(\pi\gamma^2 \dot{P}_0^2). \tag{28}$$

In Table 2, we give the time (in 10^6 yr) at which the linear approximation does no longer apply when sedimentation at constant rate induces motion of a Mohorovicic discontinuity phase change. The sedimentation rate and the difference between the geothermal and

Table 2. Range of validity of the linear approximation (in 10^6 yr) for a Mohorovicic discontinuity phase change moving under the effect of sedimentation at constant rate. The sedimentation rate (in mm/yr) and the difference between the geothermal and transition temperature gradient ($\gamma g\rho - \beta$ in K/km) are varied. The thermal diffusivity $\mathscr{K}$ is assumed 0.01 cm^2/s, the sediment density 2.5 g/cm^3, γ 60 K/kbar.

Diff. in slope	Sedimentation rate in mm/yr	1.5	1	0.3	0.1	0.03
10		7.1	16	64	1600	14 400
8		4.6	10.2	41	1024	9216
6		2.56	5.76	23.04	576	5184
4		1.14	2.56	10.24	256	2304
2		0.28	0.64	2.56	64	576
1		0.07	0.16	0.64	16	144

transition temperature gradients are varied and the other parameters are given their most likely value. It can be observed that the range of validity of the approximation is large only if either the amplitude of the phase boundary motion is small (i.e. $(\gamma g\rho - \beta)$ is large) or the rate of sedimentation is low. The practical effect of these limitations is to restrict the application of the approximation to motions of the phase boundary that are small. This evaluation of the importance of the non-linearities is in contradiction with Gjevik (1972) who assumed that the non-linear effects would always be negligible.

4 Conclusions

Despite its relatively restricted applicability, the linear approximation derived here is useful. For any phase transitions, the critical parameters and the conditions that produce a specific behaviour of the phase boundary can be determined in a very general way.

Whether or not such transitions actually occur in the Earth is a problem that we shall not try to answer here. However, the results derived earlier would call the following comments:

(1) As far as the post-glacial rebound is concerned, the relaxation time for phase boundary motion is two or three orders of magnitude larger than the relaxation time observed for post-glacial uplift. It can thus be concluded that the effect of even shallow phase transitions on post-glacial uplift is negligible and that the viscosity of the mantle determined from the post-glacial rebound data is likely to be accurate.

(2) Because of the low viscosity of the upper mantle (McConnell 1968), the phase transitions in the mantle will not be affected by changes in surface pressure. These phase transitions are too deep to be affected by changes in surface temperature in a reasonably short time. Local undulations of these phase changes cannot be the consequence of changing surface pressure or temperature, but may be the effect of deep changes in geothermal regime or mineral composition. Note also that because of the low mantle viscosity, local motions of these phase boundaries would induce no motion of the Earth's surface.

(3) The only phase changes that would be affected by changes in surface temperature and pressure should be present in the outer 100 km of the Earth. Although several phase transitions may occur in that region, the author's opinion is that a phase change Mohorovicic discontinuity would be the most significant for two major reasons: (a) The relatively important density contrast between basalt and eclogite (Ito & Kennedy 1971) would induce significant uplifts and subsidences. Combined with isostatic adjustments, these phase boundary motions could explain the continuous deposition of sediment in shallow water, (b) the characteristic thermal time for propagation of heat from the surface to that boundary is of the order of a geosyncline's duration, i.e. 10^8 yr. This would explain the uplift of the Earth's surface after a deposition cycle. Furthermore, the order of magnitude of the relaxation time for the basalt–eclogite phase transition correlates well with the characteristic time associated with orogeny (Wellman 1972).

Although the basalt–eclogite transition still raises controversy among geophysicists, the results reported here indicate that the amplitude and the time constant of the motion of such a phase transition is in accordance with the geophysical and geological observations of Earth's surface motions with long characteristic time.

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Appendix

A simpler but slightly less general derivation of the solution (14) can be obtained by the integral equation method (Lightfoot 1930). Let us assume that a phase boundary is present at depth $x_m(t)(x_m(0) = x_0)$ and that as long as the heat equation is concerned, both phases have the same density, ρ, thermal diffusivity, $\mathscr{K}$, and thermal conductivity, k. The initial temperature is given by $T(x, t = 0) = \beta x$. The temperature at the phase boundary must at any time be equal to the transition temperature given by equation (3).

The initial temperature at the phase boundary is given by equation (7). Therefore, we must have

$$z_m(t) = x_m(t) - x_0 = \frac{T(x_m, t) - \beta x_m(t) - \gamma P_0(t)}{(\gamma g \rho - \beta)} \tag{A1}$$

$T(x_m, t) - \beta x_m$ is the temperature perturbation $\theta(x_m, t)$ at $x = x_m(t)$. This temperature perturbation has two components: $\theta'(x, t)$ due to the boundary conditions at the surface which is given by (Carslaw & Jaeger 1959, p. 63)

$$\theta'(x, t) = \frac{x}{2\sqrt{(\pi\mathscr{K})}} \int_0^t \frac{T_0(t') \exp[-x^2/4\mathscr{K}(t - t')]\, dt'}{(t - t')^{3/2}} \tag{A2}$$

$\theta''(x, t)$ due to the moving heat source at the phase boundary. This component is given by (Carslaw & Jaeger 1959, p. 357)

$$\theta''(x, t) = \frac{L}{c} \int_0^t \dot{x}_m(t')\, G(x, t; x_m', t')\, dt' \tag{A3}$$

where the Green's function is defined by

$$G(x, t; x', t') = \frac{\exp[-(x - x')^2/4\mathscr{K}(t - t')] - \exp[-(x + x')/4\mathscr{K}(t - t')]}{2\sqrt{(\pi\mathscr{K}(t - t'))}} \tag{A4}$$

Introducing θ' and θ'' in equation (A1), we obtain the non-linear integral equation that must be solved for $x_m(t)$

$$x_m(t) - x_0 = -\gamma P_0(t)/(\gamma g\rho - \beta)$$

$$+ x_m(t)/(\gamma g\rho - \beta) \times \int_0^t \frac{T_0(t')\exp\,[-x_m^2(t)/4\mathscr{K}(t-t')]\,dt'}{2\sqrt{(\pi\mathscr{K})}\,(t-t')^{3/2}}$$

$$+ L/c\,(\gamma g\rho - \beta) \times \int_0^t \dot{x}_m(t')\,G\,(x_m(t), t; x_m(t'), t')\,dt'. \tag{A5}$$

This non-linear equation cannot be solved directly; it could be solved by iterations. However, a linear approximation to its solution can be obtained if it is assumed that the phase boundary motion is so small that $x_m(t) \approx x_0$ in the left-hand side of (A5) but $\dot{x}_m(t) \neq 0$. In that case, equation (A5) is replaced by a linear integral equation of the convolution type

$$x_m(t) - x_0 = -\gamma P_0(t)/(\gamma g\rho - \beta)$$

$$+ x_0/(\gamma g\rho - \beta) \times \int_0^t \frac{T_0(t')\exp\,[-x_0^2/4\mathscr{K}(t-t')]\,dt'}{2\sqrt{(\pi\mathscr{K})}\,(t-t')^{3/2}}$$

$$+ L/c\,(\gamma g\rho - \beta) \times \int_0^t dt'\,\dot{x}_0(t')\,G\,(x_0, t; x_0, t'). \tag{A6}$$

The solution given by equation (19) is obtained by Laplace transforming equation A6.