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*Research Article*

# A Linear Functional Equation of Third Order Associated with the Fibonacci Numbers

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Given a vector space $X$, we investigate the solutions $f : \mathbb{R} \to X$ of the linear functional equation of third order $f(x) = pf(x-1) + qf(x-2) + rf(x-3)$, which is strongly associated with a well-known identity for the Fibonacci numbers. Moreover, we prove the Hyers-Ulam stability of that equation.

## 1. Introduction

The problem of stability of functional equations was motivated by a question of Ulam [1] and a solution to it by Hyers [2]. Since then, numerous papers have been published on that subject and we refer to [3–6] for more details, some discussions, and further references; for examples of very recent results, see, for example, [7].

In this paper, as usual, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ stand for the sets of complex numbers, real numbers, integers, and positive integers, respectively. For a nonempty subset $S$ of a vector space, let $\xi : S \to S$ be a function. Moreover, $\xi^0(x) = x$, $\xi^{n+1}(x) = \xi(\xi^n(x))$, and (only for bijective $\xi$) $\xi^{-n-1}(x) = \xi^{-1}(\xi^{-n}(x))$ for $x \in S$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Jung has proved in [3] (see also [8]) some results on solutions and Hyers-Ulam stability of the functional equation

$$f(x) = pf(\xi(x)) - qf(\xi^2(x)), \qquad (1)$$

in the case where $S = \mathbb{R}$ and $\xi(x) = x - 1$ for $x \in \mathbb{R}$.

If $S := \mathbb{N}_0$ and $p, q \in \mathbb{Z}$, then solutions $x : \mathbb{N}_0 \to \mathbb{Z}$ of the difference equation $f(x) = pf(x-1) - qf(x-2)$ are called the Lucas sequences (see, e.g., [9]). In some special cases they are called with specific names, for example, the Fibonacci numbers ($p = 1$, $q = -1$, $x(0) = 0$, and $x(1) = 1$), the Lucas numbers ($p = 1$, $q = -1$, $x(0) = 2$, and $x(1) = 1$), the Pell numbers ($p = 2$, $q = -1$, $x(0) = 0$, and $x(1) = 1$), the Pell-Lucas (or companion Lucas) numbers ($p = 2$, $q = -1$, $x(0) = 2$, and $x(1) = 2$), and the Jacobsthal numbers ($p = 1$, $q = -2$, $x(0) = 0$, and $x(1) = 1$).

For some information and further references concerning the functional equations in a single variable, we refer to [10–12]. Let us mention yet that the problem of Hyers-Ulam stability of functional equations is connected to the notions of controlled chaos and shadowing (see [13]).

We remark that if $\xi : S \to S$ is bijective, then (1) can be written in the following equivalent form:

$$f(\eta^2(x)) = pf(\eta(x)) - qf(x), \qquad (2)$$

where $\eta := \xi^{-1}$.

In view of the last remark, the following Hyers-Ulam stability result concerning (1) can be derived from [14, Theorem 2] (see also [15]).

**Theorem 1.** *Let $p, q \in \mathbb{R}$ be given with $q \neq 0$ and let $S$ be a nonempty subset of a vector space. Assume that $a_1$, $a_2$ are the complex roots of the quadratic equation $x^2 - px + q = 0$ with $|a_i| \neq 1$ for $i \in \{1, 2\}$. Moreover, assume that $X$ is either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$.*

*Let $\xi : S \to S$ be bijective. If a function $f : S \to X$ satisfies the inequality*

$$\left\| f(x) - pf(\xi(x)) + qf\left(\xi^2(x)\right)\right\| \leq \varepsilon \tag{3}$$

*for all $x \in S$ and for some $\varepsilon \geq 0$, then there exists a unique solution $F : S \to X$ of* (1) *with*

$$\| f(x) - F(x)\| \leq \frac{\varepsilon}{\left|\left(|a_1| - 1\right)\left(|a_2| - 1\right)\right|} \tag{4}$$

*for all $x \in S$.*

In [16, Theorem 1.4], the method presented in [3] was modified so as to prove a theorem which is a complement of Theorem 1. Note that, for bijective $\xi$, the following theorem improves the estimation (4) in some cases (e.g., $a_1 = 3/2$, $a_2 = -3/2$, or $a_1 = 1/2$, $a_2 = -1/2$). However, in some other situations (e.g., $a_1 = 3$, $a_2 = -3$), the estimation (4) is better than (5). The following theorem also complements Theorem 1, because $\xi$ can be quite arbitrary in the case of $(\alpha)$.

**Theorem 2.** *Given $p, q \in \mathbb{R}$ with $q \neq 0$, assume that the distinct complex roots $a_1, a_2$ of the quadratic equation $x^2 - px + q = 0$ satisfy one of the following two conditions:*

($\alpha$) *$|a_i| < 1$ for $i \in \{1, 2\}$;*

($\beta$) *$|a_i| \neq 1$ for $i \in \{1, 2\}$ and $\xi : S \to S$ is bijective.*

*Moreover, assume that $X$ is either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$. If a function $f : S \to X$ satisfies inequality* (3)*, then there exists a solution $F : S \to X$ of* (1) *such that*

$$\| f(x) - F(x)\| \leq \frac{\varepsilon}{|a_1 - a_2|}\left(\frac{|a_1|}{\left||a_1| - 1\right|} + \frac{|a_2|}{\left||a_2| - 1\right|}\right) \tag{5}$$

*for all $x \in S$. Moreover, if the condition* ($\beta$) *is true, then the $F$ is the unique solution of* (1) *satisfying* (5)*.*

In this paper, we investigate the solutions of the functional equation

$$f(x) = pf(x-1) + qf(x-2) + rf(x-3), \tag{6}$$

where $p, q, r$ are real constants. Moreover, we also prove the Hyers-Ulam stability of that equation. Equation (6) is a kind of linear functional equations of third order because it is of the form

$$f(x) = a_1(x) f(\xi(x)) + a_2(x) f\left(\xi^2(x)\right) + a_3(x) f\left(\xi^3(x)\right) \tag{7}$$

for the case of $a_1(x) = p$, $a_2(x) = q$, $a_3(x) = r$, and $\xi(x) = x - 1$.

## 2. General Solution

In the following theorem, we apply [16, Theorem 1.1] for the investigation of general solutions of the functional equation (6).

**Theorem 3.** *Let $p, q, r$ be real constants such that the cubic equation*

$$x^3 + px^2 - qx + r = 0 \tag{8}$$

*has the following properties:*

(i) *$\alpha_1$ and $\alpha_2$ are two distinct nonzero real roots of the cubic equation* (8)*;*

(ii) *it holds true that either $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for $i \in \{1, 2\}$ or $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for $i \in \{1, 2\}$.*

*Let $X$ be either a real vector space if $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for $i \in \{1, 2\}$ or a complex vector space if $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for $i \in \{1, 2\}$. Then, a function $f : \mathbb{R} \to X$ is a solution of the functional equation* (6) *if and only if there exist functions $h_1, h_2 : [-1, 1) \to X$ such that*

$$\begin{aligned} f(x) = & \frac{\alpha_1}{\alpha_1 - \alpha_2} V_{[x]+1} h_2(x - [x]) \\ & + \frac{\alpha_1 r}{\alpha_2(\alpha_1 - \alpha_2)} V_{[x]} h_2(x - [x] - 1) \\ & - \frac{\alpha_2}{\alpha_1 - \alpha_2} U_{[x]+1} h_1(x - [x]) \\ & - \frac{\alpha_2 r}{\alpha_1(\alpha_1 - \alpha_2)} U_{[x]} h_1(x - [x] - 1), \end{aligned} \tag{9}$$

*where $[x]$ denotes the largest integer not exceeding $x$, and $U_n$, $V_n$ are defined in* (13) *and* (23)*.*

*Proof.* Assume that $f : \mathbb{R} \to X$ is a solution of (6). If we define an auxiliary function $g_1 : \mathbb{R} \to X$ by

$$g_1(x) := f(x) + \alpha_1 f(x-1), \tag{10}$$

then it follows from (6) that $g_1$ satisfies

$$g_1(x) = \left(\alpha_1 + p\right) g_1(x-1) + \frac{r}{\alpha_1} g_1(x-2) \tag{11}$$

for any $x \in \mathbb{R}$. According to [16, Theorem 1.1] or [3, Theorem 2.1], there exists a function $h_1 : [-1, 1) \to X$ such that

$$\begin{aligned} g_1(x) &= f(x) + \alpha_1 f(x-1) \\ &= U_{[x]+1} h_1(x - [x]) + \frac{r}{\alpha_1} U_{[x]} h_1(x - [x] - 1) \end{aligned} \tag{12}$$

for all $x \in \mathbb{R}$, where

$$U_n = \frac{a^n - b^n}{a - b} \quad (n \in \mathbb{Z}) \tag{13}$$

and $a, b$ are the distinct roots of the quadratic equation

$$x^2 - \left(\alpha_1 + p\right) x - \frac{r}{\alpha_1} = 0, \tag{14}$$

that is,

$$\begin{aligned} a &= \frac{\alpha_1 + p}{2} + \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}, \\ b &= \frac{\alpha_1 + p}{2} - \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}. \end{aligned} \tag{15}$$

Since $a$ is a root of the quadratic equation (14), we have

$$a^2 = (\alpha_1 + p)\,a + \frac{r}{\alpha_1}. \tag{16}$$

We multiply both sides of (16) with $a$ and make use of (16) and (i) to get

$$\begin{aligned} a^3 &= pa^2 + \alpha_1 a^2 + \frac{r}{\alpha_1}a \\ &= pa^2 + \alpha_1\left((\alpha_1 + p)\,a + \frac{r}{\alpha_1}\right) + \frac{r}{\alpha_1}a \\ &= pa^2 + \frac{a}{\alpha_1}\left(\alpha_1^3 + p\alpha_1^2 + r\right) + r \\ &= pa^2 + qa + r. \end{aligned} \tag{17}$$

Similarly, we also obtain

$$b^3 = pb^2 + qb + r. \tag{18}$$

Using (13), (17), and (18), we have

$$\begin{aligned} pU_{n-1} &+ qU_{n-2} + rU_{n-3} \\ &= \frac{\left(pa^2 + qa + r\right)a^{n-3} - \left(pb^2 + qb + r\right)b^{n-3}}{a-b} \\ &= \frac{a^n - b^n}{a-b} = U_n \end{aligned} \tag{19}$$

for all $n \in \mathbb{Z}$.

If we define an auxiliary function $g_2 : \mathbb{R} \to X$ by

$$g_2(x) := f(x) + \alpha_2 f(x-1), \tag{20}$$

then it follows from (6) that $g_2$ satisfies

$$g_2(x) = (\alpha_2 + p)\,g_2(x-1) + \frac{r}{\alpha_2}g_2(x-2) \tag{21}$$

for any $x \in \mathbb{R}$. According to [16, Theorem 1.1] or [3, Theorem 2.1], there exists a function $h_2 : [-1, 1) \to X$ such that

$$\begin{aligned} g_2(x) &= f(x) + \alpha_2 f(x-1) \\ &= V_{[x]+1}h_2(x-[x]) + \frac{r}{\alpha_2}V_{[x]}h_2(x-[x]-1) \end{aligned} \tag{22}$$

for all $x \in \mathbb{R}$, where

$$V_n = \frac{c^n - d^n}{c-d} \quad (n \in \mathbb{Z}) \tag{23}$$

and $c$, $d$ are the distinct roots of the quadratic equation

$$x^2 - (\alpha_2 + p)\,x - \frac{r}{\alpha_2} = 0, \tag{24}$$

that is,

$$\begin{aligned} c &= \frac{\alpha_2 + p}{2} + \sqrt{\left(\frac{\alpha_2 + p}{2}\right)^2 + \frac{r}{\alpha_2}}, \\ d &= \frac{\alpha_2 + p}{2} - \sqrt{\left(\frac{\alpha_2 + p}{2}\right)^2 + \frac{r}{\alpha_2}}. \end{aligned} \tag{25}$$

As in the first part, we verify that

$$V_n = pV_{n-1} + qV_{n-2} + rV_{n-3} \tag{26}$$

for all $n \in \mathbb{Z}$.

We now multiply (12) with $\alpha_2$ and (22) with $\alpha_1$, we subtract the former from the latter, and we then divide the resulting equation by $(\alpha_1 - \alpha_2)$ to get (9).

We assume that a function $f : \mathbb{R} \to X$ is given by (9), where $h_1, h_2 : [-1, 1) \to X$ are arbitrarily given functions and $U_n$, $V_n$ are given by (13) and (23), respectively. Then, by (9), (19), and (26), we have

$$\begin{aligned} pf&(x-1) + qf(x-2) + rf(x-3) \\ &= \frac{\alpha_1}{\alpha_1 - \alpha_2}\left(pV_{[x]} + qV_{[x]-1} + rV_{[x]-2}\right)h_2(x-[x]) \\ &\quad + \frac{\alpha_1 r}{\alpha_2(\alpha_1 - \alpha_2)}\left(pV_{[x]-1} + qV_{[x]-2} + rV_{[x]-3}\right) \\ &\quad \times h_2(x-[x]-1) \\ &\quad - \frac{\alpha_2}{\alpha_1 - \alpha_2}\left(pU_{[x]} + qU_{[x]-1} + rU_{[x]-2}\right)h_1(x-[x]) \\ &\quad - \frac{\alpha_2 r}{\alpha_1(\alpha_1 - \alpha_2)}\left(pU_{[x]-1} + qU_{[x]-2} + rU_{[x]-3}\right) \\ &\quad \times h_1(x-[x]-1) \\ &= \frac{\alpha_1}{\alpha_1 - \alpha_2}V_{[x]+1}h_2(x-[x]) \\ &\quad + \frac{\alpha_1 r}{\alpha_2(\alpha_1 - \alpha_2)}V_{[x]}h_2(x-[x]-1) \\ &\quad - \frac{\alpha_2}{\alpha_1 - \alpha_2}U_{[x]+1}h_1(x-[x]) \\ &\quad - \frac{\alpha_2 r}{\alpha_1(\alpha_1 - \alpha_2)}U_{[x]}h_1(x-[x]-1) = f(x) \end{aligned} \tag{27}$$

for all $x \in \mathbb{R}$, which implies that $f$ is a solution of (6). □

According to [17, p. 92], the Fibonacci numbers $F_n$ satisfy the identity

$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2 \tag{28}$$

for all integers $n > 3$. We can easily notice that the linear equation of third order

$$f(x) = 2f(x-1) + 2f(x-2) - f(x-3) \tag{29}$$

is strongly related to identity (28).

**Corollary 4.** *Let $X$ be a real vector space. A function $f : \mathbb{R} \to X$ is a solution of the functional equation* (29) *if and only if there exist functions $h_1, h_2 : [-1, 1) \to X$ such that*

$$\begin{aligned} f(x) = {} & \frac{5+3\sqrt{5}}{10} U_{[x]+1} h_1(x-[x]) \\ & + \frac{15+7\sqrt{5}}{10} U_{[x]} h_1(x-[x]-1) \\ & + \frac{5-3\sqrt{5}}{10} V_{[x]+1} h_2(x-[x]) \\ & + \frac{15-7\sqrt{5}}{10} V_{[x]} h_2(x-[x]-1), \end{aligned} \tag{30}$$

*where $U_n$ and $V_n$ are defined in* (33).

*Proof.* If we set $p = 2$, $q = 2$, and $r = -1$ in (8), then the cubic equation

$$x^3 + 2x^2 - 2x - 1 = 0 \tag{31}$$

has three distinct nonzero roots including

$$\alpha_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}, \qquad \alpha_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}. \tag{32}$$

Moreover, it holds that $(\alpha_1 + p)^2 + 4r/\alpha_1 > 0$ and $(\alpha_2 + p)^2 + 4r/\alpha_2 > 0$. By (13), (15), (23), and (25), we have

$$U_n = \frac{a^n - b^n}{a-b}, \qquad V_n = \frac{c^n - d^n}{c-d}, \tag{33}$$

where we make use of (15) and (25) to calculate

$$a = \frac{3+\sqrt{5}}{2}, \qquad b = -1, \qquad c = \frac{3-\sqrt{5}}{2}, \qquad d = -1. \tag{34}$$

Finally, in view of Theorem 3, we conclude that the assertion of our corollary is true. □

**Corollary 5.** *If a function $f : \mathbb{R} \to \mathbb{R}$ is a solution of functional equation* (29)*, then there exist real constants $\mu_1$, $\mu_2$, $\nu_1$, and $\nu_2$ such that*

$$\begin{aligned} f(n) = {} & \frac{5+3\sqrt{5}}{10} \mu_1 U_{n+1} + \frac{15+7\sqrt{5}}{10} \mu_2 U_n \\ & + \frac{5-3\sqrt{5}}{10} \nu_1 V_{n+1} + \frac{15-7\sqrt{5}}{10} \nu_2 V_n \end{aligned} \tag{35}$$

*for all $n \in \mathbb{Z}$, where $U_n$ and $V_n$ are defined in* (33).

## 3. Hyers-Ulam Stability

We apply the classical direct method to the proof of the following theorem. The classical direct method was first proposed by Hyers [2].

**Theorem 6.** *Let $p$, $q$, $r$ be real constants with $r \neq 0$, let $\alpha$ be a nonzero root of the cubic equation* (8)*, and let $a$, $b$ be the roots of the quadratic equation $x^2 - (\alpha+p)x - r/\alpha = 0$ with $|a| > 1$ and $0 < |b| < 1$. Let $X$ be either a real Banach space if $(\alpha+p)^2 + 4r/\alpha > 0$ or a complex Banach space if $(\alpha+p)^2 + 4r/\alpha < 0$. If a function $f : \mathbb{R} \to X$ satisfies the inequality*

$$\| f(x) - pf(x-1) - qf(x-2) - rf(x-3) \| \le \varepsilon \tag{36}$$

*for all $x \in \mathbb{R}$ and for some $\varepsilon \ge 0$, then there exists a solution $G : \mathbb{R} \to X$ of* (6) *such that*

$$\| f(x) + \alpha f(x-1) - G(x) \| \le \frac{|a| - |b|}{|a-b|} \frac{\varepsilon}{(|a|-1)(1-|b|)} \tag{37}$$

*for all $x \in \mathbb{R}$.*

*Proof.* If we define an auxiliary function $g : \mathbb{R} \to X$ by

$$g(x) := f(x) + \alpha f(x-1), \tag{38}$$

then, as we did in (11), it follows from (36) that $g$ satisfies the inequality

$$\left\| g(x) - (\alpha+p) g(x-1) - \frac{r}{\alpha} g(x-2) \right\| \le \varepsilon \tag{39}$$

or

$$\| g(x) - ag(x-1) - b[g(x-1) - ag(x-2)] \| \le \varepsilon \tag{40}$$

for any $x \in \mathbb{R}$.

If we replace $x$ with $x-k$ in the last inequality, then we have

$$\begin{aligned} & \| g(x-k) - ag(x-k-1) \\ & \quad -b[g(x-k-1) - ag(x-k-2)] \| \le \varepsilon \end{aligned} \tag{41}$$

for all $x \in \mathbb{R}$. Furthermore, we get

$$\begin{aligned} & \| b^k [g(x-k) - ag(x-k-1)] \\ & \quad -b^{k+1} [g(x-k-1) - ag(x-k-2)] \| \le |b|^k \varepsilon \end{aligned} \tag{42}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (42), we obviously have

$$\begin{aligned} & \| g(x) - ag(x-1) - b^n [g(x-n) - ag(x-n-1)] \| \\ & \le \sum_{k=0}^{n-1} \| b^k [g(x-k) - ag(x-k-1)] \\ & \qquad -b^{k+1} [g(x-k-1) - ag(x-k-2)] \| \\ & \le \sum_{k=0}^{n-1} |b|^k \varepsilon \end{aligned} \tag{43}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (42) implies that the sequence $\{b^n[g(x-n) - ag(x-n-1)]\}$ is a Cauchy sequence (note that $0 < |b| < 1$). Therefore, we can define a function $G_1 : \mathbb{R} \to X$ by

$$G_1(x) := \lim_{n \to \infty} b^n [g(x-n) - ag(x-n-1)], \tag{44}$$

since $X$ is complete. In view of the definition of $G_1$ and using the relations, $a+b=\alpha+p$ and $ab=-r/\alpha$, we obtain

$$\begin{aligned}
&(\alpha+p)G_1(x-1)+\frac{r}{\alpha}G_1(x-2)\\
&\quad=(a+b)G_1(x-1)-abG_1(x-2)\\
&\quad=\frac{a+b}{b}\lim_{n\to\infty}b^{n+1}\left[g(x-(n+1))-ag(x-(n+1)-1)\right]\\
&\qquad-\frac{ab}{b^2}\lim_{n\to\infty}b^{n+2}\left[g(x-(n+2))-ag(x-(n+2)-1)\right]\\
&\quad=\frac{a+b}{b}G_1(x)-\frac{a}{b}G_1(x)=G_1(x)
\end{aligned}\tag{45}$$

for all $x\in\mathbb{R}$. Since $\alpha$ is a nonzero root of the cubic equation (8), it follows from (45) that

$$\begin{aligned}
&G_1(x)-pG_1(x-1)-qG_1(x-2)-rG_1(x-3)\\
&\quad=(\alpha+p)G_1(x-1)+\frac{r}{\alpha}G_1(x-2)-pG_1(x-1)\\
&\qquad-qG_1(x-2)-rG_1(x-3)\\
&\quad=\alpha G_1(x-1)+\left(-q+\frac{r}{\alpha}\right)G_1(x-2)-rG_1(x-3)\\
&\quad=\alpha G_1(x-1)+\left(-\alpha^2-p\alpha\right)G_1(x-2)-rG_1(x-3)\\
&\quad=\alpha\left((\alpha+p)G_1(x-2)+\frac{r}{\alpha}G_1(x-3)\right)\\
&\qquad-\alpha(\alpha+p)G_1(x-2)-rG_1(x-3)=0
\end{aligned}\tag{46}$$

for all $x\in\mathbb{R}$. Hence, we conclude that $G_1$ is a solution of (6).

If $n$ tends to infinity, then (43) yields that

$$\left\|g(x)-ag(x-1)-G_1(x)\right\|\le\frac{\varepsilon}{1-|b|}\tag{47}$$

for every $x\in\mathbb{R}$.

On the other hand, it also follows from (36) that

$$\left\|g(x)-bg(x-1)-a\left[g(x-1)-bg(x-2)\right]\right\|\le\varepsilon\tag{48}$$

for all $x\in\mathbb{R}$. Analogously to (42), replacing $x$ by $x+k$ in the last inequality and then dividing by $|a|^k$ both sides of the resulting inequality, then we have

$$\begin{aligned}
&\Big\|a^{-k}\left[g(x+k)-bg(x+k-1)\right]\\
&\quad-a^{-k+1}\left[g(x+k-1)-bg(x+k-2)\right]\Big\|\le|a|^{-k}\varepsilon
\end{aligned}\tag{49}$$

for all $x\in\mathbb{R}$ and $k\in\mathbb{Z}$. By using (49), we further obtain

$$\begin{aligned}
&\left\|a^{-n}\left[g(x+n)-bg(x+n-1)\right]-\left[g(x)-bg(x-1)\right]\right\|\\
&\quad\le\sum_{k=1}^{n}\Big\|a^{-k}\left[g(x+k)-bg(x+k-1)\right]\\
&\qquad\qquad-a^{-k+1}\left[g(x+k-1)-bg(x+k-2)\right]\Big\|\\
&\quad\le\sum_{k=1}^{n}|a|^{-k}\varepsilon
\end{aligned}\tag{50}$$

for $x\in\mathbb{R}$ and $n\in\mathbb{N}$.

On account of (49), we see that the sequence $\{a^{-n}[g(x+n)-bg(x+n-1)]\}$ is a Cauchy sequence for any fixed $x\in\mathbb{R}$ (note that $|a|>1$). Hence, we can define a function $G_2:\mathbb{R}\to X$ by

$$G_2(x):=\lim_{n\to\infty}a^{-n}\left[g(x+n)-bg(x+n-1)\right].\tag{51}$$

Due to the definition of $G_2$ and the relations, $a+b=\alpha+p$ and $ab=-r/\alpha$, we get

$$\begin{aligned}
&(\alpha+p)G_2(x-1)+\frac{r}{\alpha}G_2(x-2)\\
&\quad=(a+b)G_2(x-1)-abG_2(x-2)\\
&\quad=\frac{a+b}{a}\lim_{n\to\infty}a^{-(n-1)}\left[g(x+n-1)-bg(x+n-2)\right]\\
&\qquad-\frac{ab}{a^2}\lim_{n\to\infty}a^{-(n-2)}\left[g(x+n-2)-bg(x+n-3)\right]\\
&\quad=\frac{a+b}{a}G_2(x)-\frac{b}{a}G_2(x)=G_2(x)
\end{aligned}\tag{52}$$

for any $x\in\mathbb{R}$. Similarly as in the first part, we can show that $G_2$ is a solution of (6).

If we let $n$ tend to infinity, then it follows from (50) that

$$\left\|G_2(x)-g(x)+bg(x-1)\right\|\le\frac{\varepsilon}{|a|-1}\tag{53}$$

for $x\in\mathbb{R}$.

It follows from (47) and (53) that

$$\begin{aligned}
&\left\|g(x-1)-\frac{1}{a-b}G_2(x)+\frac{1}{a-b}G_1(x)\right\|\\
&\quad\le\left\|\frac{1}{a-b}G_1(x)-\frac{1}{a-b}g(x)+\frac{a}{a-b}g(x-1)\right\|\\
&\qquad+\left\|\frac{1}{a-b}g(x)-\frac{b}{a-b}g(x-1)-\frac{1}{a-b}G_2(x)\right\|\\
&\quad\le\frac{|a|-|b|}{|a-b|}\frac{\varepsilon}{(|a|-1)(1-|b|)}
\end{aligned}\tag{54}$$

for any $x\in\mathbb{R}$.

Finally, if we define a function $G:\mathbb{R}\to X$ by

$$G(x):=\frac{1}{a-b}G_2(x+1)-\frac{1}{a-b}G_1(x+1)\tag{55}$$

for all $x \in \mathbb{R}$, then $G$ is also a solution of (6). Moreover, the validity of (37) follows from the last inequality. □

The following theorem is the main theorem of this paper.

**Theorem 7.** *Given real constants $p$, $q$, $r$ with $r \neq 0$, let $\alpha_1$ and $\alpha_2$ be distinct nonzero roots of cubic equation* (8) *and let $a_i$, $b_i$ be the roots of the quadratic equation $x^2 - (\alpha_i + p)x - r/\alpha_i = 0$ with $|a_i| > 1$ and $0 < |b_i| < 1$ for $i \in \{1, 2\}$. Assume that either $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for all $i \in \{1, 2\}$ or $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for all $i \in \{1, 2\}$. Let $X$ be either a real Banach space if $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ or a complex Banach space if $(\alpha_i + p)^2 + 4r/\alpha_i < 0$. If a function $f : \mathbb{R} \to X$ satisfies inequality* (36) *for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a solution $F : \mathbb{R} \to X$ of* (6) *such that*

$$\|f(x) - F(x)\| \leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_2|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_1| - 1)(1 - |b_1|)} + \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_2| - 1)(1 - |b_2|)} \quad (56)$$

*for all $x \in \mathbb{R}$.*

*Proof.* According to Theorem 6, there exists a solution $F_i : \mathbb{R} \to X$ of (6) such that

$$\|f(x) + \alpha_i f(x-1) - F_i(x)\| \leq \frac{|a_i| - |b_i|}{|a_i - b_i|} \frac{\varepsilon}{(|a_i| - 1)(1 - |b_i|)} \quad (57)$$

for any $x \in \mathbb{R}$ and $i \in \{1, 2\}$. In view of the last inequalities, we have

$$\begin{aligned} &\left\| f(x) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) + \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \right\| \\ &\leq \left\| \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} f(x) - \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x-1) \right\| \\ &\quad + \left\| \frac{\alpha_1}{\alpha_1 - \alpha_2} f(x) + \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x-1) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) \right\| \\ &\leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_2|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_1| - 1)(1 - |b_1|)} \\ &\quad + \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_2| - 1)(1 - |b_2|)} \end{aligned} \quad (58)$$

for all $x \in \mathbb{R}$.

If we define a function $F : \mathbb{R} \to X$ by

$$F(x) := \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \quad (59)$$

for each $x \in \mathbb{R}$, then $F$ is also a solution of (6), and inequality (56) follows from the last inequality. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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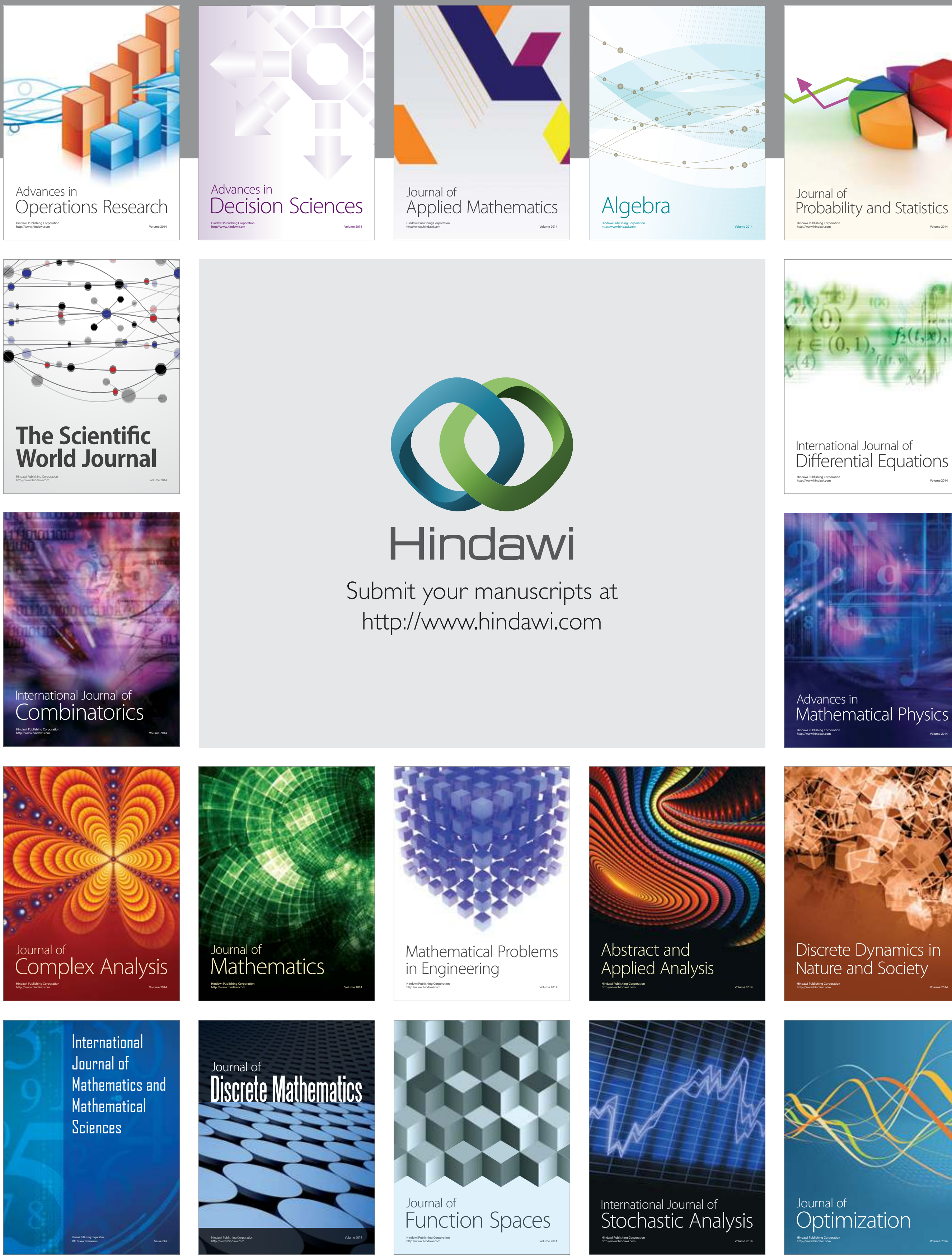
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