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A LINEAR PFAFFIAN SYSTEM AT AN IRREGULAR SINGULARITY

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction. Let A be a discrete valuation ring with the unique maximal ideal $\mathfrak{p} = (\pi)$, where π is an element in A. For an element a in A we denote by $\bar{a}$ the corresponding element in the residue class field $k = A/\mathfrak{p}$.

Let $\delta: A \to A$ be a mapping from A to A such that

(i) $\delta(a + b) = \delta(a) + \delta(b)$ for $a, b \in A$,

(ii) $\delta(ab) = a\delta(b) + b\delta(a)$ for $a, b \in A$, and

(iii) $\delta(A) \subset \mathfrak{p}^2 = (\pi^2)$.

Since $\delta(\mathfrak{p}) \subset \mathfrak{p}^2$, we get $\delta(\mathfrak{p}^m) \subset \mathfrak{p}^{m+1}$, where $\mathfrak{p}^m = (\pi^m)$.

Let M be an A-module, and let M_1 and M_2 be two submodules of M. We consider a mapping $L: M \to M$ such that

(i) $L(f + g) = L(f) + L(g)$ for $f, g \in M$,

(ii) $L(af) = \delta(a)P(f) + aL(f)$ for $a \in A$, $f \in M$, where P is an A-module endomorphism of M, and

(iii) $L(M_1) \subset M_2$.

Note that $\delta(a)P(f) \in M_2$ if $f \in M_1$.

Set $N_1 = M/M_1$ and $N_2 = M/M_2$, and let $\varphi_1: M \to N_1$ and $\varphi_2: M \to N_2$ be the canonical A-module homomorphisms which send elements of M to the corresponding elements in N_1 and N_2, respectively. Then we can define a unique mapping $H: N_1 \to N_2$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{L} & M \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ N_1 & \xrightarrow{H} & N_2 \end{array}$$

commutes, i.e., $H \circ \varphi_1 = \varphi_2 \circ L$. The mapping H has the following properties:

(i) $H(\varphi_1(f) + \varphi_1(g)) = H(\varphi_1(f)) + H(\varphi_1(g))$ for $f, g \in M$,

(ii) $H(a\varphi_1(f)) = \delta(a)\varphi_2(P(f)) + aH(\varphi_1(f))$ for $a \in A$, $f \in M$.

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Note that $\delta(a)\varphi_2(P(f))=0$ if $\varphi_1(f)=0$.

It is clear that $H(\varphi_1(f))=0$ if and only if $L(f)\in M_2$. Therefore, H *is injective if and only if* $L(f)\in M_2$ *implies* $f\in M_1$.

Since $\delta(A)\subset\mathfrak{p}^2$, we get $L(\mathfrak{p}M)\subset\mathfrak{p}M$. Set $V=M/\mathfrak{p}M$ and let φ be the canonical A-module homomorphism $\varphi\colon M\to V$ which sends elements of M to the corresponding elements in V. The module V becomes a k-vector space if we define $\bar{a}\varphi(f)=a\varphi(f)=\varphi(af)$ for $a\in A$ and $f\in M$.

Let $l\colon V\to V$ be the unique mapping which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{L} & M \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V & \xrightarrow{l} & V \end{array}$$

commutative, i.e., $l\circ\varphi=\varphi\circ L$. This mapping is actually k-linear, since $l(\bar{a}\varphi(f))=l(\varphi(af))=\varphi(L(af))=\varphi(aL(f))=\bar{a}\varphi(L(f))=\bar{a}l(\varphi(f))$. Note that $\varphi(\delta(a)P(f))=0$.

Set $U_1=V/\varphi(M_1)$ and $U_2=V/\varphi(M_2)$, and let $\psi_1\colon V\to U_1$ and $\psi_2\colon V\to U_2$ be the canonical k-linear mappings which send elements of V to the corresponding elements in U_1 and U_2, respectively. Since $l(\varphi(M_1))\subset\varphi(M_2)$, there exists a unique k-linear mapping $h\colon U_1\to U_2$ which makes the diagram

$$\begin{array}{ccc} V & \xrightarrow{l} & V \\ {\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi_2} \\ U_1 & \xrightarrow{h} & U_2 \end{array}$$

commutative, i.e., $h\circ\psi_1=\psi_2\circ l$.

The main results of this paper are the following two theorems.

THEOREM 1. *Assume that*

(Hyp. 1) *h is injective;*

(Hyp. 2) *$\pi f\in M_2$ implies $f\in M_2$ for any given $f\in M$;*

(Hyp. 3) $\bigcap_{m=1}^{\infty}\mathfrak{p}^m N_1=\{0\}$.

Then H is injective.

THEOREM 2. *Assume that*

(Hyp. 1) *h is injective;*

(Hyp. 2) *$\pi f\in M_2$ implies $f\in M_2$ for any given $f\in M$;*

(Hyp. 3′) *for every element f in M, there exists a submodule $N(f)$ of N_1 such that*

(i) $\varphi_1(f)\in N(f)$;

(ii) *$\pi\varphi_1(g) \in N(f)$ implies $\varphi_1(g) \in N(f)$ for any given $g \in M$;*

(iii) *$N(f)$ is a finitely generated A-module.*

Then H is injective.

REMARK 1. (Hyp. 2) implies that $M_2 \cap \mathfrak{p}^m M = \mathfrak{p}^m M_2$ for every positive integer m. Note that $\pi^m f \in M_2$ implies $\pi^{m-1} f \in M_2$ if (Hyp. 2) is satisfied.

REMARK 2. Condition (ii) of (Hyp. 3′) implies $N(f) \cap \mathfrak{p}^m N_1 = \mathfrak{p}^m N(f)$ for every positive integer m. Hence $N(f) \cap (\bigcap_{m=1}^{\infty} \mathfrak{p}^m N_1) = \bigcap_{m=1}^{\infty} \mathfrak{p}^m N(f)$. Note that the discrete valuation ring A is Noetherian (cf. [1; p. 94]). Since $N(f)$ is a finitely generated A-module, we get $\bigcap_{m=1}^{\infty} \mathfrak{p}^m N(f) = \{0\}$ (cf. [1; p. 110]). This means that Theorem 2 follows from Theorem 1.

In §§2 and 3, we shall explain an application of the main results to a linear Pfaffian system at an irregular singularity. The proof of Theorem 1 will be given in §4.

2. An application. We consider two n-by-n matrices $A(x, y) = \sum_{h,k=0}^{\infty} A_{hk} x^h y^k$ and $B(x, y) = \sum_{h,k=0}^{\infty} B_{hk} x^h y^k$ whose components are convergent power series in two variables (x, y), where the A_{hk} and B_{hk} are n-by-n (complex) constant matrices. Let p and q be two positive integers, and set $D_1 = x^{p+1}(\partial/\partial x)$ and $D_2 = y^{q+1}(\partial/\partial y)$. Let $\boldsymbol{C}^n\langle\langle x, y\rangle\rangle$ be the set of all convergent power series $\sum_{h,k=0}^{\infty} c_{hk} x^h y^k$ in (x, y) whose coefficients c_{hk} are n dimensional constant vectors. We define two operators

$$\Delta_1(f) = D_1 f - A(x, y)f \quad \text{and} \quad \Delta_2(f) = D_2 f - B(x, y)f\,, \qquad f \in \boldsymbol{C}^n\langle\langle x, y\rangle\rangle\,.$$

In one of our previous papers [7], we proved the following theorem.

THEOREM A. *Suppose that*

(i) *$A_{00} \in \mathrm{GL}(n; \boldsymbol{C})$ and $B_{00} \in \mathrm{GL}(n; \boldsymbol{C})$, and*

(ii) *$\Delta_1\Delta_2 = \Delta_2\Delta_1$.*

Then, for any given $f \in \boldsymbol{C}^n\langle\langle x, y\rangle\rangle$, we have $f \in$ range (Δ_1) if and only if $\Delta_2(f) \in$ range (Δ_1).

The proof was based on the method due to Harris-Sibuya-Weinberg [6]. We call this method the H-S-W method.

In the same paper, as an application, we also considered a linear Pfaffian system

$$\text{(E)} \qquad \begin{cases} D_1 u = A(x, y)u + f(x, y)\,, \\ D_2 u = B(x, y)u + g(x, y)\,, \end{cases}$$

where $f \in \boldsymbol{C}^n\langle\langle x, y\rangle\rangle$ and $g \in \boldsymbol{C}^n\langle\langle x, y\rangle\rangle$, and we proved, by utilizing Theorem A, the following theorem.

THEOREM B. *If Pfaffian system* (E) *is completely integrable, and if* $A_{00} \in \mathrm{GL}(n; \boldsymbol{C})$ *and* $B_{00} \in \mathrm{GL}(n; \boldsymbol{C})$, *then system* (E) *has a solution* u *in* $\boldsymbol{C}^n\langle\langle x, y\rangle\rangle$. *Moreover, this solution is unique.*

Note that Pfaffian system (E) is completely integrable if and only if (i) $\Delta_1\Delta_2 = \Delta_2\Delta_1$ and (ii) $\Delta_2(f) = \Delta_1(g)$.

Theorem B is also a special case (i.e., the linear case) of a theorem which was proved by Gérard and Sibuya [3 and 4]. Their proof was based on the theory of asymptotic solutions of ordinary differential equations containing parameters at an irregular singular point.

We shall explain, in this section and the next section, two proofs of Theorem A by means of Theorem 1 and Theorem 2 respectively. An application of Theorem 1 will give a proof of Theorem A which is based on the theory of asymptotic solutions of differential equations, while the H-S-W method corresponds to Theorem 2 (cf. §3).

To utilize Theorems 1 and 2, let us set

$A = \boldsymbol{C}\langle\langle y\rangle\rangle$ = the set of all convergent power series in y whose coefficients are complex numbers;

$\mathfrak{p} = (y) = yA$;

$\delta = y^{q+1}(d/dy)$ $(=D_2)$;

$M = \boldsymbol{C}^n\langle\langle x, y\rangle\rangle$;

$L = \Delta_2$ and $P = \mathrm{id}_M(=$ identity);

$M_1 = M_2 = \mathrm{range}\,(\Delta_1)$.

Then the residue class field $k = A/\mathfrak{p}$ is isomorphic to $\boldsymbol{C}$. Furthermore, since $\Delta_1: M \to M$ is an A-module homomorphism, range(Δ_1) is a submodule of M, and $N_1 = N_2 = M/\mathrm{range}\,(\Delta_1) = \mathrm{coker}\,(\Delta_1)$.

We assume that $\Delta_1\Delta_2 = \Delta_2\Delta_1$. Hence $L = \Delta_2$ maps range(Δ_1) into itself. Let $\rho: M \to \mathrm{coker}\,(\Delta_1)$ be the canonical A-module homomorphism which sends elements of M to the corresponding elements in coker(Δ_1). Then $\rho = \varphi_1 = \varphi_2$. The mapping $H: \mathrm{coker}\,(\Delta_1) \to \mathrm{coker}\,(\Delta_1)$ is defined by $H \circ \rho = \rho \circ L$ $(=\rho \circ \Delta_2)$. H is injective if and only if $\Delta_2(f) \in \mathrm{range}\,(\Delta_1)$ implies $f \in \mathrm{range}\,(\Delta_1)$. Therefore, *the conclusion of Theorem* A *is the injectivity of* H.

On the other hand, note that $\mathfrak{p}M = y\boldsymbol{C}^n\langle\langle x, y\rangle\rangle$; $V = M/\mathfrak{p}M \cong \boldsymbol{C}^n\langle\langle x\rangle\rangle$, where $\boldsymbol{C}^n\langle\langle x\rangle\rangle$ denotes the set of all convergent power series in x whose coefficients are n dimensional constant vectors. It is clear that V is a $\boldsymbol{C}$-vector space. (Note that $k = A/\mathfrak{p} \cong \boldsymbol{C}$.) Let us identify V with $\boldsymbol{C}^n\langle\langle x\rangle\rangle$. Then, the canonical A-module homomorphism $\varphi: M \to V$ is given by $\varphi(f) = f|_{y=0}$ for $f \in M$.

Since the $\boldsymbol{C}$-linear mapping $l: V \to V$ is defined by $l \circ \varphi = \varphi \circ L$, we get $l(u) = -B(x, 0)u$ for $u \in V$. Set $\Delta(u) = x^{p+1}(du/dx) - A(x, 0)u$ for

$u \in V$. Then $\Delta: V \to V$ is $\boldsymbol{C}$-linear, and $\Delta l = l\Delta$, since $\Delta_1\Delta_2 = \Delta_2\Delta_1$. Furthermore, $\varphi(\text{range}(\Delta_1)) = \text{range}(\Delta)$ and $l(\text{range}(\Delta)) \subset \text{range}(\Delta)$. Note that $U_1 = U_2 = V/\text{range}(\Delta) = \text{coker}(\Delta)$.

Let $\psi: V \to \text{coker}(\Delta)$ be the canonical $\boldsymbol{C}$-linear mapping which sends elements of V to the corresponding elements in $\text{coker}(\Delta)$. Then the $\boldsymbol{C}$-linear mapping $h: \text{coker}(\Delta) \to \text{coker}(\Delta)$ is defined by $h \circ \psi = \psi \circ l$. Therefore, h is injective if and only if

$$l(u) \in \text{range}(\Delta) \quad \text{implies} \quad u \in \text{range}(\Delta)\,. \tag{2.1}$$

If

$$B_{00} = B(0, 0) \in \text{GL}(n; \boldsymbol{C})\,, \tag{2.2}$$

then l is bijective, and $l^{-1}\Delta = \Delta l^{-1}$. Therefore (2.1) follows from (2.2). Thus we conclude that *h is injective if $B_{00} \in \text{GL}(n; \boldsymbol{C})$*.

To investigate (Hyp. 2) of Theorems 1 and 2, let $yf \in \text{range}(\Delta_1)$. This means that

$$yf(x, y) = x^{p+1}\frac{\partial g}{\partial x}(x, y) - A(x, y)g(x, y)$$

for some $g \in M$. Hence

$$0 = x^{p+1}\frac{\partial g}{\partial x}(x, 0) - A(x, 0)g(x, 0)\,.$$

Now assume that $A_{00} = A(0, 0) \in \text{GL}(n; \boldsymbol{C})$. Then $g(x, 0) = 0$ since $p > 0$. Hence $g \in y\boldsymbol{C}^n\langle\langle x, y\rangle\rangle$ and $f \in \text{range}(\Delta_1)$. Thus we conclude that *(Hyp. 2) is satisfied if $A_{00} \in \text{GL}(n; \boldsymbol{C})$*.

3. (Hyp. 3) and (Hyp. 3′). The main part of our proof of Theorem A which was given in [7] was to verify that (Hyp. 3′) is satisfied. This was done by utilizing the H-S-W method. We would not repeat it here again. Recently, Gérard [2] investigated a Pfaffian system

$$\begin{cases} P(x, y)D_1u = A(x, y)u + f(x, y) \\ P(x, y)D_2u = B(x, y)u + g(x, y)\,, \end{cases} \tag{3.1}$$

where P is a Weierstrass polynomial in x (cf. [5; p. 68]). Gérard's ideas are similar to the H-S-W method, and Theorem 2 applies to system (3.1).

To investigate (Hyp. 3), let $\rho(f) \in \bigcap_{m=1}^{\infty} \mathfrak{p}^m \,\text{coker}(\Delta_1)$. This means that, for every positive integer m, there exist two elements v_m and g_m in M such that $f = y^m g_m + \Delta_1(v_m)$. Then $\Delta_1(v_m - v_{m'}) = y^{m'}g_{m'} - y^m g_m$. If we assume that $m > m'$ and that

$$(3.2) \qquad A_{00} \in \mathrm{GL}(n; \boldsymbol{C}) ,$$

then we get $v_m - v_{m'} \in \mathfrak{p}^{m'}M$. This means that there exists a formal power series in y,

$$(3.3) \qquad v = \sum_{m=0}^{\infty} u_m y^m$$

such that

(i) $u_m \in \boldsymbol{C}^n\langle\langle x \rangle\rangle$, and

(ii) $f = \Delta_1(v)$ as formal power series in y.

If $v \in M$ (i.e., v is convergent), then we get $f \in \mathrm{range}\,(\Delta_1)$ (i.e., $\rho(f) = 0$).

We can prove the convergence of the formal solution (3.3) of the equation $f = \Delta_1(v)$, under the assumption (3.2), by utilizing the existence and uniqueness of asymptotic solutions of ordinary differential equations containing parameters at an irregular singular point (cf. [4]). Hence, under the assumption (3.2), we have

$$(3.4) \qquad \bigcap_{m=1}^{\infty} \mathfrak{p}^m \operatorname{coker}(\Delta_1) = \{0\} .$$

We can also prove (3.4) by utilizing the H-S-W method (cf. Remark 2 of §1).

4. **Proof of Theorem 1.** The proof of Theorem 1 is divided into two lemmas.

LEMMA 1. *If h is injective, then $H(\varphi_1(f)) \in \mathfrak{p}N_2$ implies $\varphi_1(f) \in \mathfrak{p}N_1$.*

PROOF. Note that $\varphi_1(f) \in \mathfrak{p}N_1$ if and only if $\varphi(f) \in \varphi(M_1)$ and that $\varphi_2(f) \in \mathfrak{p}N_2$ if and only if $\varphi(f) \in \varphi(M_2)$. Now observe that $H(\varphi_1(f)) = \varphi_2(L(f)) \in \mathfrak{p}N_2$ implies $\varphi(L(f)) \in \varphi(M_2)$. This means that $l(\varphi(f)) \in \varphi(M_2)$. Hence $\psi_2(l(\varphi(f))) = 0$, and then $h(\psi_1(\varphi(f))) = 0$. Since h is injective, we have $\psi_1(\varphi(f)) = 0$, i.e., $\varphi(f) \in \varphi(M_1)$. Therefore, $\varphi_1(f) \in \mathfrak{p}N_1$. This completes the proof of Lemma 1.

LEMMA 2. *If* (Hyp. 1) *and* (Hyp. 2) *of Theorem* 1 *are satisfied, then*

$$\ker(H) \subset \bigcap_{m=1}^{\infty} \mathfrak{p}^m N_1 .$$

PROOF. Assume that $H(\varphi_1(f)) = 0$. Then, it follows from Lemma 1 that $\varphi_1(f) \in \mathfrak{p}N_1$. Set $\varphi_1(f) = \pi^m \varphi_1(g)$, where m is a positive integer and $g \in M$. Observe that $0 = H(\varphi_1(f)) = \delta(\pi^m)\varphi_2(P(g)) + \pi^m H(\varphi_1(g))$. By virtue of (Hyp. 2), we get $H(\varphi_1(g)) \in \mathfrak{p}N_2$. Then, it follows from Lemma 1 that $\varphi_1(g) \in \mathfrak{p}N_1$. Hence $\varphi_1(f) \in \mathfrak{p}^{m+1}N_1$. This completes the proof of Lemma 2.

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