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A LIOUVILLE-TYPE THEOREM FOR SUBSONIC FLOWS AROUND AN INFINITE LONG RAMP

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Abstract. In this paper, we focus on the two-dimensional subsonic flow problem for polytropic gases around an infinite long ramp, which is motivated by a description in Section 111 of Courant-Friedrichs' book *Supersonic flow and shock waves*. The flow is assumed to be steady, isentropic and irrotational; namely, the movement of the flow is described by a second-order steady potential equation. By the complex methods together with some properties on quasi-conformal mappings, we show that a nontrivial subsonic flow around the infinite long ramp does not exist if the flow is uniformly subsonic.

1. Introduction and main result. In this paper, we will focus on the global subsonic flow problem for the polytropic gases around an infinite long 2-D ramp with cornered point or smooth convex curve (see Figure 1 and 2 below). This problem is motivated by the following descriptions given in Section 111 of [8]: For the flow around a sharp corner or body, if the oncoming flow is subsonic, then the problem involves potential flow, governed by an elliptic differential equation whose solution at any point depends on the boundary conditions even at remote parts of the boundary, and is more difficult to treat than in the case of supersonic flow. Indeed, for the supersonic flows past sharp bodies, thus far there have been many local or global existence results under various cases; one can see [6], [15], [17], [21], [25] and the references therein. However, there are few results for the subsonic flow past a sharp body although there are extensive works for the subsonic flow past a bounded body or an almost half-plane (see [2], [4], [9], [10],

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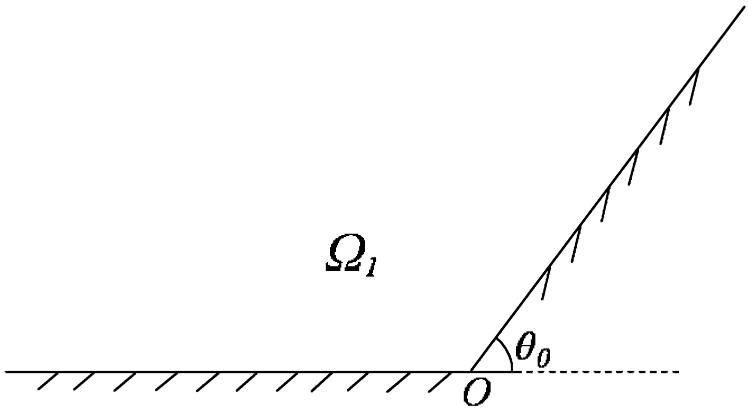


FIG. 1

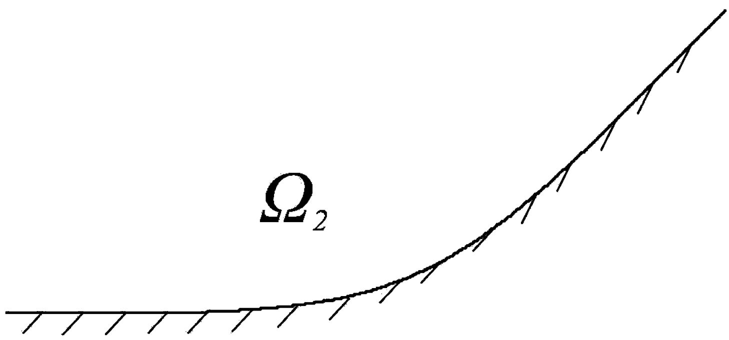


FIG. 2

[11], [20], [22], [26] and so on). Recently, under the assumptions that the Mach number of the subsonic flow is suitably low and the flow field together with its derivatives are appropriately behaved at infinity, the authors in [16] show that such a subsonic flow does not move actually. In the present paper, for the 2-D infinite ramp we will prove the same result as in [16] only under the hypothesis that the flow is uniformly subsonic. From our result, we know that the phenomenon of the subsonic flow around an infinite ramp is obviously different than that of a subsonic flow around a bounded body or an almost half-plane due to the essentially geometrical differences among these obstacles. In the latter case, the nontrivial subsonic flows can be shown to exist (see [2], [4], [9], [10], [11], [20], [22] and [26]).

Now we use the potential flow equation to describe the motion of subsonic polytropic gas in the domains $\Omega_1 = \{(x,y) : x \leq 0, y > 0; x > 0, y > tan\theta_0 x\}$ and $\Omega_2 = \{(x,y) : y > f(x), x \in \mathbb{R}\}$ respectively. Here $f(x) \in C^2(\mathbb{R})$, $f(x) = 0$ for $x \leq -1$, and $f(x) = tan\theta_0 x$ for $x \geq 1$, and is convex and strictly increasing for $-1 \leq x \leq 1$. Let $\phi(x,y)$ be the potential of velocity (u,v), i.e., $u = \partial_x\phi, v = \partial_y\phi$. Then it follows from Bernoulli's law that

$$\frac{1}{2}|\nabla\phi|^2 + h(\rho) = C_0. \tag{1.1}$$

Here $h(\rho) = \dfrac{c^2(\rho)}{\gamma - 1}$ is the specific enthalpy for the polytropic gas with the state equation $P = A\rho^\gamma$ ($1 < \gamma < 3$, $A > 0$ is a constant), $c(\rho) = \sqrt{P'(\rho)}$ is the local sound speed, and $\nabla = (\partial_x, \partial_y)$, $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ is the Bernoulli's constant.

From (1.1) and the implicit function theorem, the density function ρ can be expressed as

$$\rho = \rho(|\nabla\phi|^2). \tag{1.2}$$

Substituting (1.2) into the mass conservation equation $\partial_x(\rho u) + \partial_y(\rho v) = 0$ yields the following potential equation

$$A_{11}(u, v)\phi_{xx} + 2A_{12}(u, v)\phi_{xy} + A_{22}(u, v)\phi_{yy} = 0, \tag{1.3}$$

where

$$A_{11} = 1 - \frac{u^2}{c^2}, \qquad A_{12} = -\frac{uv}{c^2}, \qquad A_{22} = 1 - \frac{v^2}{c^2}.$$

Due to the fixed wall condition, we have on the boundary of Ω_i ($i = 1, 2$)

$$\partial_{\vec{n}}\phi = 0 \qquad \text{on} \qquad \partial\Omega_i, \tag{1.4}$$

where $\vec{n}$ represents the unit outward normal to $\partial\Omega_i$.

In addition, we assume the flow is uniformly subsonic in the whole domain Ω_i; namely, the Mach number satisfies

$$M = \frac{|\nabla\phi|}{c(\rho)} \le \lambda_0 \qquad \text{for } (x, y) \in \Omega_i, \tag{1.5}$$

where $0 < \lambda_0 < 1$ is any fixed constant.

We now state the main result in this paper.

Theorem 1.1 (Nonexistence of a global nontrivial uniformly subsonic flow). Under the assumption (1.5), if $\phi \in C^{1,\alpha}(\bar{\Omega}_i) \cap C^2(\Omega_i)$ ($0 < \alpha < 1$) is a solution to (1.3) together with the boundary condition (1.4), then $\phi \equiv C$ in $\Omega_i (i = 1, 2)$. Namely, $(u, v) \equiv (0, 0)$ in Ω_i.

Remark 1.2. The assumption on $\phi(x, y) \in C^{1,\alpha}(\bar{\Omega}_1) \cap C^2(\Omega_1)$ is plausible from the regularity theory of solutions to the second-order elliptic equations in the cornered domains (one can see [1, 14, 18] and so on). In fact, at the corner point O of Ω_1, the nonlinear equation (1.3) becomes the Laplacian equation $\Delta\phi = 0$ due to $u = v = 0$ at O. Then it follows from the result in [1] that $\phi \in C^{1,\alpha}(\bar{\Omega}_1)$ with $\alpha = \frac{\theta_0}{\pi - \theta_0}$ holds. With respect to $\phi \in C^2(\Omega_i) (i = 1, 2)$, this comes from the Schauder interior estimate (one can see [12]). In particular, for the case of Ω_2, we can further have $\phi(x, y) \in C^{2,\alpha}(\bar{\Omega}_2)$ due to the Schauder interior and boundary estimates.

Remark 1.3. In the case of Ω_1, compared with the result in [16], we have successfully removed the crucial restrictions on the smallness of λ_0 in (1.5) and some artificial assumptions of $\partial^\alpha_{x,y}\phi (0 \le |\alpha| \le 2)$ at infinity in the present paper.

REMARK 1.4. When $\theta_0 = 0$ in Ω_1, that is, when Ω_1 is an upper-half-plane, then the result similar to the Bernstein theorem on minimal surface established in [3] holds, which illustrates that the C^2-solution $\phi(x, y)$ of subsonic potential equation (1.3) in $\mathbb{R}^2_+$ must be a linear function. This means that the uniform subsonic flow in $\mathbb{R}^2_+$ should keep a constant state.

REMARK 1.5. By the dimensionless treatment, the equation (1.3) can be rewritten as

$$\partial_x(\rho\partial_x\varphi) + \partial_y(\rho\partial_y\varphi) = 0 \qquad \text{with } \rho = (1 - \frac{\gamma-1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}} \text{ and } \gamma > 1. \tag{1.6}$$

In this case, (1.6) is elliptic when $|\nabla\varphi| < \sqrt{\frac{2}{\gamma+1}}$. We note that (1.6) becomes the minimal surface equation when $\gamma = -1$. The classical Bernstein theorem states that a $C^2(\mathbb{R}^2)$ solution of the minimal surface equation in $\mathbb{R}^2$ must be a linear function. This is also true in the case when $n \leq 7$ for an n-dimensional minimal surface equation $\sum_{k=1}^{n} \partial_k(\frac{\partial_k\varphi}{\sqrt{1+|\nabla\varphi|^2}}) = 0$ (one can see [5], [13], [23] and so on). However, in the unbounded domain of $\mathbb{R}^n$ $(n \leq 7)$ with boundaries (not the whole $\mathbb{R}^n$), the classical Bernstein theorem sometimes is incorrect; one can see [7, 19] and the references therein.

REMARK 1.6. From Theorem 1.1, we can at least conclude that when the subsonic flow problem (1.3)-(1.4) really has a nontrivial subsonic solution, the flow must attain its sonic speed at some place of infinity. However, we guess that even if the uniformly subsonic condition (1.5) is replaced by the subsonic condition $M = \frac{|\nabla\phi|}{c(\rho)} < 1$ in Ω_i, the problem (1.3)-(1.4) still admits only trivial solution $\phi \equiv C$; namely, the velocity of gas $(u, v) \equiv (0, 0)$ holds.

Now we comment on the proof procedure of Theorem 1.1. Motivated by the complex methods, in particular the theory of quasi-conformal transformations and conformal mappings with respect to the Riemann metric in [2], [3], and by a careful analysis on a suitably chosen analytic and homeomorphic function between the domain Ω_i and the unit disk, we can derive that $(u(x, y), v(x, y)) \equiv 0$ holds in Ω_i. Thus, Theorem 1.1 is established.

Our paper is organized as follows: In §2, we list some basic properties of quasi-conformal mappings and give some necessary illustrations for the reader's convenience. Based on the properties given in §2, we will complete the proof of Theorem 1.1 in §3.

2. Some properties of quasi-conformal mappings.

As in [2], [3], we will use the following concepts.

DEFINITION 2.1 (Smooth Mapping). A homeomorphism mapping $\zeta(z) = \xi(x, y) + i\eta(x, y)$ of a domain D in the $z-$plane $\mathbb{C}$ is called smooth if the derivatives $\xi_x, \xi_y, \eta_x, \eta_y$ exist and are continuous, and the Jacobian $J = \xi_x\eta_y - \xi_y\eta_x > 0$.

DEFINITION 2.2 (Quasi-conformal). A smooth mapping $\zeta(z)$ is called quasi-conformal in the domain D if $\xi_x^2 + \xi_y^2 + \eta_x^2 + \eta_y^2 \leq C(\xi_x\eta_y - \xi_y\eta_x)$ holds in D for some positive constant C.

Definition 2.3 (Conformal). A smooth mapping $\zeta(z)$ in D is said to be conformal with respect to a Riemann metric $g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2$ if

$$d\xi^2 + d\eta^2 = \lambda(x,y)\big(g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2\big), \tag{2.1}$$

where $\lambda(x,y), g_{11}(x,y)$ and $g_{11}(x,y)g_{22}(x,y) - g_{12}^2(x,y)$ are positive in D.

Here we point out a useful fact that if $g_{ij} \in C^\alpha(D)(0<\alpha<1)$, then there exists a smooth mapping $\zeta(z)$ such that $\zeta(z)$ is conformal to the Riemann metric $g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2$ (one can see Lemma 2.5 of [2]).

For later use, we now list some basic properties on the conformal (or quasi-conformal) mappings and give proofs if necessary for the reader's convenience.

Lemma 2.4. If the smooth mapping $\zeta(z)$ in D is conformal with respect to Riemann metric $g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$, and the C^1-mapping $\tilde{\zeta}(z) = \tilde{\xi}(x,y) + \tilde{\eta}(x,y)$ in D satisfies $d\tilde{\xi}^2 + d\tilde{\eta}^2 = \tilde{\lambda}(x,y)(g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2)$ with $\tilde{\lambda}(x,y) \geq 0$ and $\dfrac{\partial(\tilde{\xi},\tilde{\eta})}{\partial(x,y)} \geq 0$, then $\tilde{\zeta}$ is an analytic function of ζ.

Proof. By $d\xi^2 + d\eta^2 = \lambda(g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2)$ with $\lambda > 0$ and $d\tilde{\xi}^2 + d\tilde{\eta}^2 = \tilde{\lambda}(x,y)(g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2)$, we have

$$d\tilde{\xi}^2 + d\tilde{\eta}^2 = \frac{\tilde{\lambda}}{\lambda}(d\xi^2 + d\eta^2). \tag{2.2}$$

On the other hand, one has

$$\begin{aligned} d\tilde{\xi}^2 + d\tilde{\eta}^2 &= \left(\frac{\partial\tilde{\xi}}{\partial\xi}d\xi + \frac{\partial\tilde{\xi}}{\partial\eta}d\eta\right)^2 + \left(\frac{\partial\tilde{\eta}}{\partial\xi}d\xi + \frac{\partial\tilde{\eta}}{\partial\eta}d\eta\right)^2 \\ &= \left((\frac{\partial\tilde{\xi}}{\partial\xi})^2 + (\frac{\partial\tilde{\eta}}{\partial\xi})^2\right)d\xi^2 + 2\left(\frac{\partial\tilde{\xi}}{\partial\xi}\frac{\partial\tilde{\xi}}{\partial\eta} + \frac{\partial\tilde{\eta}}{\partial\xi}\frac{\partial\tilde{\eta}}{\partial\eta}\right)d\xi d\eta + \left((\frac{\partial\tilde{\xi}}{\partial\eta})^2 + (\frac{\partial\tilde{\eta}}{\partial\eta})^2\right)d\eta^2. \end{aligned} \tag{2.3}$$

(2.3) together with (2.2) yields

$$\begin{cases} (\dfrac{\partial\tilde{\xi}}{\partial\xi})^2 + (\dfrac{\partial\tilde{\eta}}{\partial\xi})^2 = (\dfrac{\partial\tilde{\xi}}{\partial\eta})^2 + (\dfrac{\partial\tilde{\eta}}{\partial\eta})^2, \\ \dfrac{\partial\tilde{\xi}}{\partial\xi}\dfrac{\partial\tilde{\xi}}{\partial\eta} + \dfrac{\partial\tilde{\eta}}{\partial\xi}\dfrac{\partial\tilde{\eta}}{\partial\eta} = 0. \end{cases} \tag{2.4}$$

Set $A = \dfrac{\partial\tilde{\xi}}{\partial\xi}$, $B = \frac{\partial\tilde{\xi}}{\partial\eta}$, $C = \frac{\partial\tilde{\eta}}{\partial\xi}$, $D = \frac{\partial\tilde{\eta}}{\partial\eta}$. Then (2.4) can be simplified as

$$\begin{cases} A^2 + C^2 = B^2 + D^2, \\ AB + CD = 0. \end{cases}$$

By a direct computation, we can arrive at $(D^2 + B^2)(D^2 - A^2) = 0$. Hence, if $D^2 + B^2 > 0$, then $D^2 = A^2$; if $D^2 + B^2 = 0$, then $A = B = C = D = 0$; this also means $D^2 = A^2$. That is, $D^2 = A^2$ actually holds.

We now assert $A = D$. Indeed, if not, then there exists a point in D such that $A = -D \neq 0$. It follows from $AB + CD = 0$ that $B = C$. Due to $J = \dfrac{\partial(\tilde{\xi},\tilde{\eta})}{\partial(\xi,\eta)} =$

$\dfrac{\partial(\tilde{\xi},\tilde{\eta})}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(\xi,\eta)} \geq 0$, but $J = -D^2 - B^2 < 0$, this is a contradiction. Thus, $A = D$ and further $B^2 = C^2$.

Next, we assert $B = -C$. Indeed, if not, there exists some point in D such that $B = C \neq 0$. At this time, $A = -D$ holds by $AB + CD = 0$. This, together with $A = D$, yields $A = D = 0$. Hence, $J = -B^2 < 0$ is contradictory with $J \geq 0$. Namely, $B = -C$.

It follows from $A = D$ and $B = -C$ that

$$\frac{\partial\tilde{\xi}}{\partial\xi} - \frac{\partial\tilde{\eta}}{\partial\eta} = 0, \quad \frac{\partial\tilde{\xi}}{\partial\eta} + \frac{\partial\tilde{\eta}}{\partial\xi} = 0.$$

Namely, the Cauchy-Riemann equations of $(\tilde{\xi},\tilde{\eta})$ on the variables (ξ,η) hold. Thus $\tilde{\zeta}$ is an analytic function of ζ and we complete the proof of Lemma 2.4. $\square$

LEMMA 2.5. If $g_{ij}(x,y) \in C^\alpha(D)$ for the Riemann metric $g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$ and satisfies in D

$$\frac{g_{11}+g_{22}}{\sqrt{g_{11}g_{22}-g_{12}^2}} \leq C, \tag{2.5}$$

with $C > 0$ a constant, then each conformal mapping $\zeta(z) = \xi(x,y) + i\eta(x,y)$ with respect to this Riemann metric is quasi-conformal in D.

Proof. Since $\zeta(z)$ is conformal with respect to $g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$, then we obtain

$$\begin{cases} \xi_x^2 + \eta_x^2 = \lambda g_{11}, \\ \xi_y^2 + \eta_y^2 = \lambda g_{22}, \\ \xi_x\xi_y + \eta_x\eta_y = \lambda g_{12}. \end{cases}$$

This derives

$$\lambda^2 g_{11}g_{22} = \lambda^2 g_{12}^2 + J^2$$

and

$$J = \lambda\sqrt{g_{11}g_{22}-g_{12}^2} = \frac{\sqrt{g_{11}g_{22}-g_{12}^2}}{g_{11}+g_{22}}(\xi_x^2 + \xi_y^2 + \eta_x^2 + \eta_y^2).$$

Therefore, we have $\xi_x^2 + \xi_y^2 + \eta_x^2 + \eta_y^2 \leq C(\xi_x\eta_y - \xi_y\eta_x)$, and we complete the proof of Lemma 2.5. $\square$

The following result comes from Lemma 2.10 of [2]; here we omit its proof.

LEMMA 2.6. Under a quasi-conformal homeomorphism of a domain D onto a domain Δ, a non-degenerate boundary continuum of D corresponds to a non-degenerate boundary continuum of Δ.

Set $w^* = q^* e^{-i\theta}$ (which is called the distorted velocity in [2]), where $q^*(q) = q \cdot exp\left(\int_0^q \frac{\sqrt{1-M^2(s)}-1}{s}ds\right)$ with $q = \sqrt{u^2+v^2}$, $M(q) = \dfrac{q}{c(\rho(q^2))}$, $\theta = arctan\dfrac{v}{u}$ and (u,v) is a subsonic solution to (1.3).

Let $u^* = Rew^*$ and $v^* = -Imw^*$; then one has

LEMMA 2.7.

$$(du^*)^2 + (dv^*)^2 = \tilde{\lambda}(x,y)\left(A_{22}dx^2 - 2A_{12}dxdy + A_{11}dy^2\right), \tag{2.6}$$

where $\tilde{\lambda}(x,y) \geq 0$ and $\dfrac{\partial(u^*, -v^*)}{\partial(x,y)} \geq 0$ for $(x,y) \in \Omega_i$ $(i = 1,2)$.

Proof. Due to

$$\frac{\partial(u^*,v^*)}{\partial(x,y)} = \frac{\partial(u^*,v^*)}{\partial(q^*,\theta)}\frac{\partial(q^*,\theta)}{\partial(q,\theta)}\frac{\partial(q,\theta)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)},$$

we obtain

$$\begin{aligned}
\frac{\partial u^*}{\partial x} &= \left(\frac{u^2\beta\, q^*}{q^3} + \frac{q^* v^2}{q^3}\right)\phi_{xx} + \left(\frac{u\beta\, q^* v}{q^3} - \frac{q^* vu}{q^3}\right)\phi_{xy},\\
\frac{\partial u^*}{\partial y} &= \left(\frac{u^2\beta\, q^*}{q^3} + \frac{q^* v^2}{q^3}\right)\phi_{xy} + \left(\frac{u\beta\, q^* v}{q^3} - \frac{q^* vu}{q^3}\right)\phi_{yy},\\
\frac{\partial v^*}{\partial x} &= \left(\frac{u\beta\, q^* v}{q^3} - \frac{q^* vu}{q^3}\right)\phi_{xx} + \left(\frac{v^2\beta\, q^*}{q^3} + \frac{q^* u^2}{q^3}\right)\phi_{xy},\\
\frac{\partial v^*}{\partial y} &= \left(\frac{u\beta\, q^* v}{q^3} - \frac{q^* vu}{q^3}\right)\phi_{xy} + \left(\frac{v^2\beta\, q^*}{q^3} + \frac{q^* u^2}{q^3}\right)\phi_{yy}
\end{aligned}$$

with $\beta = \beta(q) = \sqrt{1 - M^2(q)}$.

This, together with the equation (1.3) and a direct computation, yields

$$\begin{aligned}
&(du^*)^2 + (dv^*)^2 = \left(\frac{\partial u^*}{\partial x}dx + \frac{\partial u^*}{\partial y}dy\right)^2 + \left(\frac{\partial v^*}{\partial x}dx + \frac{\partial v^*}{\partial y}dy\right)^2\\
&= \left(\frac{q^*}{q^2}\right)^2 \Big[\Big(\left(v^2 + u^2\beta^2\right)\phi_{xx}^2 + 2\,uv\left(\beta^2 - 1\right)\phi_{xx}\phi_{xy} + \left(u^2 + v^2\beta^2\right)\phi_{xy}^2\Big)dx^2\\
&\quad + \Big(\left(2\,v^2 + 2\,u^2\beta^2\right)\phi_{xy}\phi_{xx} + 2\,uv\left(\beta^2 - 1\right)\phi_{yy}\phi_{xx}\\
&\quad + 2\,uv\left(\beta^2 - 1\right)\phi_{xy}^2 + \left(2\,u^2 + 2\,v^2\beta^2\right)\phi_{yy}\phi_{xy}\Big)dxdy\\
&\quad + \Big(\left(v^2 + u^2\beta^2\right)\phi_{xy}^2 + 2\,uv\left(\beta^2 - 1\right)\phi_{yy}\phi_{xy} + \left(u^2 + v^2\beta^2\right)\phi_{yy}^2\Big)dy^2\Big]\\
&= (\frac{q^*}{q})^2\Bigg[\left(\frac{(c^2 - u^2)\phi_{xx}^2}{c^2} - \frac{2uv\phi_{xx}\phi_{xy}}{c^2} + \frac{(c^2 - v^2)\phi_{xy}^2}{c^2}\right)dx^2\\
&\quad + 2\left(\frac{(c^2 - u^2)\phi_{xx}\phi_{xy}}{c^2} - \frac{uv\phi_{xx}\phi_{yy}}{c^2} - \frac{uv\phi_{xy}^2}{c^2} + \frac{(c^2 - v^2)\phi_{yy}\phi_{xy}}{c^2}\right)dxdy\\
&\quad + \left(\frac{(c^2 - u^2)\phi_{xy}^2}{c^2} - \frac{2uv\phi_{xy}\phi_{yy}}{c^2} + \frac{(c^2 - v^2)\phi_{yy}^2}{c^2}\right)dy^2\Bigg]\\
&= (\frac{q^*}{q})^2\Big[\left(A_{11}\phi_{xx}^2 + 2A_{12}\phi_{xx}\phi_{xy} + A_{22}\phi_{xy}^2\right)dx^2 + \left(A_{11}\phi_{xy}^2 + 2A_{12}\phi_{xy}\phi_{yy} + A_{22}\phi_{yy}^2\right)dy^2\\
&\quad + 2\left(A_{11}\phi_{xx}\phi_{xy} + A_{12}\phi_{xx}\phi_{yy} + A_{12}\phi_{xy}^2 + A_{22}\phi_{xy}\phi_{yy}\right)dxdy\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{q^*}{q}\right)^2 \Bigg[\left(\frac{A_{22}\left(A_{22}\phi_{yy}^2 + 2\,A_{12}\phi_{xy}\phi_{yy} + A_{11}\phi_{xy}^2\right)}{A_{11}}\right) dx^2 \\
&\quad + (A_{22}\phi_{yy}^2 + 2\,A_{12}\phi_{xy}\phi_{yy} + A_{11}\phi_{xy}^2) dy^2 \\
&\quad - \left(\frac{2A_{12}(A_{22}\phi_{yy}^2 + 2\,A_{12}\phi_{xy}\phi_{yy} + A_{11}\phi_{xy}^2)}{A_{11}}\right) dxdy\Bigg] \\
&= (\frac{q^*}{q})^2 \left(\frac{A_{22}\phi_{yy}^2 + 2\,A_{12}\phi_{xy}\phi_{yy} + A_{11}\phi_{xy}^2}{A_{11}}\right)\left(A_{22}dx^2 - 2A_{12}dxdy + A_{11}dy^2\right) \\
&= \left(\frac{q^*}{q}\right)^2 \left(\phi_{xy}^2 - \phi_{xx}\phi_{yy}\right)\left(A_{22}dx^2 - 2A_{12}dxdy + A_{11}dy^2\right) \\
&= \left(exp\int_0^q \frac{\sqrt{1-M^2}-1}{s} ds\right)^2 \left(\phi_{xy}^2 - \phi_{xx}\phi_{yy}\right)\left(A_{22}dx^2 - 2A_{12}dxdy + A_{11}dy^2\right).
\end{aligned}
$$

Set $\tilde{\lambda}(x,y) = \left(exp\int_0^q \frac{\sqrt{1-M^2}-1}{s} ds\right)^2 \left(\phi_{xy}^2 - \phi_{xx}\phi_{yy}\right)$; then (2.6) is proved.

In addition,

$$
\phi_{xy}^2 - \phi_{xx}\phi_{yy} = \frac{A_{11}^2\phi_{xx}^2 + 2(A_{11}A_{22} - 2A_{12}^2)\phi_{xx}\phi_{yy} + A_{22}^2\phi_{yy}^2}{4{A_{12}}^2} \geq 0 \tag{2.7}
$$

holds due to the positivity of the matrix $\begin{pmatrix} A_{11}^2 & A_{11}A_{22} - 2A_{12}^2 \\ A_{11}A_{22} - 2A_{12} & A_{22}^2 \end{pmatrix}$ by the subsonic property of (u,v). This yields $\tilde{\lambda}(x,y) \geq 0$ in Ω_i $(i = 1,2)$.

On the other hand, we have by a direct computation and (2.7) that

$$
\frac{\partial(u^*, -v^*)}{\partial(x,y)} = \beta\left(\frac{q^*}{q}\right)^2 \left(\phi_{xy}^2 - \phi_{xx}\phi_{yy}\right) \geq 0.
$$

Therefore, we complete the proof of Lemma 2.7. □

3. The proof of Theorem 1.1. Based on some preparations in §2, we now start to give the proof of Theorem 1.1.

Proof. (i). In the case of Ω_1, since the coefficients of the matric $A_{22}(u,v)dx^2 - 2A_{12}(u,v)dxdy + A_{11}(u,v)dy^2$ is $C^\alpha(\bar{\Omega}_1)$, it follows from Lemma 2.5 in [2] that there exists a homeomorphism mapping

$$
\zeta(z) : \Omega_1 \longrightarrow D
$$

such that ζ is conformal with respect to the metric $A_{22}(u,v)dx^2 - 2A_{12}(u,v)dxdy + A_{11}(u,v)dy^2$ and the domain D is convex. By the hypothesis of uniformly subsonic flow in Theorem 1.1 and Lemma 2.5, we conclude that ζ is a quasi-conformal mapping (due to $\frac{A_{11}+A_{12}}{\sqrt{A_{11}A_{22}-A_{12}^2}} \leq \frac{2}{\sqrt{1-\lambda_0^2}}$ by the condition (1.5)). On the other hand, by the fact that the composite function of the quasi-conformal mapping and the conformal mapping is still quasi-conformal, it follows from Lemma 2.6 that $D = \{\zeta : |\zeta| < 1\}$ and $\partial\Omega_1 = \{(x,y) : x \leq 0, y = 0\} \cup \{(x,y) : x \geq 0, y = tan\theta_0 x\} \cup \{(x,y) : x^2 + y^2 = \infty, x \geq 0, y \geq 0\} \to \partial D = \{\zeta : |\zeta| = 1\}$ can be assumed without loss of generality (noting that the domain D is conformal to a unit disk by the Riemann mapping theorem). This, together with Lemma 2.7 and Lemma 2.5, yields that the distorted velocity $\widetilde{w^*}(\zeta) = w^*(z(\zeta))$ is an analytic function for $\zeta \in D$. For notational convenience, we can assume $\zeta(0) = 1 \,\text{and}\, \zeta(\infty) = -1$. It is noted that $\widetilde{w}^*$ is continuous up to $\partial D \setminus \{-1\}$ by the assumption on $\phi \in C^{1,\alpha}(\bar{\Omega}_1)$. Thus, in order to use the Schwartz reflection principle (see [24] or so on) and further prove Theorem 1.1 in the case of Ω_1, the key point is to show $\widetilde{w^*} \in C(\bar{D})$ and $Im\big(\widetilde{w^*}(\zeta)\big) = 0$ on ∂D. To this end, we will give the following argument.

Suppose that the parameter equation of $\partial\Omega_1$ in complex plane $\mathbb{C}$ is

$$z = p(s) = \begin{cases} s & \text{if } s \le 0, \\ se^{i\theta_0} & \text{if } s > 0. \end{cases}$$

Let the function $s = g(\alpha)$ be determined by

$$\zeta(p(s)) = e^{i\alpha}, \qquad \alpha \in (0, 2\pi),$$

which is plausible since $\zeta(z)$ is a quasi-conformal mapping from Ω_1 onto D.

The fixed wall condition (1.4) can be rewritten as

$$Im(w^*(p(s))e^{i\theta(s)}) = 0 \quad \text{on} \quad \partial\Omega_1 \tag{3.1}$$

with

$$\theta(s) = \begin{cases} 0 & \text{if } s \le 0, \\ \theta_0 & \text{if } s > 0. \end{cases}$$

This also means

$$Im(\widetilde{w^*}(e^{i\alpha})e^{i\theta(g(\alpha))}) = 0 \quad \text{on} \quad \partial D \setminus \{-1\}. \tag{3.2}$$

To prove $\widetilde{w^*}(\zeta) \equiv 0$ in D, we intend to find a suitable analytic function $F(\zeta) = e^{\mu(\zeta)}\widetilde{w^*}(\zeta)$ such that $F(\zeta) \equiv 0$ in D.

At first, we look for an analytic function $\mu(\zeta) = \mu_1(\xi, \eta) + i\mu_2(\xi, \eta)$ with $\zeta = \xi + i\eta$ such that $Im(F(\zeta)) = 0$ on ∂D and $\mu_2 = \theta(g(\alpha))$ on ∂D.

It is noted that $\theta(g(\alpha)) = \theta_0$ for $0 < \alpha < \pi$ and $\theta(g(\alpha)) = 0$ for $\pi < \alpha < 2\pi$. Moreover, $\mu_2(\xi, \eta)$ is harmonic in D. Then by Poisson's formulae, we have

$$\mu_2(\rho, \theta) = \frac{1-\rho^2}{2\pi}\int_0^{2\pi} \frac{\theta(g(\alpha))}{\rho^2 + 1 - 2\rho cos(\alpha - \theta)} d\alpha = \frac{1-\rho^2}{2\pi}\int_0^{\pi} \frac{\theta_0}{\rho^2 + 1 - 2\rho cos(\alpha - \theta)} d\alpha;$$

here $\zeta = \rho e^{i\theta}$ with $\theta \in [0, 2\pi]$.

Due to $\int \dfrac{dx}{a - cosx} = \dfrac{2}{\sqrt{a^2-1}} arctan(\sqrt{\dfrac{a+1}{a-1}} tan\dfrac{x}{2})$ for the constant $a > 1$, in the case of $0 < \theta < \pi$, we have

$$\begin{aligned} \mu_2(\rho, \theta) &= \frac{1-\rho^2}{2\pi} \frac{2\theta_0}{\sqrt{(\rho^2+1)^2 - (2\rho)^2}} arctan \frac{(1-\rho^2)tan\frac{\alpha-\theta}{2}}{(\rho-1)^2}\Bigg|_{\alpha=0}^{\alpha=\pi} \\ &= \frac{\theta_0}{\pi}\big[arctan(\frac{1+\rho}{1-\rho}cot\frac{\theta}{2}) + arctan(\frac{1+\rho}{1-\rho}tan\frac{\theta}{2})\big]. \end{aligned}$$

In the case of $\pi < \theta < 2\pi$, we have

$$\begin{aligned}\mu_2(\rho,\theta) &= \frac{1-\rho^2}{2\pi}\Big[\int_0^{\theta-\pi}\frac{\theta_0}{\rho^2+1-2\rho cos(\alpha-\theta)}d\alpha + \int_{\theta-\pi}^{\pi}\frac{\theta_0}{\rho^2+1-2\rho cos(\alpha-\theta)}d\alpha\Big]\\ &= \frac{1-\rho^2}{2\pi}\Big[\lim_{\varepsilon\to 0+}\int_0^{\theta-\pi-\varepsilon}\frac{\theta_0}{\rho^2+1-2\rho cos(\alpha-\theta)}d\alpha + \lim_{\varepsilon\to 0+}\int_{\theta-\pi+\varepsilon}^{\pi}\frac{\theta_0}{\rho^2+1-2\rho cos(\alpha-\theta)}d\alpha\Big]\\ &= \frac{\theta_0}{\pi}\Big[\lim_{\varepsilon\to 0+} arctan(\frac{1+\rho}{1-\rho}tan\frac{\alpha-\theta}{2})\Big|_{\alpha=0}^{\alpha=\theta-\pi-\varepsilon} + \lim_{\varepsilon\to 0+} arctan(\frac{1+\rho}{1-\rho}tan\frac{\alpha-\theta}{2})\Big|_{\alpha=\theta-\pi+\varepsilon}^{\alpha=\pi}\Big]\\ &= \theta_0 + \frac{\theta_0}{\pi}\left(arctan(\frac{1+\rho}{1-\rho}cot\frac{\theta}{2}) + arctan(\frac{1+\rho}{1-\rho}tan\frac{\theta}{2})\right).\end{aligned}$$

On the other hand, $\mu_1(\xi,\eta)$ can be determined as follows:

$$\begin{aligned}\mu_1(\xi,\eta) &= \int_{(0,0)}^{(\xi,\eta)}(\mu_1)_\xi d\xi + (\mu_1)_\eta d\eta\\ &= \int_{(0,0)}^{(\xi,\eta)}(\mu_2)_\eta d\xi - (\mu_2)_\xi d\eta\\ &= \theta_0\int_0^{\rho}\frac{(\mu_2)_\theta(r,\theta)}{r}dr\\ &= \frac{2\theta_0 cos\theta}{\pi}\int_0^{\rho}\frac{r^2-1}{4r^2cos^2\theta - r^4 - 2r^2 - 1}dr\\ &= \frac{\theta_0}{2\pi}ln\left(\frac{\rho^2+1+2\rho\cos\theta}{\rho^2+1-2\rho\cos\theta}\right).\end{aligned}$$

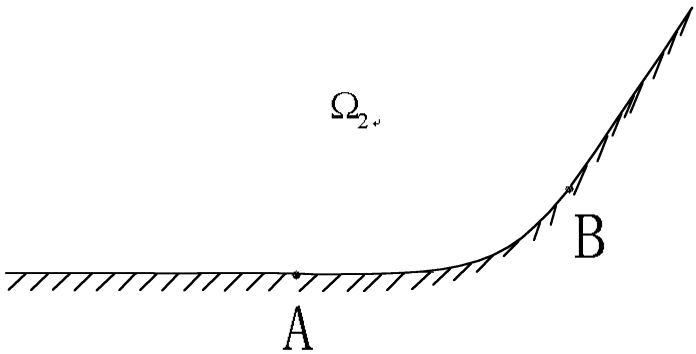
Next, we show $Im(F(\zeta)) = 0$ on ∂D and $F(\zeta)\in C(\bar{D})$.

Indeed, in terms of (3.2) and the expression of $F(\zeta)$, we have $Im(F(\zeta)) = 0$ on $\partial D\setminus\{-1\}$ and $F(\zeta)\in C(\bar{D}\setminus\{-1\})$. In addition, due to $e^{\mu_1} = \left(\frac{\rho^2+1+2\rho\cos\theta}{\rho^2+1-2\rho\cos\theta}\right)^{\frac{\theta_0}{2\pi}}$ and $|F(\zeta)|\le Ce^{\mu_1}$, we have $\lim\limits_{\zeta\to -1,|\zeta|\le 1}F(\zeta) = 0$. This means that $Im(F(\zeta)) = 0$ on ∂D and $F(\zeta)\in C(\bar{D})$ hold. Consequently, by the Schwartz reflection principle we obtain that $F(\zeta)$ can be analytically extended to the whole complex plane $\mathbb{C}$ and, further, $F(\zeta)\equiv C$ in D holds due to $Im(F(\zeta)) = 0$ on ∂D. It follows from $\lim\limits_{\zeta\to -1}F(\zeta) = 0$ that $F(\zeta)\equiv 0$. This implies $\widetilde{w^*}(\zeta)\equiv 0$ in D and $w^*(z)\equiv 0$ in Ω_1. Namely, $u = v\equiv 0$ in Ω_1. Then we complete the proof of Theorem 1.1 for the domain Ω_1.

(ii). In the case of Ω_2, due to the convexity of Ω_2, as in (i), it follows from the Riemann mapping theorem and Lemma 2.6 that Ω_2 is quasi-conformal to the unit disk $D = \{\zeta : |\zeta| < 1\}$. For convenience, we describe the domain Ω_2 as follows with $A = (-1,0)$ and $B = (1, tan\theta_0)$.

Without loss of generality, suppose that the quasi-conformal mapping $\zeta(z):\Omega_2\longrightarrow D$ satisfies $\zeta(\infty) = -1$, $\zeta(A) = e^{-i\frac{\pi}{4}}$, $\zeta(B) = e^{i\frac{\pi}{4}}$.

As in (i), in order to show Theorem 1.1 in the case of Ω_2, we intend to find a suitable analytic function $F(\zeta) = e^{\mu(\zeta)}\widetilde{w^*}(\zeta)$ such that $F(\zeta)\equiv 0$ in D with the same notations of $\widetilde{w^*}(\zeta)$ and $\mu(\zeta)$ in (i).



Suppose that the parameter equation of $\partial\Omega_2$ in the complex plane is $z = p(s) = s + if(s)$ for $s \in (-\infty, \infty)$. Let the function $s = g(\alpha)$ be determined by

$$\zeta(p(s)) = e^{i\alpha}, \qquad \alpha \in (0, 2\pi).$$

Set $\mu_2(\xi, \eta) = \arctan f'(g(\alpha))$ on ∂D, and then it follows from the harmonic property of μ_2 that we have

$$\mu_2(\rho, \theta) = \frac{1-\rho^2}{2\pi} \int_0^{2\pi} \frac{\arctan f'(g(\alpha))}{1 + \rho^2 - 2\rho\cos(\alpha - \theta)} d\alpha.$$

For notational convenience, we denote by $\varphi(\alpha) \equiv \arctan f'(g(\alpha))$. Meanwhile, $\mu_1(\xi, \eta)$ can be expressed as

$$\begin{aligned}
\mu_1(\rho,\theta) &= \int_0^\rho \frac{(\mu_2)_\theta(r, \theta)}{r} dr \\
&= \int_0^\rho \int_0^{2\pi} \frac{(1-r^2)sin(\alpha-\theta)\varphi(\alpha)}{\pi(1+r^2-2rcos(\alpha-\theta))^2} dr d\alpha \\
&= \int_0^{2\pi} \frac{sin(\alpha-\theta)\varphi(\alpha)}{\pi} d\alpha \int_0^\rho \frac{1-r^2}{(1+r^2-2rcos(\alpha-\theta))^2} dr \\
&= \int_0^{2\pi} \frac{sin(\alpha-\theta)\varphi(\alpha)}{\pi} d\alpha \int_0^\rho \left(\frac{2-2rcos(\alpha-\theta)}{(1+r^2-2rcos(\alpha-\theta))^2} - \frac{1}{1+r^2-2rcos(\alpha-\theta)} \right) dr.
\end{aligned} \tag{3.3}$$

Due to $\int \dfrac{dx}{(a^2+x^2)^2} = \dfrac{1}{2a^2}\dfrac{x}{a^2+x^2} + \dfrac{1}{2a^3}arctan(\dfrac{x}{a})$ for the constant $a > 0$, we have

$$\begin{aligned}
&\int \frac{2-2rcos(\alpha-\theta)}{(1+r^2-2rcos(\alpha-\theta))^2}dr \\
&= \int \frac{2-2rcos(\alpha-\theta)}{\big((r-cos(\alpha-\theta))^2+sin^2(\alpha-\theta)\big)^2}dr \\
&= \int \frac{2-2\tau cos(\alpha-\theta)-2cos^2(\alpha-\theta)}{(\tau^2+sin^2(\alpha-\theta))^2}d\tau \qquad (\text{set } \tau = r - cos(\alpha-\theta)) \\
&= \frac{\tau}{\tau^2+sin^2(\alpha-\theta)} + \frac{1}{sin(\alpha-\theta)}arctan\Big(\frac{\tau}{sin(\alpha-\theta)}\Big) + \frac{cos(\alpha-\theta)}{\tau^2+sin^2(\alpha-\theta)} \\
&= \frac{r}{r^2+sin^2(\alpha-\theta)} + \frac{1}{sin(\alpha-\theta)}arctan\Big(\frac{r}{sin(\alpha-\theta)}\Big). \qquad (3.4)
\end{aligned}$$

On the other hand, one has

$$\int \frac{dr}{1+r^2-2rcos(\alpha-\theta)} = \frac{1}{sin(\alpha-\theta)}arctan\Big(\frac{r-cos(\alpha-\theta)}{sin(\alpha-\theta)}\Big). \qquad (3.5)$$

Substituting (3.4)-(3.5) into (3.3) yields

$$\begin{aligned}
\mu_1(\rho,\theta) &= \frac{\rho}{\pi}\int_0^{2\pi} \frac{sin(\alpha-\theta)\varphi(\alpha)}{1+\rho^2-2\rho cos(\alpha-\theta)}d\alpha \\
&= \frac{\rho}{\pi}\int_0^{2\pi} \frac{\partial}{\partial\alpha}\Big(\frac{ln(1+\rho^2-2\rho cos(\alpha-\theta))}{2\rho}\Big)\varphi(\alpha)d\alpha \\
&= \frac{1}{2\pi}\ln(1+\rho^2+2\rho cos\theta)(\varphi(\pi+0)-\varphi(\pi-0)) \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi} \ln(1+\rho^2-2\rho cos(\alpha-\theta))\varphi'(\alpha)d\alpha \\
&= \frac{\theta_0}{2\pi}\ln(1+\rho^2+2\rho cos\theta) - \frac{1}{2\pi}\int_0^{\frac{\pi}{4}} \ln(1+\rho^2-2\rho cos(\alpha-\theta))\varphi'(\alpha)d\alpha \\
&\quad - \frac{1}{2\pi}\int_{\frac{7\pi}{4}}^{2\pi} \ln(1+\rho^2-2\rho cos(\alpha-\theta))\varphi'(\alpha)d\alpha.
\end{aligned}$$

Here we use the fact that $\varphi'(\alpha) = 0$ for $\alpha \in [\frac{\pi}{4}, \frac{7\pi}{4}]$.

It is noted that $1+\rho^2+2\rho cos\theta = |\zeta+1|^2$ and $\varphi'(\alpha)$ is bounded for $\alpha \in [0,\frac{\pi}{4}]\cup[\frac{7\pi}{4},2\pi]$. Then we obtain

$$e^{\mu_1} \le C|\zeta+1|^{\frac{\theta_0}{\pi}} \qquad \text{and } |F(\zeta)| \le C|\zeta+1|^{\frac{\theta_0}{\pi}}.$$

As in (i), one has $Im(F(\zeta)) = 0$ on ∂D and $F(\zeta) \in C(\bar{D})$ with $\lim\limits_{\zeta\to-1,|\zeta|\le 1} F(\zeta) = 0$, and we further derive $u = v \equiv 0$ in Ω_2. Consequently, we complete the proof of Theorem 1.1 in the case of Ω_2. □

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