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A List of $1+1$ Dimensional Integrable Equations and Their Properties

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Abstract

This paper contains a list of known integrable systems. It gives their recursion-, Hamiltonian-, symplectic- and cosymplectic operator, roots of their symmetries and their scaling symmetry.

1 Introduction

In this paper we give a list of known integrable systems. This list is far from being complete, even if we restrict ourselves to

- What is known today,
- Systems with less than four components, and
- Systems referred to in the literature with a name.

Originally this list was part of the author's thesis [Wan98]. Since this thesis only has a limited distribution and the reactions to this list were very favorable, we decided to publish it in a more accessible way.

The theoretical background for our list can be found in [Olv93, Dor93]. A more encyclopedic approach to this subject is taken in [Ibr96], where lists of integrable equations with their properties, mainly special solutions, Lie symmetries and conservation laws, are given. For every equation we aim to give a table containing:

- the equation itself,
- its cosymplectic operator (Hamiltonian operator, cf. [Mag78]),
- its Hamiltonian function corresponding to the cosymplectic operator,
- its symplectic operator,
- its recursion operator (possibly resulting from the cosymplectic and symplectic operators, cf. [FF80, FF81]), or its master symmetry,

- roots of its symmetries,
- its scaling symmetry.

There are also other important properties such as Miura transformations between the systems and Lax pair (cf. [Lax68]), but these are mainly ignored here. We refer to the source where we learned about each system; no attempt has been made to track the historical origins. At this point, we do not claim any new results though we did put in an effort to compute all the subjects in the list; the decomposition of the recursion operator in symplectic and cosymplectic operators mainly relies on the theory of recursion operators (cf. [SW01a]).

We hope that this material can serve as a source of motivation for future research, since it allows one to quickly formulate or dismiss general statements.

First we give the definitions of the subjects in our lists.

Definition 1. *Given an evolution equation $u_t = K(x,t,u,\cdots,u_n)$, where $u_i = \frac{\partial^i u}{\partial x^i}$, we define*

$$\begin{array}{ll} h \textit{ is a symmetry if} & \pounds_K h = \frac{\partial h}{\partial t} + D_h[K] - D_K[h] = 0, \\ \mathfrak{H} \textit{ is a cosymplectic operator if} & \pounds_K \mathfrak{H} = \frac{\partial \mathfrak{H}}{\partial t} + D_{\mathfrak{H}}[K] - D_K\mathfrak{H} - \mathfrak{H}D_K^\star = 0, \\ \mathfrak{I} \textit{ is a symplectic operator if} & \pounds_K \mathfrak{I} = \frac{\partial \mathfrak{I}}{\partial t} + D_{\mathfrak{I}}[K] + \mathfrak{I}D_K + D_K^\star \mathfrak{I} = 0, \\ \mathfrak{R} \textit{ is a recursion operator if} & \pounds_K \mathfrak{R} = \frac{\partial \mathfrak{R}}{\partial t} + D_{\mathfrak{R}}[K] - D_K\mathfrak{R} + \mathfrak{R}D_K = 0, \end{array}$$

where $\pounds_K$ denotes the Lie derivative and $\star$ means conjugation. Moreover, for the operators, the formulae are only valid on the domain of the left hand sides of the identities. A symmetry is called a **root** *of a hierarchy if it is in the domain, but not in the image, of the recursion operator. A vectorfield $\Sigma = f(t)\partial_t + S\partial_u$ is called a* **scaling symmetry** *of the equation if $\pounds_\Sigma K = \lambda K, \lambda \in \mathbb{C}$.*

Remark 2. *Here we still use the standard definition of recursion operator in the literature. We refer the reader to [SW01b], where the term* weak recursion operator *is coined, for a discussion of the problems with this definition.*

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2 List of integrable systems and some of their properties

2.1 Burgers' equation

Reference: [Olv93, p. 315], [Oev84, p. 38];

Equation	$u_t = u_2 + uu_1$	
Hamiltonian	None	[Fuc79]
Cosymplectic	None	[FF81]
Symplectic	None	[FF81]
Recursion	$\mathfrak{R}_1 = D_x + \frac{1}{2}u + \frac{1}{2}u_1 D_x^{-1}$	[Olv77]
	$\mathfrak{R}_2 = tD_x + \frac{1}{2}(tu + x) + \frac{1}{2}(tu_1 + 1)D_x^{-1}$	[Olv93]
Root	$u_1,\ tu_1 + 1$	$\star$
Scaling	$-2t\partial_t + (xu_1 + u)\partial_u$	

$\star$ Since $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are both recursion operators of the equation, we obtain a double infinity of the symmetries, by applying $\mathfrak{R}_1$ or $\mathfrak{R}_2$ successively to u_1 and tu_1+1. Note that since $\mathfrak{R}_1\mathfrak{R}_2 = \mathfrak{R}_2\mathfrak{R}_1 + \frac{1}{2}$ and $\mathfrak{R}_1(tu_1+1) = \mathfrak{R}_2(u_1)$, if we are only interested in independent symmetries, it does not matter in which order $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are applied.

Notice that there is a difference between the root of an operator and the root of symmetries for an equation. For $\mathfrak{R}_1$, we can take tu_1+1 as a root of symmetries for Burgers' equation since $\mathfrak{R}_1 = D_x + \frac{1}{2}D_x(uD_x^{-1}\cdot)$, but it is not a root of $\mathfrak{R}_1$ since it is not its symmetry.

We refer to [Ma93] for coupled Burgers systems.

2.2 Potential Burgers' equation

Reference: [Olv93, pp. 311, 317];

Equation	$u_t = u_2 + u_1^2$	
Hamiltonian	None	[Fuc79]
Cosymplectic	None	[FF81]
Symplectic	None	[FF81]
Recursion	$\mathfrak{R}_1 = D_x + u_1$	
	$\mathfrak{R}_2 = tD_x + tu_1 + \frac{1}{2}x$	[Olv93]
Root	1	
Scaling	xu_1	[Olv93]

The same arguments hold here as Burgers' equation since $\mathfrak{R}_1\mathfrak{R}_2 = \mathfrak{R}_2\mathfrak{R}_1 + \frac{1}{2}$.

The author of [Fok80] found the 2^{nd}-order equations of the form $u_t = u_2 + f(u, u_1)$, which possess a 3^{rd}-order symmetry and obtained the following equations:

$$u_t = u_2 + \frac{f''(u)}{f'(u)}u_1^2 + \alpha f(u)u_1, \tag{2.1}$$

where α is constant and $f(u)$ is an arbitrary function, with the recursion operator $D_x + \frac{f''(u)}{f'(u)}u_1 + \frac{1}{2}\alpha f(u) + \frac{1}{2}\alpha u_1 D_x^{-1} f'(u)$.

$$u_t = u_2 + \frac{\gamma - f'(u)}{f(u)}u_1^2 + \alpha f(u), \tag{2.2}$$

where $f(u)$ is an arbitrary function and α, γ are constant, with the recursion operator $D_x + \frac{\gamma - f'(u)}{f(u)}u_1$.

Notice that the (potential) Burgers' equation is a particular case of (2.1). If we take $\alpha = 1$ and $\frac{\gamma - f'(u)}{f(u)} = 1$, i.e., $f(u) = \beta\exp(-u) + \gamma$, it leads to the nontrivial equation:

$$u_t = u_2 + u_1^2 + \beta\exp(-u) + \gamma.$$

2.3 Diffusion equation

Reference: [Oev84, p. 39];

Equation	$u_t = u^2 u_2$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Recursion	$uD_x + u^2 u_2 D_x^{-1} \frac{1}{u^2}$
Root	$u^2 u_2$
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}.$

2.4 Nonlinear diffusion equation

Reference: [Olv93, Ex. 5.10];

Equation	$u_t = D_x(\frac{u_1}{u^2})$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Recursion	$D_x^2 \frac{1}{u} D_x^{-1} = \frac{1}{u} D_x - \frac{2u_1}{u^2} - u_t D_x^{-1}$
Root	u_t
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathcal{C}.$

2.5 Korteweg–de Vries equation

Reference: [Olv93, p. 312], [Oev84, pp. 18, 67, 78, 84, 97], [Dor93, pp. 85, 151, 158, 162], [Oev90, pp. 27, 60];

Equation	$u_t = u_3 + u u_1$	[KdV95]
Hamiltonian	$\frac{u^2}{2}$	
Cosymplectic	$D_x^3 + \frac{1}{3}(uD_x + D_x u)$	
Symplectic	D_x^{-1}	
Recursion	$D_x^2 + \frac{2}{3} u + \frac{1}{3} u_1 D_x^{-1}$	[Olv77]
Root	u_1	
Scaling	$x u_1 + 2u$	

2.6 Potential Korteweg–de Vries equation

Reference: [Dor93, p. 125];

Equation	$u_t = u_3 + 3u_1^2$
Hamiltonian	$\frac{1}{2} u u_4 + 2 u u_1 u_2$
Cosymplectic	D_x^{-1}
Symplectic	$D_x^3 + 2(u_1 D_x + D_x u_1)$
Recursion	$D_x^2 + 4u_1 - 2D_x^{-1} u_2$
Root	1
Scaling	$x u_1 + u$

2.7 Modified Korteweg–de Vries equation

Reference: [Olv93, Ex. 5.11], [Oev84, p. 97], [Oev90, pp. 29, 60];

Equation	$u_t = u_3 + u^2 u_1$
Hamiltonian	$\frac{u^2}{2}$
Cosymplectic	$D_x^3 + \frac{2}{3} D_x u D_x^{-1} u D_x$
Symplectic	D_x^{-1}
Recursion	$D_x^2 + \frac{2}{3}u^2 + \frac{2}{3}u_1 D_x^{-1} u$ [Olv77]
Root	u_1
Scaling	$xu_1 + u$

2.8 Potential modified Korteweg–de Vries equation

Reference: [Olv93, Ex.5.11];

Equation	$u_t = u_3 + \frac{1}{3}u_1^3$
Hamiltonian	$\frac{1}{2}uu_4 + \frac{1}{4}uu_1^2 u_2$
Cosymplectic	D_x^{-1}
Symplectic	$D_x^3 + \frac{2}{3} D_x u_1 D_x^{-1} u_1 D_x$
Recursion	$D_x^2 + \frac{2}{3}u_1^2 - \frac{2}{3}u_1 D_x^{-1} u_2$
Root	u_1
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathbb{C}$.

In the paper [Fok80], the author found the 3rd-order equations, not involving 2nd-order derivatives, i.e., of the form $u_t = u_3 + f(u, u_1)$, which possess a 5th-order symmetry and obtained the following equations:

$$u_t = u_3 + \alpha u_1^3 + \beta u_1^2 + \gamma u_1, \tag{2.3}$$

where α, β, γ are constant, with the recursion operator $D_x^2 + 2\alpha u_1^2 + \frac{4}{3}\beta u_1 - \frac{2}{3}(3\alpha u_1 + \beta) D_x^{-1} u_2 + \gamma$.

$$u_t = u_3 + \alpha u_1^3 + f(u) u_1, \tag{2.4}$$

where $f(u)$ satisfies $f''' + 8\alpha f' = 0$, with the recursion operator $D_x^2 + 2\alpha u_1^2 + \frac{2}{3} f(u) - \frac{1}{3} u_1 D_x^{-1}(6\alpha u_2 - f')$.

pKdV ($\alpha = 0$) and pmKdV ($\beta = 0$) are particular cases of (2.3). pmKdV ($f(u) = 0$), KdV ($\alpha = 0$ and $f(u) = u$) and mKdV ($\alpha = 0$ and $f(u) = u^2$) are particular cases of (2.4) including Calogero–Degasperis–Fokas equation [CD81]:

$$u_t = u_3 - \frac{1}{8}u_1^3 + (a\exp(u) + b\exp(-u) + c)u_1.$$

2.9 Cylindrical Korteweg–de Vries equation

Reference: [OF84, ZC86, Cho87a];

Equation	$u_t = u_3 + uu_1 - \frac{u}{2t}$
Hamiltonian	None
Cosymplectic	$D_x^3 + \frac{1}{3}(uD_x + D_x u) + \frac{1}{6t}(xD_x + D_x x)$
Symplectic	tD_x^{-1}
Recursion	$t(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}) + \frac{1}{3}x + \frac{1}{6}D_x^{-1}$
Root	$\sqrt{t}(\frac{u_1}{3} + \frac{1}{6t})$
Scaling	$-3t\partial_t + (2u + xu_1)\partial_u$

Consider the generalized Korteweg–de Vries equation

$$u_t + u_3 + 6uu_1 + 6f(t)u - x(f'_t + 12f^2) = 0,$$

where f is an arbitrary function of t. It possesses recursion operator:

$$\mathfrak{R} = \frac{1}{g(t)}(D_x^2 + 4(u - xf(t)) + 2(u_1 - f(t))D_x^{-1})$$

with the root $\frac{1}{\sqrt{g}}(u_1 - f)$, where $g(t) = \exp(-\int 12f dt)$, cf. [Cho87a] (the author also studied generalized mKdV in the same way [Cho87b]).

If we take $f(t) = \frac{1}{12t}$ and then do transformation $\tilde{u} = \frac{1}{6}u$ and $\tilde{t} = -t$, we get the cylindrical Korteweg–de Vries equation.

2.10 Ibragimov–Shabat equation

Reference: [IŠ80, Cal87];

Equation	$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Root	u_1
Scaling	$xu_1 + \frac{1}{2}u$
Master Symmetries	$xu_t + \frac{3}{2}u_2 + 5u_1u^2 + \frac{1}{2}u^5$

No recursion operator seems to be known for this equation.

We should mention that this equation possesses infinitely many symmetries [IŠ80], but only one local conserved density u^2 [Kap82]. The transformation $u = \sqrt{\frac{w_1}{2w}}$ [Cal87] transforms it into $w_t = w_3 - \frac{3}{4}\frac{w_2^2}{w_1}$ and the master symmetry becomes $xw_t + \frac{1}{2}w_2$, which is the master symmetry for this new equation. Notice that the new equation has a recursion operator $\mathfrak{R} = D_x^2 - \frac{w_2}{w_1}D_x + \frac{w_3}{2w_1} - \frac{w_2^2}{4w_1^2} - D_x^{-1}(\frac{w_4}{2w_1} - \frac{w_2w_3}{w_1^2} + \frac{w_2^3}{2w_1^3}) = \left(D_x - D_x^{-1}\frac{w_2}{2w_1}D_x\right)^2$ with the root w_1.

2.11 Harry Dym equation

Reference: [Olv93, Ex.5.15], [Oev84, p. 107];

Equation	$u_t = u^3u_3$	
Hamiltonian	$-\frac{1}{u}$	
Cosymplectic	$u^3D_x^3u^3$	
Symplectic	$\frac{1}{u^2}D_x^{-1}\frac{1}{u^2}$	
Recursion	$u^3D_x^3uD_x^{-1}\frac{1}{u^2}$	[LLS+83]
	$= u^2D_x^2 - uu_1D_x + uu_2 + u^3u_3D_x^{-1}\frac{1}{u^2}$	
Root	u^3u_3	
Scaling	$\alpha xu_1 + \beta u, \alpha, \beta \in \mathbb{C}$.	

Sometimes the equation is written as $u_t = D_x^3(\frac{1}{\sqrt{u}})$, cf. [Dor93, p. 85].

2.12 Schwarzian KdV equation

Reference: [Dor93, p. 121];

Equation	$u_t = u_3 - \frac{3}{2}\frac{u_2^2}{u_1}$
Hamiltonian	$\frac{u_2^2}{2u_1^2}$
Cosymplectic	$2(\frac{1}{u_1^2}D_x + D_x\frac{1}{u_1^2})^{-1}$
Symplectic	$\frac{1}{2}(\frac{1}{u_1^2}D_x^3 + D_x^3\frac{1}{u_1^2}) + (\frac{u_3}{u_1^3} - \frac{3u_2^2}{u_1^4})D_x + D_x(\frac{u_3}{u_1^3} - \frac{3u_2^2}{u_1^4})$
Recursion	$\begin{cases} D_x^2 - \frac{2u_2}{u_1}D_x + (\frac{u_3}{u_1} - \frac{u_2^2}{u_1^2}) - u_1 D_x^{-1}\xi, \\ \xi = \frac{3u_2^3}{u_1^4} - \frac{4u_2u_3}{u_1^3} + \frac{u_4}{u_1^2} \end{cases}$
Root	u_1
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathbb{C}.$

2.13 Cavalcante–Tenenblat equation

Reference: [CT88];

Equation	$u_t = D_x^2(u_1^{-\frac{1}{2}}) + u_1^{\frac{3}{2}}$
Hamiltonian	$-2\sqrt{u_1}$
Cosymplectic	$D_x - u_1 D_x^{-1} u_1$
Symplectic	$u_1^{-\frac{1}{2}} D_x u_1^{-\frac{1}{2}} - \frac{1}{4} u_1^{-\frac{3}{2}} u_2 D_x^{-1} u_1^{-\frac{3}{2}} u_2$
Recursion	$\frac{1}{u_1}D_x^2 - \frac{3u_2}{2u_1^2}D_x - \frac{u_3}{2u_1^2} + \frac{3u_2^2}{4u_1^3} - u_1 + \frac{u_t}{2}D_x^{-1}u_1^{-\frac{3}{2}}u_2$
Root	u_t
Scaling	xu_1

2.14 Liouville equation

Reference: [Dor93, pp. 134, 164];

Equation	$u_{xt} = \exp(u)$
Hamiltonian	$\exp(u)$ $\star$
Cosymplectic	D_x^{-1}
Symplectic	$D_x^3 - D_x u_1 D_x^{-1} u_1 D_x$
Recursion	$D_x^2 - u_1^2 + u_1 D_x^{-1} u_2$
Root	u_1
Scaling	$\alpha x u_1 + \beta u, \alpha, \beta \in \mathbb{C}.$

$\star$ Actually, the equation is treated as an evolution equation $u_t = D_x^{-1}\exp(u)$.

The **Sinh–Gordon equation** $u_{xt} = \sinh u$ has exactly the same geometric structure, cf. [AC91].

We refer to [ZS01] for recent developments in integrable hyperbolic equations of Liouville type.

2.15 Sine–Gordon equation

Reference: [Olv93, Ex.5.12], [Dor93, pp. 133, 163];

Equation	$u_{xt}=\sin u$	Oevel[16]
Hamiltonian	$-\cos u$	$\star$
Cosymplectic	D_x^{-1}	
Symplectic	$D_x^3+D_xu_1D_x^{-1}u_1D_x$	
Recursion	$D_x^2+u_1^2-u_1D_x^{-1}u_2$	[Olv77]
Root	u_1	
Scaling	$\alpha xu_1+\beta u, \alpha,\beta\in\mathbb{C}$.	

$\star$ As we mentioned for the Liouville equation, this equation is also treated as an evolution equation $u_t=D_x^{-1}\sin u$.

2.16 Klein–Gordon equations

Reference: [AC91, p. 366], [Kon87, p. 41], [FG80];

Equation	$u_{xt}=\alpha\exp(-2u)+\beta\exp(u)$
Hamiltonian	$-\frac{\alpha}{2}\exp(-2u)+\beta\exp(u)$
Cosymplectic	D_x^{-1}
Symplectic	$\mathfrak{I}$
Recursion	$\mathfrak{R}$
Root	$u_1, u_5+5u_2u_3-5u_1^2u_3-5u_1u_2^2+u_1^5$
Scaling	xu_1

$$
\begin{aligned}
\mathfrak{I} \;=\;& D_x^7+3(u_2D_x^5+D_x^5u_2)-3(u_1^2D_x^5+D_x^5u_1^2)-8(u_4D_x^3+D_x^3u_4)\\
&+10(u_1u_3D_x^3+D_x^3u_1u_3)+\frac{29}{2}(u_2^2D_x^3+D_x^3u_2^2)-3(u_1^2u_2D_x^3+D_x^3u_1^2u_2)\\
&+\frac{9}{2}(u_1^4D_x^3+D_x^3u_1^4)+5(u_6D_x+D_xu_6)-6(u_1u_5D_x+D_xu_1u_5)\\
&-25(u_2u_4D_x+D_xu_2u_4)+3(u_1^2u_4D_x+D_xu_1^2u_4)-21(u_3^2D_x+D_xu_3^2)\\
&+8(u_1u_2u_3D_x+D_xu_1u_2u_3)-8(u_1^3u_3D_x+D_xu_1^3u_3)\\
&+6(u_2^3D_x+D_xu_2^3)-44(u_1^2u_2^2D_x+D_xu_1^2u_2^2)-2(u_1^6D_x+D_xu_1^6)\\
&+2u_2D_x^{-1}(u_6+5u_2u_4+5u_3^2-5u_1^2u_4-20u_1u_2u_3-5u_2^3+5u_1^4u_2)\\
&+2(u_6+5u_2u_4+5u_3^2-5u_1^2u_4-20u_1u_2u_3-5u_2^3+5u_1^4u_2)D_x^{-1}u_2\\
\mathfrak{R} \;=\;& D_x^6+6(u_2-u_1^2)D_x^4+9(u_3-2u_1u_2)D_x^3\\
&+(5u_4-22u_1u_3-13u_2^2-6u_1^2u_2+9u_1^4)D_x^2\\
&+(u_5-8u_1u_4-15u_2u_3-3u_1^2u_3-6u_1u_2^2+18u_1^3u_2)D_x\\
&-4u_1u_5+20u_1^3u_3-20u_1u_2u_3+20u_1^2u_2^2-4u_1^6\\
&+2u_1D_x^{-1}(u_6+5u_2u_4+5u_3^2-5u_1^2u_4-20u_1u_2u_3-5u_2^3+5u_1^4u_2)\\
&+2(u_5+5u_2u_3-5u_1^2u_3-5u_1u_2^2+u_1^5)D_x^{-1}u_2
\end{aligned}
$$

It shares its recursion operator [Bil93] with the Potential Kupershmidt equation, i.e., $u_t=u_5+5u_2u_3-5u_1^2u_3-5u_1u_2^2+u_1^5$ (equation (4.2.7) in [MSS91]).

Klein–Gordon equations $u_{xt}=f(u)$ possess a nontrivial symmetry if and only if $f(u)=\alpha\exp(-\lambda u)+\beta\exp(\lambda u)$ or $f(u)=\alpha\exp(-2\lambda u)+\beta\exp(\lambda u)$, cf. [ZS79].

2.17 Kupershmidt equation

Reference: [MSS91, Eq. (4.2.6)], [FG80, Bil93];

Equation	$u_t = u_5 + 5u_1u_3 + 5u_2^2 - 5u^2u_3 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1$
Hamiltonian	$\frac{u_2^2}{2} - \frac{5u_1^3}{6} + \frac{5u^2u_1^2}{2} + \frac{u^6}{6}$
Cosymplectic	D_x
Symplectic	$\mathfrak{I}$
Recursion	$\mathfrak{R}$
Root	u_1, u_t
Scaling	$xu_1 + u$

$$
\begin{aligned}
\mathfrak{I} &= D_x^5 + 3(u_1D_x^3 + D_x^3u_1) - 3(u^2D_x^3 + D_x^3u^2) - 3(u_1u^2D_x + D_xu_1u^2) \\
&\quad + \frac{5}{2}(u_1^2D_x + D_xu_1^2) - 2(u_3D_x + D_xu_3) + \frac{9}{2}(u^4D_x + D_xu^4) \\
&\quad - 2(uu_2D_x + D_xuu_2) - 2(u_4 - 5u^2u_2 - 5uu_1^2 + 5u_1u_2 + u^5)D_x^{-1}u \\
&\quad - 2uD_x^{-1}(u_4 - 5u^2u_2 - 5uu_1^2 + 5u_1u_2 + u^5) \\
\mathfrak{R} &= D_x^6 + 6u_1D_x^4 - 6u^2D_x^4 - 30uu_1D_x^3 + 15u_2D_x^3 + 9u^4D_x^2 - 6u^2u_1D_x^2 \\
&\quad - 40uu_2D_x^2 - 31u_1^2D_x^2 + 14u_3D_x^2 - 9u^2u_2D_x + 54u^3u_1D_x - 18uu_1^2D_x \\
&\quad - 30uu_3D_x - 63u_1u_2D_x + 6u_4D_x - 4u^6 + 38u^3u_2 + 74u^2u_1^2 \\
&\quad - 3u^2u_3 - 12uu_4 - 38uu_1u_2 + u_5 - 6u_1^3 - 23u_1u_3 - 15u_2^2 \\
&\quad - 2u_tD_x^{-1}u - 2u_1D_x^{-1}\left(u_4 - 5u^2u_2 - 5uu_1^2 + 5u_1u_2 + u^5\right)
\end{aligned}
$$

2.18 Sawada–Kotera equation

Reference: [SK74, CDG76, FO82, FOW87, Bil93], [Oev84, p. 105], [Oev90, p. 30], [MSS91, Eq. (4.2.2)];

Equation	$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1$
Hamiltonian	$\frac{u^3}{6} - \frac{u_1^2}{2}$
Cosymplectic	$D_x\left(D_x + 2(D_x^{-1}u + uD_x^{-1})\right)D_x$
Symplectic	$\left(D_x + D_x^{-1}u\right)D_x\left(D_x + uD_x^{-1}\right)$
Recursion	$\mathfrak{R}$
Root	u_1, u_t
Scaling	$xu_1 + 2u$

$$
\begin{aligned}
\mathfrak{R} &= D_x^6 + 6uD_x^4 + 9u_1D_x^3 + 9u^2D_x^2 + 11u_2D_x^2 + 10u_3D_x + 21uu_1D_x \\
&\quad + 4u^3 + 16uu_2 + 6u_1^2 + 5u_4 + u_1D_x^{-1}(2u_2 + u^2) + u_tD_x^{-1}
\end{aligned}
$$

2.19 Potential Sawada–Kotera equation

Reference: [MSS91, Eq. (4.2.4)], [Bil93];

Equation	$u_t = u_5 + 5u_1u_3 + \frac{5}{3}u_1^3$
Hamiltonian	$\frac{u_2^2}{2} - \frac{u_1^3}{6}$
Cosymplectic	$D_x + 2(u_1D_x^{-1} + D_x^{-1}u_1)$
Symplectic	$\left(D_x + u_1D_x^{-1}\right)D_x^3\left(D_x + D_x^{-1}u_1\right)$
Recursion	$\mathfrak{R}$
Root	$u_1, 1$
Scaling	$xu_1 + u$

$$
\mathfrak{R} = D_x^6 + 6u_1D_x^4 + 3u_2D_x^3 + 8u_3D_x^2 + 9u_1^2D_x^2 + 2u_4D_x + 3u_2u_1D_x
$$

$$+3u_5+13u_3u_1+3u_2^2+4u_1^3-2u_1D_x^{-1}\left(u_4+u_2u_1\right)$$
$$-2D_x^{-1}\left(u_6+3u_4u_1+6u_3u_2+2u_2u_1^2\right)$$

2.20 Kaup–Kupershmidt equation

Reference: [Kau80, FO82, FOW87, Bil93],[MSS91, Eq. (4.2.3)];

Equation	$u_t=u_5+5uu_3+\frac{25}{2}u_1u_2+5u^2u_1$
Hamiltonian	$\frac{2u^3}{3}-\frac{u_1^2}{2}$
Cosymplectic	$D_x\left(D_x+\frac{1}{2}(uD_x^{-1}+D_x^{-1}u)\right)D_x$
Symplectic	$D_x^3+\frac{3}{2}(uD_x+D_xu)+D_x^2uD_x^{-1}+D_x^{-1}uD_x^2$ $+2(u^2D_x^{-1}+D_x^{-1}u^2)$
Recursion	$\mathfrak{R}$
Root	u_1, u_t
Scaling	xu_1+2u

$$\mathfrak{R} = D_x^6+6uD_x^4+18u_1D_x^3+9u^2D_x^2+\frac{49}{2}u_2D_x^2+30uu_1D_x+\frac{35}{2}u_3D_x$$
$$+4u^3+\frac{41}{2}uu_2+\frac{69}{4}u_1^2+\frac{13}{2}u_4+\frac{1}{2}u_1D_x^{-1}(u_2+2u^2)+u_tD_x^{-1}$$

2.21 Potential Kaup–Kupershmidt equation

Reference: [MSS91, Eq. (4.2.5)], [Bil93];

Equation	$u_t=u_5+10u_1u_3+\frac{15}{2}u_2^2+\frac{20}{3}u_1^3$
Hamiltonian	$\frac{u_2^2}{2}-\frac{4u_1^3}{3}$
Cosymplectic	$D_x+u_1D_x^{-1}+D_x^{-1}u_1$
Symplectic	$D_x^5+5(u_1D_x^3+D_x^3u_1)-3(u_3D_x+D_xu_3)$ $+8(u_1^2D_x+D_xu_1^2)$
Recursion	$\mathfrak{R}$
Root	$u_1, 1$
Scaling	xu_1+u

$$\mathfrak{R} = D_x^6+12u_1D_x^4+24u_2D_x^3+25u_3D_x^2+36u_1^2D_x^2+10u_4D_x+48u_1u_2D_x$$
$$+3u_5+21u_2^2+34u_1u_3+32u_1^3-2u_1D_x^{-1}\left(u_4+8u_1u_2\right)$$
$$-D_x^{-1}\left(u_6+12u_1u_4+24u_2u_3+32u_1^2u_2\right)$$

2.22 Diffusion system

Reference: [Oev84, p. 41];

Equation	$\begin{cases} u_t=u_2+v^2 \\ v_t=v_2 \end{cases}$
Hamiltonian	None
Cosymplectic	None
Symplectic	None
Recursion	$\begin{pmatrix} D_x & vD_x^{-1} \\ 0 & D_x \end{pmatrix}$
Root	$\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$

Scaling	$\begin{pmatrix} xu_1+2u \\ xv_1+2v \end{pmatrix} + \alpha \begin{pmatrix} 2u \\ v \end{pmatrix}, \alpha \in \mathbb{C}$

2.23 Dispersiveless Long Wave system

Reference: [AC91, Gök98];

Equation	$\begin{cases} u_t = u_1 v + u v_1 \\ v_t = u_1 + v v_1 \end{cases}$
Hamiltonian	$\frac{u^2+uv^2}{2}$
Cosymplectic	$\begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} 2D_x^{-1} & vD_x^{-1} \\ D_x^{-1}v & uD_x^{-1}+D_x^{-1}u \end{pmatrix}$
Recursion	$\begin{pmatrix} v & 2u+u_1D_x^{-1} \\ 2 & v+v_1D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \end{pmatrix} + \alpha \begin{pmatrix} 2u \\ v \end{pmatrix}, \alpha \in \mathbb{C}$

2.24 Sine–Gordon equation in the laboratory coordinates

Reference: [CLL87];

Equation	$\begin{cases} u_t = v \\ v_t = u_2 - \sin(u) \end{cases}$
Hamiltonian	$\frac{1}{2}(u_1^2+v^2) - \cos(u)$
Cosymplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} -\mathfrak{R}_{21} & -\mathfrak{R}_{22} \\ \mathfrak{R}_{11} & \mathfrak{R}_{12} \end{pmatrix}$
Recursion	$\mathfrak{R} = \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	None

$$\begin{aligned}
\mathfrak{R}_{11} &= 4D_x^2 - 2\cos(u) + (u_1+v)^2 - (u_1+v)D_x^{-1}(u_2+v_1-\sin(u)), \\
\mathfrak{R}_{12} &= 4D_x + (u_1+v)D_x^{-1}(u_1+v), \\
\mathfrak{R}_{21} &= 4D_x^3 + (u_1+v)^2 D_x - 4\cos(u)D_x + 2u_1\sin(u) + (u_2+v_1)(u_1+v) \\
&- (u_2+v_1-\sin(u))D_x^{-1}(u_2+v_1-\sin(u)), \\
\mathfrak{R}_{22} &= 4D_x^2 + (u_1+v)^2 - 2\cos(u) + (u_2+v_1-\sin(u))D_x^{-1}(u_1+v).
\end{aligned}$$

2.25 AKNS equation

Reference: [Oev84, p. 100];

Equation	$\begin{cases} u_t = -u_2 + 2u^2 v \\ v_t = v_2 - 2v^2 u \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1 - vu_1)$
Cosymplectic	$\begin{pmatrix} 2uD_x^{-1}u & D_x - 2uD_x^{-1}v \\ D_x - 2vD_x^{-1}u & 2vD_x^{-1}v \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} -D_x + 2uD_x^{-1}v & 2uD_x^{-1}u \\ -2vD_x^{-1}v & D_x - 2vD_x^{-1}u \end{pmatrix}$
Root	$\begin{pmatrix} -u \\ v \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + u \\ xv_1 + v \end{pmatrix}$

2.26 Nonlinear Schrödinger equation

Reference: [Oev84, p. 102], [Dor93, p. 135], [Oev90, pp. 31, 61];

Equation	$\begin{cases} u_t = v_2 \mp v(u^2 + v^2) \\ v_t = -u_2 \pm u(u^2 + v^2) \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1 - vu_1)$
Cosymplectic	$\begin{pmatrix} D_x \mp 2vD_x^{-1}v & \pm 2vD_x^{-1}u \\ \pm 2uD_x^{-1}v & D_x \mp 2uD_x^{-1}u \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} \mp 2vD_x^{-1}u & D_x \mp 2vD_x^{-1}v \\ -D_x \pm 2uD_x^{-1}u & \pm 2uD_x^{-1}v \end{pmatrix}$
Root	$\begin{pmatrix} -v \\ u \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + u \\ xv_1 + v \end{pmatrix}$

The system can be written as $iq_t = q_2 \mp q^2 q^\star$, where $i^2 = -1$, cf. [AC91].

2.27 Derivative Schrödinger system

Reference: [Oev84, p. 103];

Equation	$\begin{cases} u_t = -v_2 - (u^2 + v^2)u_1 \\ v_t = u_2 - (u^2 + v^2)v_1 \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1 - vu_1)$
Cosymplectic	$\begin{pmatrix} -D_x & \frac{u^2+v^2}{2} \\ -\frac{u^2+v^2}{2} & -D_x \end{pmatrix} - \begin{pmatrix} v \\ -u \end{pmatrix} D_x^{-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
	$- \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} v \\ -u \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Recursion	$\begin{pmatrix} -\frac{u^2+v^2}{2} & -D_x \\ D_x & -\frac{u^2+v^2}{2} \end{pmatrix} + \begin{pmatrix} v \\ -u \end{pmatrix} D_x^{-1} \begin{pmatrix} v_1 \\ -u_1 \end{pmatrix}^\dagger - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} u \\ v \end{pmatrix}^\dagger$
Root	$\begin{pmatrix} v \\ -u \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + \frac{u}{2} \\ xv_1 + \frac{v}{2} \end{pmatrix}$

2.28 Modified derivative Schrödinger system

Reference: [WHV95];

Equation	$\begin{cases} u_t = D_x(u^3 + uv^2 + \beta u - v_1) \\ v_t = D_x(vu^2 + v^3 + u_1) \end{cases}$
Hamiltonian	$\frac{1}{2}(u^2+v^2)$
Cosymplectic	$\begin{pmatrix} \beta D_x + 2uD_x u & -D_x^2 + 2vD_x u \\ D_x^2 + 2uD_x v & 2vD_x v \end{pmatrix} -2 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} \beta + 2u^2 & -D_x + 2uv \\ D_x + 2uv & 2v^2 \end{pmatrix} +2 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_x^{-1} \begin{pmatrix} u \\ v \end{pmatrix}^\dagger$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 + \frac{u}{2} \\ xv_1 + \frac{v}{2} \\ x\beta_x + \beta \end{pmatrix}.$

2.29 Boussinesq system

Reference: [Olv93, p. 459];

Equation	$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{3}u_3 + \frac{8}{3}uu_1 \end{cases}$
Hamiltonian	$\frac{1}{2}v$
Cosymplectic	$\begin{pmatrix} D_x^3 + 2uD_x + u_1 & 3vD_x + 2v_1 \\ 3vD_x + v_1 & \mathfrak{H}_{22} \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & D_x^{-1} \\ D_x^{-1} & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} 3v + 2v_1 D_x^{-1} & D_x^2 + 2u + u_1 D_x^{-1} \\ \mathfrak{R}_{21} & 3v + v_1 D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$

Scaling $\begin{pmatrix} xu_1+2u \\ xv_1+3v \end{pmatrix}$

$$
\begin{aligned}
\mathfrak{H}_{22} &= \frac{1}{3}D_x^5+\frac{5}{3}(uD_x^3+D_x^3u)-(u_2D_x+D_xu_2)+\frac{16}{3}uD_xu \\
\mathfrak{R}_{21} &= \frac{1}{3}D_x^4+\frac{10}{3}uD_x^2+5u_1D_x+3u_2+\frac{16}{3}u^2+2v_tD_x^{-1}
\end{aligned}
$$

2.30 Modified Boussinesq system

Reference: [FG81];

Equation	$\begin{cases} u_t=3v_2+6uv_1+6u_1v \\ v_t=-u_2-6vv_1+2uu_1 \end{cases}$
Hamiltonian	$\frac{1}{2}(uv_1-u_1v-2v^3+2vu^2)$
Cosymplectic	$\begin{pmatrix} 3D_x & 0 \\ 0 & D_x \end{pmatrix}$
Symplectic	$\begin{pmatrix} \frac{1}{3}D_x^{-1}\mathfrak{R}_{11} & \frac{1}{3}D_x^{-1}\mathfrak{R}_{12} \\ D_x^{-1}\mathfrak{R}_{21} & D_x^{-1}\mathfrak{R}_{22} \end{pmatrix}$
Recursion	$\begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1+u \\ xv_1+v \end{pmatrix}$

$$
\begin{aligned}
\mathfrak{R}_{11} &= 6vD_x^2+9v_1D_x+3v_2-12uv_1-24u^2v-2u_tD_x^{-1}u-6u_1D_x^{-1}(2uv+v_1), \\
\mathfrak{R}_{12} &= 3D_x^3+6uD_x^2+9u_1D_x-3u^2D_x-9v^2D_x+3u_2-6u^3-36vv_1 \\
&- 18uv^2-6u_tD_x^{-1}v+6u_1D_x^{-1}(u_1-u^2+3v^2), \\
\mathfrak{R}_{21} &= -D_x^3+2uD_x^2+u^2D_x+3u_1D_x+3v^2D_x+u_2-6uv^2-2u^3+4uu_1 \\
&- 2v_tD_x^{-1}u-6v_1D_x^{-1}(v_1+2uv), \\
\mathfrak{R}_{22} &= -6vD_x^2-9v_1D_x-12u^2v+12u_1v-3v_2+36v^3 \\
&- 6v_tD_x^{-1}v+6v_1D_x^{-1}(u_1-u^2+3v^2).
\end{aligned}
$$

2.31 Landau–Lifshitz system

Reference: [vBK91];

Equation	$\begin{cases} u_t=-\sin(u)v_2-2\cos(u)u_1v_1+(J_1-J_2)\sin(u)\cos(v)\sin(v) \\ v_t=\frac{u_2}{\sin(u)}-\cos(u)v_1^2+\cos(u)(J_1\cos^2(v)+J_2\sin^2(v)-J_3) \end{cases}$
Hamiltonian	$\frac{1}{2}(\sin^2(u)(J_1\cos^2(v)+J_2\sin^2(v)-J_3)+J_3-u_1^2-\sin^2(u)v_1^2)$
Cosymplectic	$\frac{1}{\sin(u)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Symplectic	$\sin(u)\begin{pmatrix} \mathfrak{R}_{21} & \mathfrak{R}_{22} \\ -\mathfrak{R}_{11} & -\mathfrak{R}_{12} \end{pmatrix}$

Recursion	$\begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \\ xJ_x + 2J \end{pmatrix}$, where $J = (J_1, J_2, J_3)$.

where

$$\begin{aligned}
\mathfrak{R}_{11} &= -D_x^2 - 2\sin^2(u)v_1^2 - u_1^2 + v_1^2 - (J_1 - J_2)\sin^2(u)\sin^2(v) \\
&\quad +(J_1 - J_3)\sin^2(u) + J_3 - J_2 + u_t D_x^{-1} \cdot (\sin(u)v_1) - u_1 D_x^{-1} \cdot S_1 \\
\mathfrak{R}_{12} &= 2\cos(u)\sin(u)v_1 D_x + \cos(u)\sin(u)v_2 - 3\sin^2(u)u_1 v_1 + 2u_1 v_1 \\
&\quad +u_t D_x^{-1} \cdot (-\sin(u)u_1) - u_1 D_x^{-1} \cdot S_2 \\
\mathfrak{R}_{21} &= -2\cos(u)v_1 D_x - \cos(u)v_2 + u_1 v_1 \\
&\quad +v_t D_x^{-1} \cdot (\sin(u)v_1) - v_1 D_x^{-1} \cdot S_1 \\
\mathfrak{R}_{22} &= -D_x^2 - 2\cos(u)u_1 D_x - \cos(u)u_2 - (J_1 - J_2)\sin(u)\sin^2(v) \\
&\quad -2\sin^2(u)v_1^2 + v_1^2 + (J_1 - J_3)\sin^2(u) + J_3 - J_2 \\
&\quad +v_t D_x^{-1} \cdot (-\sin(u)u_1) - v_1 D_x^{-1} \cdot S_2 \\
S_1 &= (J_1 - J_2)\cos(u)\sin(u)\sin^2(v) - (J_1 - J_3)\cos(u)\sin(u) \\
&\quad +\cos(u)\sin(u)v_1^2 - u_2, \\
S_2 &= (J_1 - J_2)\cos(v)\sin^2(u)\sin(v) - 2\cos(u)\sin(u)u_1 v_1 - \sin^2(u)v_2.
\end{aligned}$$

2.32 Wadati–Konno–Ichikawa system

Reference: [WKI79, BPT83], [Kon87, p. 88];

Equation	$\begin{cases} u_t = D_x^2(\frac{u}{\sqrt{1+uv}}) \\ v_t = -D_x^2(\frac{v}{\sqrt{1+uv}}) \end{cases}$
Hamiltonian	$2\sqrt{1+uv}$
Cosymplectic	$\begin{pmatrix} 0 & D_x^2 \\ -D_x^2 & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & \frac{2}{1+uv} \\ -\frac{2}{1+uv} & 0 \end{pmatrix} - \begin{pmatrix} \frac{v}{\sqrt{1+uv}} \\ \frac{u}{\sqrt{1+uv}} \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} \frac{v_1}{(1+uv)^{\frac{3}{2}}} \\ -\frac{u_1}{(1+uv)^{\frac{3}{2}}} \end{pmatrix}^\dagger$ $- \begin{pmatrix} \frac{v_1}{(1+uv)^{\frac{3}{2}}} \\ -\frac{u_1}{(1+uv)^{\frac{3}{2}}} \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} \frac{v}{\sqrt{1+uv}} \\ \frac{u}{\sqrt{1+uv}} \end{pmatrix}^\dagger$
Root	$\begin{pmatrix} u_t \\ v_t \end{pmatrix}, \begin{pmatrix} D_x^2(\frac{u_1}{(1+uv)^{\frac{3}{2}}}) \\ D_x^2(\frac{v_1}{(1+uv)^{\frac{3}{2}}}) \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \end{pmatrix}$

2.33 Hirota–Satsuma system

Reference: [HS81, Fuc82], [Kon87, p. 207], [Oev90, pp. 32, 61], [Oev84, pp. 31, 84];

Equation $\begin{cases} u_t = \frac{1}{2}u_3 + 3uu_1 - 6vv_1 \\ v_t = -v_3 - 3uv_1 \end{cases}$

Hamiltonian $\frac{1}{2}u^2 - v^2$

Cosymplectic $\begin{pmatrix} \frac{1}{2}D_x^3 + uD_x + D_xu & vD_x + D_xv \\ vD_x + D_xv & \frac{1}{2}D_x^3 + uD_x + D_xu \end{pmatrix}$

Symplectic $\begin{pmatrix} \frac{1}{2}D_x + uD_x^{-1} + D_x^{-1}u & -2D_x^{-1}v \\ -2vD_x^{-1} & -2D_x \end{pmatrix}$

Recursion $\mathfrak{R}$

Root $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}$

Scaling $\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

$$\begin{aligned} \mathfrak{R}(u,v) &= \begin{pmatrix} \frac{1}{2}D_x^3 + D_x \cdot u + uD_x & D_x \cdot v + vD_x \\ D_x \cdot v + vD_x & \frac{1}{2}D_x^3 + D_x \cdot u + uD_x \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} \frac{1}{2}D_x + D_x^{-1} \cdot u + uD_x^{-1} & -2D_x^{-1} \cdot v \\ -2vD_x^{-1} & -2D_x \end{pmatrix} \\ &\sim \begin{pmatrix} \frac{1}{2}u_3 + 3uu_1 - 6vv_1 \\ -v_3 - 3uv_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 1, & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} u, & -2v \end{pmatrix} \end{aligned}$$

2.34 The Symmetrically–coupled Korteweg–de Vries system

Reference: [Fuc82];

Equation $\begin{cases} u_t = u_3 + v_3 + 6uu_1 + 4uv_1 + 2u_1v \\ v_t = u_3 + v_3 + 6vv_1 + 4vu_1 + 2v_1u \end{cases}$

Hamiltonian $\frac{1}{2}(u+v)^2$

Cosymplectic $\begin{pmatrix} D_x^3 + 2(uD_x + D_xu) & 0 \\ 0 & D_x^3 + 2(vD_x + D_xv) \end{pmatrix}$

Symplectic $\begin{pmatrix} D_x^{-1} & D_x^{-1} \\ D_x^{-1} & D_x^{-1} \end{pmatrix}$

Recursion $\begin{pmatrix} D_x^2 + 4u + 2u_1D_x^{-1} & D_x^2 + 4u + 2u_1D_x^{-1} \\ D_x^2 + 4v + 2v_1D_x^{-1} & D_x^2 + 4v + 2v_1D_x^{-1} \end{pmatrix}$

Root $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$

Scaling $\begin{pmatrix} xu_1 + 2u \\ xv_1 + 2v \end{pmatrix}$

2.35 The Complexly–coupled Korteweg–de Vries system

Reference: [Fuc82];

Equation $\begin{cases} u_t = u_3 + 6uu_1 + 6vv_1 \\ v_t = v_3 + 6uv_1 + 6vu_1 \end{cases}$

Hamiltonian $\frac{1}{2}(u^2 + v^2)$

Cosymplectic $\begin{pmatrix} D_x^3 + 2(uD_x + D_xu) & 2D_xv + 2vD_x \\ 2D_xv + 2vD_x & D_x^3 + 2(uD_x + D_xu) \end{pmatrix}$

Symplectic	$\begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} D_x^2+4u+2u_1D_x^{-1} & 4v+2v_1D_x^{-1} \\ 4v+2v_1D_x^{-1} & D_x^2+4u+2u_1D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1+2u \\ xv_1+2v \end{pmatrix}$

2.36 Coupled nonlinear wave system (Ito system)

Reference: [Ito82, AF87], [Dor93, p. 94];

Equation	$\begin{cases} u_t = u_3+6uu_1+2vv_1 \\ v_t = 2uv_1+2u_1v \end{cases}$
Hamiltonian	$\frac{u^2+v^2}{2}$
Cosymplectic	$\begin{pmatrix} D_x^3+4uD_x+2u_1 & 2vD_x \\ 2vD_x+2v_1 & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix}$
Recursion	$\begin{pmatrix} D_x^2+4u+2u_1D_x^{-1} & 2v \\ 2v+2v_1D_x^{-1} & 0 \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1+2u \\ xv_1+2v \end{pmatrix}$

2.37 Drinfel'd–Sokolov system

Reference: [Gök98];

Equation	$\begin{cases} u_t = 3vv_1 \\ v_t = 2v_3+u_1v+2uv_1 \end{cases}$
Hamiltonian	$\frac{v^2}{2}$
Cosymplectic	$\begin{pmatrix} 2D_x^3+2uD_x+u_1 & 2vD_x+v_1 \\ 2vD_x+v_1 & 2D_x^3+2uD_x+u_1 \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^3+\frac{5}{2}(uD_x+D_xu) & -15vD_x+\frac{15}{2}v_1 \\ -15vD_x-\frac{45}{2}v_1 & -27D_x^3-\frac{27}{2}(uD_x+D_xu) \end{pmatrix}$ $-(\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}^\dagger + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger)$ $-18 \begin{pmatrix} 0 \\ v \end{pmatrix}^\dagger D_x^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix}^\dagger,$
Root	$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}; \begin{pmatrix} u_t \\ v_t \end{pmatrix}; \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1+2u \\ xv_1+2v \end{pmatrix}$

$$\eta_1 = -u_2-2u^2+3v^2,$$

$$\eta_2 = 9v_2 + 6uv,$$
$$h_1 = -2u_5 - 10uu_3 - 25u_1u_2 + 30vv_3 + 45v_1v_2 - 10u^2u_1 + 15v^2u_1 + 30uvv_1,$$
$$h_2 = 18v_5 + 10vu_3 + 35u_2v_1 + 45u_1v_2 + 30uv_3 + 10uu_1v + 10u^2v_1 + 15v^2v_1.$$

2.38 Benney system

Reference: [Ben73, AF87];

Equation	$\begin{cases} u_t = vv_1 + 2D_x(uw) \\ v_t = 2u_1 + D_x(vw) \\ w_t = 2v_1 + 2ww_1 \end{cases}$
Hamiltonian	$uw + \frac{v^2}{2}$
Cosymplectic	$\begin{pmatrix} uD_x + D_xu & vD_x & wD_x \\ D_xv & 0 & 2D_x \\ D_xw & 2D_x & 0 \end{pmatrix}$
Symplectic	$\begin{pmatrix} 0 & 0 & D_x^{-1} \\ 0 & D_x^{-1} & 0 \\ D_x^{-1} & 0 & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} w & v & 2u + u_1D_x^{-1} \\ 2 & 0 & v + v_1D_x^{-1} \\ 0 & 2 & w + w_1D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$
Scaling	$\begin{pmatrix} xu_1 \\ xv_1 \\ xw_1 \end{pmatrix} + \alpha \begin{pmatrix} 3u \\ 2v \\ w \end{pmatrix}; \alpha \in \mathbb{C}$

2.39 Dispersive water wave system

Reference: [AF87];

Equation	$\begin{cases} u_t = D_x(uw) \\ v_t = -v_2 + 2D_x(vw) + uu_1 \\ w_t = w_2 - 2v_1 + 2ww_1 \end{cases}$
Hamiltonian	$vw + \frac{u^2}{2}$
Cosymplectic	$\begin{pmatrix} 0 & D_xu & 0 \\ uD_x & vD_x + D_xv & -D_x^2 + wD_x \\ 0 & D_x^2 + D_xw & -2D_x \end{pmatrix}$
Symplectic	$\begin{pmatrix} D_x^{-1} & 0 & 0 \\ 0 & 0 & D_x^{-1} \\ 0 & D_x^{-1} & 0 \end{pmatrix}$
Recursion	$\begin{pmatrix} 0 & 0 & u + u_1D_x^{-1} \\ u & -D_x + w & 2v + v_1D_x^{-1} \\ 0 & -2 & D_x + w + w_1D_x^{-1} \end{pmatrix}$
Root	$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$

Scaling $\begin{pmatrix} xu_1 + \frac{3}{2}u \\ xv_1 + 2v \\ xw_1 + w \end{pmatrix}$

If $u = 0$, this system reduces to the Broer–Kaup system studied in [Gök98].

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