Faculty of Sciences and Mathematics
University of Niš

Available at:
www.pmf.ni.ac.yu/sajt/publikacije/publikacije_pocetna.html

Filomat **20:2** (2006), 101–105

A LITTLEWOOD-PALEY TYPE INEQUALITY FOR HARMONIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{R}^N$

OLIVERA DJORDJEVIĆ

Abstract. It is proved the following: If u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$, and $0 < p < 1$, then there holds the inequality

$$\sup_{0<r<1} \int_{\partial B} |u(ry)|^p \, d\sigma \leq |u(0)|^p + C_{p,N} \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x) .$$

In the case $p > (N-2)/(N-1)$, this was proved by Stević [17].

Let $\mathbb{R}^N$ $(N \geq 2)$ denote the N-dimensional Euclidean space. In [17], Stević proved that if u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$, and $\frac{N-2}{N-1} \leq p < 1$, then there holds the inequality

$$\sup_{0<r<1} M_p^p(r,u) \leq C_1 |u(0)|^p + C_2 \int_B (1-|x|)^{p-1} |\nabla u(x)|^p \, dV(x) . \tag{1}$$

Here dV denotes the Lebesgue measure in $\mathbb{R}^N$ normalized so that $V(B) = 1$, and as usual

$$M_p^p(r,u) = \int_{\partial B} |u(ry)|^p \, d\sigma ,$$

where $d\sigma$ is the normalized surface measure on the sphere ∂B. It is the aim of this note to remove the strange condition $(N-2)/(N-1) \leq p < 1$. This condition appears in [17] because the proof in the paper is based on the fact, due Stein and Weiss [16, 15], that $|\nabla u|^p$ is subharmonic for $p \geq (N-2)/(N-1)$. Our result is slightly stronger than (1):

[1]Received: October 2, 2006

2000 *Mathematics Subject Classification.* 31B05.

Key words and phrases. Littlewood-Paley inequalities, harmonic functions.

Supported by MNZŽS, Project 144010.

Theorem 1. *If u is a function harmonic in B, and $0 < p < 1$, then there holds the inequality*

$$\sup_{0<r<1} M_p^p(r,u) \leq |u(0)|^p + C\int_B (1-|x|)^{p-1}|\nabla u(x)|^p\, dV(x)\,, \tag{2}$$

where C is a constant depending only on p and N.

In the case $N = 2$, this theorem was proved by Flett [2]. Inequality (2) holds for $1 < p < 2$ as well, while if $p > 2$, then there holds the reverse inequality; these inequalities are due to Littlewood and Paley [6]. Elementary proofs of the Littewood-Paley inequalities are given in [12] and [7, 14] ($p > 2$).

Observe that if $u > 0$ in B, and $0 < p < 1$, then (2) is completely trivial because then function u^p is superharmonic and therefore

$$\sup_{0<r<1} M_p^p(r,u) \leq |u(0)|^p\,.$$

Thus (2) shows in particular how much $|u|^p$ is far from being superharmonic.

Our proof of Theorem 1 is based on a fundamental result of Hardy and Littlewood [3] and Fefferman and Stein [1] on subharmonic behavior of $|u|^p$. We state this result in the following way.

Lemma 1. *If $U \geq 0$ is a function subharmonic in $B(a,2\varepsilon)$ $(a \in \mathbb{R}^N, \varepsilon > 0)$, then there holds the inequality*

$$\sup_{x\in B(a,\varepsilon)} U(x)^p \leq C\varepsilon^{-N}\int_{B(a,2\varepsilon)} U^p dV, \qquad 0 < \varepsilon < 1\,, \tag{3}$$

where C depends only on p, N.

Here $B(a,r)$ denotes the ball of radius r centered at a. For simple proofs of Lemma 1 we refer to [9, 13], and for generalizations to various classes of functions, we refer to [4, 5, 8, 10, 11]. From Lemma 1 we shall deduce the following crucial fact:

Lemma 2. *Let $r_j = 1 - 2^{-j}$ for $j \geq 0$, and $r_{-1} = 0$. If $0 < p < 1$ and u is harmonic in B, then there holds inequality*

$$M_p^p(r_{j+1},u) - M_p^p(r_j,u) \leq C\int_{r_{j-1}\leq|x|\leq r_{j+2}} (1-|x|)^{p-1}|\nabla u(x)|^p\, dV(x)\,, \quad j \geq 0\,,$$

where C depends only on p and N.

Proof. We start from the inequality

$$M_p^p(r_{j+1},u) - M_p^p(r_j,u) \leq \int_S |u(r_{j+1}y) - u(r_jy)|^p\, d\sigma(y)\,. \tag{4}$$

By Lagrange's theorem,
(5)
$$|u(r_{j+1}y)-u(r_jy)| \le (r_{j+1}-r_j) \sup_{r_j<r<r_{j+1}} |\nabla u(ry)| \le 2^{-j} \sup_{r_j<r<r_{j+1}} |\nabla u(ry)| \,.$$

Hence, by Lemma 1 with $U = |\nabla u|, a = a_j = (r_j + r_{j+1})y/2$ and $\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2}$,

$$(6) \qquad |u(r_{j+1}y) - u(r_jy)|^p \le C2^{-jp}2^{jN} \int_{B(a_j,2^{-j-1})} |\nabla u(x)|^p \, dV(x) \,.$$

On the other hand, simple calculation shows that $|x-a_jy| \le 2^{-j-1}$ implies

$$2^{-j-2} \le 1 - |x| \,, \qquad |x - y| \le 2^{-j+1} \,.$$

Hence

$$2^{-j}2^{jN} \le 2^{N+2}P(x, y), \qquad \text{for}\, x \in B(a_j, 2^{-j-1}) \,,$$

where P denotes the Poisson kernel,

$$(7) \qquad P(x, y) = \frac{1 - |x|^2}{|x - y|^N} \,.$$

From this and (6) we get

$$(8) \quad |u(r_{j+1}y) - u(r_jy)|^p \le C2^{-j(p-1)} \int_{r_{j-1}\le|x|\le r_{j+2}} P(x, y)|\nabla u(x)|^p \, dV(x) \,,$$

where we have used the inclusion

$$\{x : |x - a_j| \le 2^{-j-1}\} \subset \{x : r_{j-1} \le |x| \le r_{j+2} \,.\}$$

Now we integrate (8) over ∂B and use the formula

$$\int_S P(x, y) \, d\sigma(y) = 1$$

to get

$$\begin{aligned} \int_S |u(r_{j+1}y) - u(r_jy)|^p \, d\sigma(y) &\le C2^{-j(p-1)} \int_{r_{j-1}\le|x|\le r_{j+2}} |\nabla u(x)|^p \, dV(x) \\ &\le C \int_{r_{j-1}\le|x|\le r_{j+2}} (1 - |x|)^{p-1}|\nabla u(x)|^p \, dV(x) \,. \end{aligned}$$

Combining this with (4) we get the desired result. □

Proof of Theorem 1. Let $n \ge 1$. By Lemma 2, we have

$$\begin{aligned} M_p^p(r_n,u)-|u(0)|^p &= M_p^p(r_n,u)-M_p^p(r_0,u)\\ &= \sum_{j=0}^{n-1} M_p^p(r_{j+1},u)-M_p^p(r_j,u)\\ &\le C\sum_{j=0}^{n-1}\int_{r_{j-1}\le|x|\le r_{j+2}}(1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\\ &\le 3C\int_{|x|\le r_{n+1}}(1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\\ &\le 3C\int_B(1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\,. \end{aligned}$$

This proves the inequality

$$M_p^p(r,u)\le|u(0)|^p+C\int_B(1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\,, \tag{9}$$

for $r=r_n$. If $r\in(0,1)$ is arbitrary, we choose n so that $r_n\le r\le r_{n+1}$. Then we have

$$|u(ry)-u(r_ny)|\le 2^{-n}\sup_{r_n<r<r_{n+1}}|\nabla u(ry)|\,.$$

Hence, by the proof of Lemma 2,

$$\begin{aligned} M_p^p(r,u)-M_p^p(r_n,u) &\le C\int_{r_{n-1}\le|x|\le r_{n+2}}(1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\\ &\le C\int_B(1-|x|)^{p-1}|\nabla u(x)|^p\,dV(x)\,. \end{aligned}$$

This completes the proof.

Acknowledgement. The author wishes to thank Professor Pavlović for introducing into the subject and for valuable comments.

References

[1] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137 - 193.

[2] T. M. Flett, *Mean values of power series*, Pacific J. Math. 25(1968), 463-494.

[3] G.H. Hardy and J.E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math. **167** (1932), 405-423.

[4] M. Jevtić, M. Pavlović, *M-Besov p-classes and Hankel operators in the Bergman spaces on the unit ball*, Arch. Math. 61(1993), 367-376.

[5] M. Jevtić, M. Pavlović, *On subharmonic behaviour of functions and their tangential derivatives on the unit ball in* $\mathbf{C}^n$, Indian J. Pure Appl. Math. 30(1999), no. 4, 407-418.

[6] J.E. Littlewood and R.E.A.C. Paley, *Theorems on Fourier series and power series. II*, Proc. Lond. Math. Soc. 42(1936), 52-89.

[7] D.H. Luecking, *A new proof of an inequality of Littlewood and Paley*, Proc. Amer. Math. Soc. 103(1988), 887-893.
[8] M. Pavlović, *Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball*, Indag. Math. (N.S.) 2(1991), no. 1, 89-98.
[9] M. Pavlović, *On subharmonic behaviour and oscillation of functions on balls in* $\mathbf{R}^n$, Publ. Inst. Math. (Beograd) (N.S.) **55(69)** (1994), 18-22.
[10] M. Pavlović, *Subharmonic behaviour of smooth functions*, Mat. Vesnik **48** (1996), no. 1-2, 15-21.
[11] M. Pavlović, *Decompositions of Lp and Hardy spaces of polyharmonic functions*, J. Math. Anal. Appl. **216**(1997), 499-509.
[12] M. Pavlović, *A LittlewoodPaley theorem for subharmonic functions with subharmonic Laplacian*, Publ. Inst. Math. (Belgrade) 68(82)(2000), 77-82.
[13] M. Pavlović, *Introduction to function spaces on the disk*, Posebna izdanja 20, Matematički Institut SANU, 2004.
[14] M. Pavlović, *A short proof of an inequality of Littlewood and Paley*, Proc. Am. Math. Soc. to appear.
[15] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
[16] E. M. Stein, G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of* H^p *- spaces*, Acta Math. 103 (1960) 25-62.
[17] S. Stević, *A Littlewood-Paley type inequality*, Bull. Braz. Math. Soc. (N.S.) **34**(2003), no. 2, 211-217.

Fakultet organizacionih nauka, Jove Ilića 154, Belgrade, Serbia
E-mail address: `oliveradj@fon.bg.ac.yu`