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A LITTLEWOOD-RICHARDSON FILTRATION AT ROOTS OF 1 FOR QUANTUM DEFORMATIONS OF SKEW SCHUR MODULES

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Abstract. It is well known that the classical skew Schur module is isomorphic to a direct sum of (non-skew) Schur modules, the multiplicities being given by the Littlewood-Richardson rule. We define a multiparameter quantum deformation of the classical skew Schur module, and show that up to a filtration, it still has a Littlewood-Richardson decomposition. The ground ring can be any commutative ring, and q is allowed to be a root of 1.

In [H-H] a quantum deformation of the classical skew Schur module is defined in the "Jimbo case", over every commutative ring R, and for every choice of a unit q in R. Let P be a multiplicatively antisymmetric matrix with entries which are integer powers of q. Denote by $R[GL(q, \mathrm{P})]$ the multiparameter quantum matrix bialgebra associated to q and P. Slightly generalizing [H-H], we define a multiparameter quantum deformation, $L_{\lambda/\mu}\mathrm{V}_{\mathrm{P}}$, of the same classical module.

In case R is a field and q is not a root of 1, arguments like those given in [H-H, Sections 7 and 8] can be used to show that $L_{\lambda/\mu}\mathrm{V}_{\mathrm{P}}$ is completely reducible, and its decomposition into irreducibles is $\sum_{\nu}\gamma(\lambda/\mu;\nu)L_{\nu}\mathrm{V}_{\mathrm{P}}$, where the coefficients $\gamma(\lambda/\mu;\nu)$ are the usual Littlewood-Richardson coefficients. The goal of this paper is to construct a filtration of $L_{\lambda/\nu}\mathrm{V}_{\mathrm{P}}$ as an $R[GL(q, \mathrm{P})]$-comodule, valid when R is *any* ring and q is allowed to be a root of 1, such that the associated graded object is precisely $\sum_{\nu}\gamma(\lambda/\mu;\nu)L_{\nu}\mathrm{V}_{\mathrm{P}}$.

The construction of the filtration closely follows the strategy employed by the first author in some previous work (cf., e.g., [B]), and is based on an easy multiparameter quantum deformation of some results contained in [A-B-W].

The paper is divided into two parts. The first one provides the necessary background, the main definitions, and some fundamental properties. The second part contains the actual construction of the desired filtration.

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1. The ingredients.

1.1. Let $N>1$ be a positive integer. Choose a unit q in a commutative ring R; fix a matrix $\mathrm{P}=(p_{ij})_{i,j=1}^{N}$ where the p_{ij}'s are non-zero elements of R with the property

$$p_{ij}p_{ji}=p_{ii}=1\,, \qquad \forall i,j=1,\dots,N\,.$$

Consider the free R-module $\mathrm{V_P}$ with basis $\{u_1,\dots,u_N\}$ and define an automorphism $\beta_{\mathrm{V_P}}$ on $\mathrm{V_P}\otimes\mathrm{V_P}$ by the following rule:

$$\beta_{\mathrm{V_P}}(u_i\otimes u_j)=\begin{cases} u_i\otimes u_i & \text{if } i=j \\ qp_{ji}u_j\otimes u_i & \text{if } i<j \\ qp_{ji}u_j\otimes u_i+(1-q^2)u_i\otimes u_j & \text{if } i>j\,. \end{cases}$$

Then $(\mathrm{V_P},\beta_{\mathrm{V_P}})$ is a YB pair in the sense of [H-H]. Moreover it satisfies Iwahori's quadratic equation

$$(\mathrm{id}_{\mathrm{V_P}\otimes\mathrm{V_P}}-\beta_{\mathrm{V_P}})\circ(\mathrm{id}_{\mathrm{V_P}\otimes\mathrm{V_P}}+q^{-2}\beta_{\mathrm{V_P}})=0\,,$$

as one can easily verify.

1.2. The *multiparameter quantum matrix bialgebra* $SE(q,\mathrm{P})$ (cf. [S]) is the algebra generated by the N^2 elements x_{ij} $(i,j=1,\dots,N)$ with relations (for $i<j$ and $k<m$):

$$x_{ik}x_{im}=qp_{mk}x_{im}x_{ik}\,, \qquad x_{ik}x_{jk}=qp_{ij}x_{jk}x_{ik}\,, \qquad p_{mk}x_{im}x_{jk}=p_{ij}x_{jk}x_{im}\,,$$

$$p_{km}x_{ik}x_{jm}-p_{ij}x_{jm}x_{ik}=(q-q^{-1})x_{im}x_{jk}\,.$$

The coalgebra structure is given by the following comultiplication and counity:

$$\Delta(x_{ij})=\sum_{k=1}^{N}x_{ik}\otimes x_{kj}\,, \qquad \varepsilon(x_{ij})=\delta_{ij}\,.$$

1.3. For $1\le i_1<\cdots<i_k\le N$, $1\le j_1<\cdots<j_k\le N$, define

$$\det_{q,\mathrm{P}}(i_1,\dots,i_k;j_1,\dots,j_k)=\sum_{\sigma\in\mathscr{S}_k}\Bigg(\prod_{r<t,\sigma(r)>\sigma(t)}(-qp_{i_{\sigma(t)}j_{\sigma(r)}})\Bigg)x_{i_{\sigma(1)}j_1}\cdots x_{i_{\sigma(k)}j_k}\,.$$

The group-like element $\det_{q,\mathrm{P}}(1,\dots,N;1,\dots,N)$ of $SE(q,\mathrm{P})$ is called the *multiparameter quantum determinant* of $SE(q,\mathrm{P})$, and is denoted by $\det_{q,\mathrm{P}}$. The *multiparameter quantum coordinate algebra of the general linear group* is the Hopf algebra $R[GL(q,\mathrm{P})]=SE(q,\mathrm{P})[\det_{q,\mathrm{P}}^{-1}]$ (see [H]). The antipode of x_{ij} is given by

$$S(x_{ij})=(-q)^{i-j}p_{i+1\,i}p_{i+2\,i}\cdots p_{Ni}p_{j\,j+1}p_{j\,j+2}\cdots p_{jN}$$
$$\times\det_{q,\mathrm{P}}(1,\dots,\hat{j},\dots,N;1,\dots,\hat{i},\dots,N)\det_{q,\mathrm{P}}^{-1}\,.$$

There is a natural $R[GL(q,\mathrm{P})]$-comodule structure on $\mathrm{V_P}$ given by

$$u_j\mapsto\sum_i u_i\otimes x_{ij}\,.$$

Consider the Hopf ideal $\mathscr{B}_{\mathrm{P}}^{+}$ of $R[GL(q,\mathrm{P})]$ generated by all x_{ij} with $i>j$ and put

$$R[B^{+}(q,\mathrm{P})]=R[GL(q,\mathrm{P})]/\mathscr{B}_{\mathrm{P}}^{+}\ .$$

The relations between the generators in $R[B^{+}(q,\mathrm{P})]$ are those given in (1.2) when we put $x_{ij}=0$ for $i>j$. In particular x_{ii} commutes with x_{jj} for all i,j and $\det_{q,\mathrm{P}}=\prod_i x_{ii}$.

1.4. Henceforth the p_{ij}'s will be integer powers of q. More precisely (cf. [R]) we shall take

$$p_{ij}=q^{2(u_{ji}-u_{j-1i}-u_{ji-1}+u_{j-1i-1})}\,,$$

where $U=(u_{ij})_{i,j=1}^{N-1}$ is an appropriate alternating integer matrix. In this way we shall be in the situation of [C-V], where in fact an integer form of the multiparameter quantum function algebra is constructed.

Notice that every $R[GL(q,\mathrm{P})]$-comodule will become in a natural way a module over an integer form of a suitable multiparameter deformation of the Drinfeld-Jimbo quantum group (see [C-V], [H]).

From now on, we shall also skip all indices q,P in our notation as long as no ambiguity is likely.

1.5. Let us begin reviewing some results of [H-H], freely adopting the notation in there. Starting from the YB pair $(\mathrm{V},\beta_{\mathrm{V}})$, we can construct some graded YB bialgebras. First of all the tensor algebra $T\mathrm{V}=\bigoplus_{i\geq 0}\mathrm{V}^{\otimes i}=\bigoplus_{i\geq 0}T_i\mathrm{V}$ with YB operator $T(\beta_{\mathrm{V}})=\bigoplus_{i,j\geq 0}\beta_{\mathrm{V}}(\chi_{ij})$, where χ_{ij} is the following element of $\mathscr{S}_{i+j}$:

$$\chi_{ij}=\begin{pmatrix} 1 & 2 & \dots & i & i+1 & i+2 & \dots & i+j\\ j+1 & j+2 & \dots & j+i & 1 & 2 & \dots & j\end{pmatrix}.$$

We recall that if $\sigma=\sigma_{i_1}\dots\sigma_{i_r}$ is a reduced expression for an element $\sigma\in\mathscr{S}_k$, then it is well defined on $T_k\mathrm{V}$ the operator $\beta_{\mathrm{V}}(\sigma)=\beta_{\mathrm{V}}(\sigma_{i_1})\circ\cdots\circ\beta_{\mathrm{V}}(\sigma_{i_r})$, $\beta_{\mathrm{V}}(\sigma_j)$ being the map $\mathrm{id}_{\mathrm{V}}^{\otimes j-1}\otimes\beta_{\mathrm{V}}\otimes\mathrm{id}_{\mathrm{V}}^{\otimes k-j-1}$. In order to describe the coproduct of $T\mathrm{V}$, for every sequence $\alpha=(\alpha_1,\dots,\alpha_s)$ of nonnegative integers such that $\sum_i\alpha_i=k$, define $\Delta_{T\mathrm{V}}^{\alpha}$ to be the composite map $T\mathrm{V}\to T_s\mathrm{V}\to T_\alpha\mathrm{V}$ of the s-th iteration of $\Delta_{T\mathrm{V}}$ and the projection onto $T_\alpha\mathrm{V}=\mathrm{V}^{\otimes\alpha_1}\otimes\cdots\otimes\mathrm{V}^{\otimes\alpha_s}$. Put

$$\mathscr{S}^{\alpha}=\Big\{\sigma\in\mathscr{S}_k\,\Big|\,\sigma(1)<\cdots<\sigma(\alpha_1),\ \sigma(\alpha_1+1)<\cdots<\sigma(\alpha_1+\alpha_2),\dots,$$
$$\sigma\Big(\sum_{i=1}^{s-1}\alpha_i+1\Big)<\cdots<\sigma\Big(\sum_{i=1}^{s}\alpha_i\Big)\Big\}.$$

Then $\Delta_{T\mathrm{V}}^{\alpha}=\sum_{\sigma\in\mathscr{S}^{\alpha}}\beta_{\mathrm{V}}(\sigma^{-1})$.

1.6. A key role in what follows is played by the symmetric and the exterior algebras $S\mathrm{V}$ and $\Lambda\mathrm{V}$ of the YB pair $(\mathrm{V},\beta_{\mathrm{V}})$. The algebra $S\mathrm{V}$ is generated by $u_1,\dots,u_N$ with relations

$$u_iu_j=p_{ji}qu_ju_i\qquad(i<j)\,,$$

while ΛV is the algebra on the same generators with relations

$$u_i \wedge u_i = 0\,, \qquad p_{ij} q u_i \wedge u_j + u_j \wedge u_i = 0 \quad (i<j)\,.$$

So for every sequence $i=(i_1, \dots, i_k)$ of elements in $[1, N]$ we have

$$u_{i_1} \wedge \cdots \wedge u_{i_k} = \begin{cases} 0 & \text{if there are repetitions in } i \\ \left(\prod_{r<t, \sigma(r)>\sigma(t)} (-q)^{-1} p_{i_{\sigma(r)} i_{\sigma(t)}} \right) u_{i_{\sigma(1)}} \wedge \cdots \wedge u_{i_{\sigma(k)}} & \text{if } i_1 < \cdots < i_k \text{ and } \sigma \in \mathcal{S}_k\,. \end{cases}$$

The R-modules S_rV and Λ_rV are free with bases, respectively,

$$\{u_{j_1} \cdots u_{j_r} \mid 1 \le j_1 \le \cdots \le j_r \le N\}\,, \qquad \{u_{j_1} \wedge \cdots \wedge u_{j_r} \mid 1 \le j_1 < \cdots < j_r \le N\}\,.$$

1.7. Put $\gamma_V = -q^{-2}\beta_V$. Then the two YB operators β_V and γ_V satisfy conditions (4.9) and (4.10) in [H-H], that is, (V, β_V, γ_V) is a YB triple. It follows that SV and ΛV are graded YB bialgebras (Theorem 4.10 in [H-H]). Moreover there exist YB operators φ_{SV} and ψ_{SV} on SV, and $\varphi_{\Lambda V}$ and $\psi_{\Lambda V}$ on ΛV, for which (SV, φ_{SV}, ψ_{SV}) and (ΛV, $\varphi_{\Lambda V}$, $\psi_{\Lambda V}$) are YB algebra triples. In particular, the operator $\varphi_{\Lambda V}$ is defined by the relation $\varphi_{\Lambda V} \circ (p \otimes p) = (p \otimes p) \circ T(-\beta_V)$ where p denotes the projection from TV onto ΛV. The multiplicative structure on ΛV is given by the fusion procedure, namely, by

$$m_{T_i(\Lambda V)} = m_{\Lambda V}^{\otimes i} \circ \varphi_{\Lambda V}(\omega_i)\,, \qquad \omega_i = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & i+3 & \dots & 2i \\ 1 & 3 & \dots & 2i-1 & 2 & 4 & \dots & 2i \end{pmatrix}.$$

Finally note that TV, SV and ΛV are $R[GL]$-equivariant as YB bialgebras with YB algebra triples, that is, all the structure morphisms (including the YB operators) are homomorphisms of $R[GL]$-comodules.

1.8. A translation into our setting of Lemma 5.3 in [H-H] gives the following very useful equality.

LEMMA. *For any $k \ge 0$ and any sequence $(i_1, \dots, i_k)$ with $1 \le i_1 < \cdots < i_k \le N$ we have*:

$$\Delta_{\Lambda V}^{(1,\dots,1)}(u_{i_1} \wedge \cdots \wedge u_{i_k}) = \sum_{\sigma \in \mathcal{S}_k} \left(\prod_{r<t, \sigma(r)>\sigma(t)} (-q) p_{i_{\sigma(r)} i_{\sigma(t)}} \right) u_{i_{\sigma(1)}} \otimes \cdots \otimes u_{i_{\sigma(k)}}\,.$$

1.9. We are now ready to introduce our multiparameter quantum deformations of the classical skew Schur modules. In fact all definitions are results in Section 6 of [H-H], stated for the "Jimbo case", still hold in our situation. For all but Lemma 6.12 can be deduced from formal properties of graded YB bialgebras which are also equipped with a structure of YB algebra triple. The proof of Lemma 6.12, which depends on the definition of the particular YB operator, can be easily modified for our purposes.

Given a skew partition λ/μ with $l(\lambda/\mu)=s$ and $\lambda_1=t$, denote by $d_{\lambda/\mu}(\mathrm{V})$ the *Schur map*, that is, the composite map

$$\Lambda_{\lambda/\mu}\mathrm{V}=\Lambda_{\lambda_1-\mu_1}\mathrm{V}\otimes\cdots\otimes\Lambda_{\lambda_t-\mu_t}\mathrm{V}$$

$$\xrightarrow{\Delta_{\Lambda\mathrm{V}}^{(1^{\lambda_1-\mu_1})}\otimes\cdots\otimes\Delta_{\Lambda\mathrm{V}}^{(1^{\lambda_t-\mu_t})}} T_{\lambda/\mu}\mathrm{V}=T_{\lambda_1-\mu_1}\mathrm{V}\otimes\cdots\otimes T_{\lambda_t-\mu_t}\mathrm{V}\xrightarrow{(-q^{-2}\beta_{\mathrm{V}})(\chi_{\lambda/\mu})}$$

$$T_{\tilde\lambda/\tilde\mu}\mathrm{V}=T_{\tilde\lambda_1-\tilde\mu_1}\mathrm{V}\otimes\cdots\otimes T_{\tilde\lambda_s-\tilde\mu_s}\mathrm{V}\xrightarrow{p\otimes\cdots\otimes p} S_{\tilde\lambda/\tilde\mu}\mathrm{V}=S_{\tilde\lambda_1-\tilde\mu_1}\mathrm{V}\otimes\cdots\otimes S_{\tilde\lambda_s-\tilde\mu_s}\mathrm{V}\,,$$

where, as usual, $\tilde\lambda$ denotes the dual partition of λ, and $\chi_{\lambda/\mu}$ is the permutation defined in Section 6 of [H-H]. We illustrate such a permutation by the following example:

$$\lambda=(5,4,2)\qquad \mu=(2,1)\qquad \chi_{\lambda/\mu}=\begin{pmatrix}1&2&3&4&5&6&7&8\\4&6&8&2&5&7&1&3\end{pmatrix}$$

$$\begin{matrix}\bullet&\bullet&1&2&3\\ \bullet&4&5&6&\\ 7&8&&&\end{matrix}\quad\xrightarrow{\chi_{\lambda/\mu}}\quad\begin{matrix}\bullet&\bullet&4&6&8\\ \bullet&2&5&7&\\ 1&3&&&\end{matrix}\quad .$$

The image of the Schur map is the *Schur module of* V *with respect to the skew partition* λ/μ, denoted by $L_{\lambda/\mu}\mathrm{V}$. It is an $R[GL]$-comodule, with coaction induced by the following coaction of SE on $T_k\mathrm{V}$:

$$u_{j_1}\otimes\cdots\otimes u_{j_k}\mapsto\sum_{i_1,\dots,i_k}(u_{i_1}\otimes\cdots\otimes u_{i_k})\otimes x_{i_1j_1}\cdots x_{i_kj_k}\,.$$

1.10. The principal properties of $L_{\lambda/\mu}\mathrm{V}$ are summarized in the following theorem, which one proves along the lines of Theorem 6.19 and Corollary 6.20 in [H-H].

THEOREM. *Let λ/μ be a skew partition with $l(\lambda/\mu)=s$. Then:*

(i) *$L_{\lambda/\mu}\mathrm{V}$ is a free R-module, and for any $\sigma\in\mathscr{S}_N$, a free basis is the set*

$$L_{\lambda/\mu}\mathrm{Y}(\sigma)=\{d_{\lambda/\mu}(\mathrm{V})(\xi_{\mathrm{S}})\,|\,\mathrm{S}\in\mathrm{St}_{\lambda/\mu}\mathrm{Y}(\sigma)\}\,.$$

Here $\mathrm{St}_{\lambda/\mu}\mathrm{Y}(\sigma)$ denotes the set of all standard tableaux in the alphabet $\mathrm{Y}(\sigma)=\{u_{\sigma(1)}<\cdots<u_{\sigma(N)}\}$, and

$$\xi_{\mathrm{S}}=\mathrm{S}(1,\mu_1+1)\wedge\cdots\wedge\mathrm{S}(1,\lambda_1)\otimes\cdots\otimes\mathrm{S}(s,\mu_s+1)\wedge\cdots\wedge\mathrm{S}(s,\lambda_s)\in\Lambda_{\lambda/\mu}\mathrm{V}\,.$$

(ii) *Let R' be a commutative ring and let $f: R\to R'$ be a homomorphism of commutative rings. Then we have an isomorphism of $R'[GL]$-comodules*

$$L_{\lambda/\mu}(R'\otimes_R\mathrm{V})\simeq R'\otimes_R L_{\lambda/\mu}\mathrm{V}\,.$$

As a consequence of (ii), it will not be restrictive for us to take $R=\boldsymbol{Z}[\mathscr{Q},\mathscr{Q}^{-1}]$, where $\mathscr{Q}$ stands for an indeterminate.

1.11. We recall that an element of $\mathrm{Tab}_{\lambda/\mu}\mathrm{Y}(\sigma)$, the set of all tableaux of shape

λ/μ with elements in $Y(\sigma)$, is said to be *row-standard* if its rows are strictly increasing, and *column-standard* if its columns are non-decreasing. A tableau is said to be *standard* if it is both row- and column-standard. Let $\mathrm{Row}_{\lambda/\mu}Y(\sigma)$ denote the set of row-standard tableaux of shape λ/μ and with elements in $Y(\sigma)$. For every $S \in \mathrm{Row}_{\lambda/\mu}Y(\sigma)$, the element $d_{\lambda/\mu}(\xi_S)$ can be expressed as a linear combination of basis elements. The algorithm, call it $\mathscr{R}_\sigma$, which does this is based on a descending induction with respect to a pseudo-order defined in $\mathrm{Tab}_{\lambda/\mu}Y(\sigma)$. Let S and S′ be elements in $\mathrm{Tab}_{\lambda/\mu}Y(\sigma)$. We say that $S \leq_\sigma S'$ if for any p, q

$$\#\{(i,j) \in \Delta_{\lambda/\mu} \mid i \leq p,\ S(i,j) \in \{u_{\sigma(1)}, \ldots, u_{\sigma(q)}\}\} \\ \geq \#\{(i,j) \in \Delta_{\lambda/\mu} \mid i \leq p,\ S'(i,j) \in \{u_{\sigma(1)}, \ldots, u_{\sigma(q)}\}\}.$$

The key steps of $\mathscr{R}_\sigma$ are the following:

1. Choose two adjacent lines in S where there is a violation of column-standardness; we are in the situation of Proposition (1.12) below, and we can use Corollary (1.13). We get certain S_i's such that $S_i <_\sigma S$ for every i.

2. Apply induction to each S_i.

$\mathscr{R}_\sigma$ is also called the "straightening law with respect to the ordering $u_{\sigma(1)} < \cdots < u_{\sigma(N)}$".

1.12. Proposition. *Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be partitions with $\lambda \supset \mu$. Define $\gamma = \lambda - \mu$ and take a, b nonnegative integers with $a + b < \lambda_2 - \mu_1$. Then the image of the composite map*

$$\overline{\square}_{(a,b)} : \Lambda_a V \otimes \Lambda_{\gamma_1 - a + \gamma_2 - b} V \otimes \Lambda_b V \xrightarrow{1 \otimes \Delta \otimes 1} \Lambda_a V \otimes \Lambda_{\gamma_1 - a} V \otimes \Lambda_{\gamma_2 - b} V \otimes \Lambda_b V$$

$$\xrightarrow{m \otimes m} \Lambda_{\gamma_1} V \otimes \Lambda_{\gamma_2} V = \Lambda_{\lambda/\mu} V$$

is contained in $\mathrm{Im}(\square_{\lambda/\mu})$, *where* $\square_{\lambda/\mu} = \sum_{v=0}^{\lambda_2 - \mu_1 - 1} \square_v$, *and* $\square_v$ *is given by*

$$\Lambda_{\gamma_1 + \gamma_2 - v} V \otimes \Lambda_v V \xrightarrow{\Delta \otimes 1} \Lambda_{\gamma_1} V \otimes \Lambda_{\gamma_2 - v} V \otimes \Lambda_v V \xrightarrow{1 \otimes m} \Lambda_{\gamma_1} V \otimes \Lambda_{\gamma_2} V.$$

Proof. Mimic the proof of Lemma 6.15 in [H-H]. □

1.13. Corollary. *Let λ/μ be a skew partition with $l(\lambda) = s$, σ be an element of S_N and S be an element of $\mathrm{Row}_{\lambda/\mu}Y(\sigma) \setminus \mathrm{St}_{\lambda/\mu}Y(\sigma)$. Then there exist $S_1, \ldots, S_r \in \mathrm{Row}_{\lambda/\mu}Y(\sigma)$ with $S_i <_\sigma S$ for any i, such that*

$$\xi_S - \sum_i c_i \xi_{S_i} \in \mathrm{Im}(\square_{\lambda/\mu}) = \mathrm{Ker}(d_{\lambda/\mu}(V)),$$

for some $c_i \in \boldsymbol{Z}[q, q^{-1}]$. Here:

$$\square_{\lambda/\mu}=\sum_{i=1}^{s-1} 1_1\otimes\cdots\otimes 1_{i-1}\otimes\square_{\lambda^i/\mu^i}\otimes 1_{i+2}\otimes\cdots\otimes 1_s\,,$$

$$\lambda^i=(\lambda_i,\lambda_{i+1})\,,\quad \mu^i=(\mu_i,\mu_{i+1})\,,\quad 1_j=\mathrm{id}_{\Lambda_{\lambda_j-\mu_j}\mathrm{V}}\,.$$

PROOF. Mimic the proof of Lemma 6.18 in [H-H]. □

1.14. We want to stress a consequence of Theorem (1.10) and of all the machinery which allows to prove it. First of all note that the subcategory of $\mathcal{YB}_R$ (cf. [H-H]) given by the YB pairs as in (1.1) is a preadditive one. Let $\mathrm{P}^1=(p_{ij}^1)_{i,j=1}^n$ and $\mathrm{P}^2=(p_{ij}^2)_{i,j=1}^m$ be two multiplicatively antisymmetric matrices, and put $\mathrm{V}_{\mathrm{P}^1}=\langle u_1^1,\dots,u_n^1\rangle$, $\mathrm{V}_{\mathrm{P}^2}=\langle u_1^2,\dots,u_m^2\rangle$. Then define a YB operator on $\mathrm{V}_\mathrm{P}=\langle u_1^1,\dots,u_n^1,u_1^2,\dots,u_m^2\rangle$ by means of the matrix $\mathrm{P}=(p_{ij})_{i,j=1}^N$, $N=n+m$, defined as follows:

$$p_{ij}=\begin{cases} p_{ij}^1 & \text{for } i,j\in[1,n]\\ p_{ij}^2 & \text{for } i,j\in[n+1,N]\\ 1 & \text{for } i\in[1,n],\ \ j\in[n+1,N] \text{ or } i\in[n+1,N],\ \ j\in[1,n]\,.\end{cases}$$

Note that $\mathrm{V}_\mathrm{P}\simeq\mathrm{V}_{\mathrm{P}^1}\oplus\mathrm{V}_{\mathrm{P}^2}$ becomes in a natural way an $R[GL(q,\mathrm{P}^1)]\otimes R[GL(q,\mathrm{P}^2)]$-comodule. Write for short $\mathrm{V}_i=\mathrm{V}_{\mathrm{P}^i}$, $\beta_i=\beta_{\mathrm{V}_{\mathrm{P}^i}}$, for $i=1,2$, and let $\mu\subset\gamma\subset\lambda$ be partitions. Following [A-B-W], define two R-modules

$$M_\gamma(\Lambda_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))=\mathrm{Im}\left(\sum_{\mu\subseteq\sigma\subseteq\lambda,\sigma\geq\gamma}\Lambda_{\sigma/\mu}\mathrm{V}_1\otimes\Lambda_{\lambda/\sigma}\mathrm{V}_2\to\Lambda_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2)\right),$$

$$\dot{M}_c(\Lambda_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))=\mathrm{Im}\left(\sum_{\mu\subseteq\sigma\subseteq\lambda,\sigma>\gamma}\Lambda_{\sigma/\mu}\mathrm{V}_1\otimes\Lambda_{\lambda/\sigma}\mathrm{V}_2\to\Lambda_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2)\right),$$

where the indicated maps are obtained by tensoring the obvious maps

$$\Lambda_{\sigma_i-\mu_i}\mathrm{V}_1\otimes\Lambda_{\lambda_i-\sigma_i}\mathrm{V}_2\to\Lambda_{\lambda_i-\mu_i}(\mathrm{V}_1\oplus\mathrm{V}_2)\,.$$

Let $M_\gamma(L_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))$ and $\dot{M}_\gamma(L_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))$ be the images of the previous modules under the Schur map $d_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2)$. The following result holds as in the classical case:

THEOREM. *The R-modules*

$$L_{\gamma/\mu}\mathrm{V}_1\otimes L_{\lambda/\gamma}\mathrm{V}_2\,,\qquad M_\gamma(L_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))/\dot{M}_\gamma(L_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))$$

are isomorphic. Hence the R-modules $M_\gamma(L_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2))$, $\mu\subseteq\gamma\subseteq\lambda$, *give a filtration of* $L_{\lambda/\mu}(\mathrm{V}_1\oplus\mathrm{V}_2)$, *whose associated graded module is isomorphic to*

$$\sum_{\mu\subseteq\gamma\subseteq\lambda}L_{\gamma/\mu}\mathrm{V}_1\otimes L_{\lambda/\gamma}\mathrm{V}_2\,.$$

PROOF. Follow *verbatim* the proof of Theorem II. 4.11 in [A-B-W]. □

Note that the isomorphism of the theorem is in fact an isomorphism of

$R[GL(q, P^1)] \otimes R[GL(q, P^2)]$-comodules.

2. The recipe.

2.1. In this section we let R be the ring $R = \mathbf{Z}[\mathcal{Q}, \mathcal{Q}^{-1}]$, $\mathcal{Q}$ an indeterminate, and take a multiplicatively antisymmetric matrix $P = (p_{ij})_{i,j=1}^N$ and the YB pair (V_P, β_{V_P}), where $V_P = \langle u_1, \ldots, u_N \rangle$ and

$$\text{(1)} \qquad \beta_{V_P}(u_i \otimes u_j) = \begin{cases} u_i \otimes u_i & \text{if } i = j \\ \mathcal{Q} p_{ji} u_j \otimes u_i & \text{if } i < j \\ \mathcal{Q} p_{ji} u_j \otimes u_i + (1 - \mathcal{Q}^2) u_i \otimes u_j & \text{if } i > j. \end{cases}$$

We are going to construct a filtration of $L_{\lambda/\mu} V_P$ as an $R[GL(\mathcal{Q}, P)]$-comodule, such that the associated graded object is isomorphic to $\sum_\nu \gamma(\lambda/\mu; \nu) L_\nu V_P$. As in the classical Littlewood-Richardson rule, here $\gamma(\lambda/\mu; \nu)$ stands for the number of standard tableaux of shape λ/μ filled with $\tilde{\mu}_1$ copies of 1, $\tilde{\mu}_2$ copies of 2, $\tilde{\mu}_3$ copies of 3 etc., such that the associated word (formed by listing all entries from bottom to top in each column, starting from the leftmost column) is a lattice permutation. The construction is a suitable "deformation" of the one used in the first author's doctoral thesis, Brandeis University 1984, as illustrated for instance in [B]. We again remark that owing to Theorem (1.10) (ii), the construction holds in fact for every commutative ring R and every choice of a unit $q \in R$.

2.2. In order to embed $L_{\lambda/\mu} V_P$ into a (non-skew) Schur module, let $M = \mu_1$ and consider another multiplicatively antisymmetric matrix $P' = (p'_{ij})_{i,j=1}^M$, together with the YB pair $(V_{P'}, \beta_{V_{P'}})$, where $V_{P'} = \langle u'_1, \ldots, u'_M \rangle$ and $\beta_{V_{P'}}$ is defined similarly to (1) above. For convenience of notation, we shall denote V_P, u_i, $V_{P'}$, and u'_i by V, i, V′, and i', respectively.

It follows from Theorem (1.14) that the $R[GL(\mathcal{Q}, P')] \otimes R[GL(\mathcal{Q}, P)]$-comodule $L_\lambda(V' \oplus V)$ is isomorphic to $\sum_{\alpha \subseteq \lambda} L_\alpha V' \otimes L_{\lambda/\alpha} V$, up to a filtration.

Let $(L_\lambda(V' \oplus V))_h$ denote the sub-R-module of $L_\lambda(V' \oplus V)$ spanned by the tableaux in which h V′-indices occur. (In this section we identify a tableaux T with ξ_T and the corresponding element of a Schur module, according to the case.) Then up to a filtration,

$$(L_\lambda(V' \oplus V))_h \simeq \sum_{\alpha \subseteq \lambda, |\alpha| = h} L_\alpha V' \otimes L_{\lambda/\alpha} V,$$

as $R[GL(\mathcal{Q}, P')] \otimes R[GL(\mathcal{Q}, P)]$-comodules.

If $(L_\lambda(V' \oplus V))_{\tilde{\mu}}$ denotes the sub-R-module of $L_\lambda(V' \oplus V)$ spanned by the tableaux in which every i' occurs exactly $\tilde{\mu}_i$ times, also:

$$\text{(2)} \qquad (L_\lambda(V' \oplus V))_{\tilde{\mu}} \simeq \sum_{\alpha \subseteq \lambda} (L_\alpha V')_{\tilde{\mu}} \otimes L_{\lambda/\alpha} V,$$

as $R[GL(\mathcal{Q}, P)]$-comodules, up to a filtration.

Since the bottom piece of the filtration relative to (2) corresponds to the (lexicographically) largest partition α, namely μ, it follows:

$$(L_\lambda V')_{\tilde{\mu}} \otimes L_{\lambda/\mu}V \overset{R[GL(\mathcal{Q}, P)]}{\hookrightarrow} (L_\lambda(V' \oplus V))_{\tilde{\mu}}.$$

And $\mathrm{rk}(L_\mu V')_{\tilde{\mu}} = 1$ implies that

$$L_{\lambda/\mu}V \overset{R[GL(\mathcal{Q}, P)]}{\hookrightarrow} (L_\lambda(V' \oplus V))_{\tilde{\mu}},$$

as wished.

Explicitly, the embedding sends the tableau $d_{\lambda/\mu}(V)(a_1 \otimes \cdots \otimes a_s)$, $s = l(\lambda)$, to

$$d_\lambda(V' \oplus V)[(b^{(\mu_1)} \wedge a_1) \otimes \cdots \otimes (b^{(\mu_r)} \wedge a_r) \otimes a_{r+1} \otimes \cdots \otimes a_s], \qquad r = l(\mu),$$

where we write $b^{(k)}$ for $1' \wedge 2' \wedge \cdots \wedge k' \in \Lambda_k V'$. Notice that $b^{(k)}$ is a relative $R[B^+(\mathcal{Q}, P')]$-invariant.

2.3. Let $t = (t_{r1}, \ldots, t_{11}; t_{r2}, \ldots, t_{12}; \ldots; t_{rs}, \ldots, t_{1s})$ be a family of nonnegative integers such that

$$\sum_{i=1}^{s} t_{ji} = \mu_j, \qquad \forall j = 1, \ldots, r.$$

Let f denote the $R[GL(\mathcal{Q}, P')]$-equivariant composite map:

$$\begin{array}{c} \Lambda_{\mu_r}V' \otimes \cdots \otimes \Lambda_{\mu_1}V' \\ \Big\downarrow \otimes_{j=r}^{1}(\Delta^{t_j}_{\Lambda V'}) \\ (\Lambda_{t_{r1}}V' \otimes \cdots \otimes \Lambda_{t_{rs}}V') \otimes \cdots \otimes (\Lambda_{t_{11}}V' \otimes \cdots \otimes \Lambda_{t_{1s}}V') \\ \Big\downarrow \varphi_{\Lambda V'}(\omega_{rs}) \\ (\Lambda_{t_{r1}}V' \otimes \Lambda_{t_{r-1,1}}V' \otimes \cdots \otimes \Lambda_{t_{11}}V') \otimes \cdots \otimes (\Lambda_{t_{rs}}V' \otimes \Lambda_{t_{r-1,s}}V' \otimes \cdots \otimes \Lambda_{t_{1s}}V') \\ \Big\downarrow (m^{(r)}_{\Lambda V'})^{\otimes s} \\ \Lambda_{t_{r1}+t_{r-1,1}+\cdots+t_{11}}V' \otimes \cdots \otimes \Lambda_{t_{rs}+t_{r-1,s}+\cdots+t_{1s}}V' \end{array}$$

where $t_j = (t_{j1}, \ldots, t_{js})$, $m^{(r)}_{\Lambda V'}: \Lambda V' \otimes \cdots \otimes \Lambda V' \to \Lambda V'$ is obtained by iterating the multiplication, and

$$\omega_{rs} = \begin{pmatrix} 1 & 2 & 3 & \ldots & s & s+1 & s+2 & \ldots & 2s+1 & \ldots & rs \\ 1 & r+1 & 2r+1 & \ldots & (s-1)r+1 & 2 & r+2 & \ldots & 3 & \ldots & rs \end{pmatrix}$$

(cf. (1.5) and (1.7)).

As $b^{(\mu_r)} \otimes \cdots \otimes b^{(\mu_1)}$ is a relative $R[B^+(\mathcal{Q}, P')]$-invariant, also $f(b^{(\mu_r)} \otimes \cdots \otimes b^{(\mu_1)})$ is so. We denote the latter by $b(t)$.

2.4. For every $\nu \subseteq \lambda$ such that $|\nu| = |\lambda| - |\mu|$, let $B(\lambda/\nu)$ denote the set of all possible $b(t)$ which satisfy the further equalities:

$$\sum_{j=1}^{r} t_{ji}=\lambda_i-\nu_i\,, \qquad \forall i=1,\dots,s\,.$$

For every $b\in B(\lambda/\nu)$, we call $\varphi(\nu, b)$ the restriction to $\Lambda_\nu V\otimes\{b\}$ of the following composite map

$$\Lambda_\nu V\otimes\Lambda_{\lambda/\nu}V' \xrightarrow{\varphi_\nu(\lambda)} \Lambda_\lambda(V'\oplus V) \xrightarrow{d_\lambda(V'\oplus V)} \Lambda_\lambda(V'\oplus V)\,,$$

where $\varphi_\nu(\lambda)$ is obtained by tensoring the morphisms

$$\Lambda_{\nu_i}V\otimes\Lambda_{\lambda_i-\nu_i}V'\to\Lambda_{\lambda_i}(V'\oplus V)\,, \qquad x\otimes y\mapsto x\wedge y\,, \quad i=1,\dots,s\,.$$

PROPOSITION. *The image of* $\varphi(\nu, b)$ *lies in* $L_{\lambda/\mu}V \hookrightarrow L_\lambda(V'\oplus V)$.

PROOF. As $\varphi(\nu, b)$ is $R[GL(\mathcal{Q}, P')]\otimes R[GL(\mathcal{Q}, P)]$-equivariant, and b is a relative $R[B^+(\mathcal{Q}, P')]$-invariant of V'-content $\tilde{\mu}$ (i.e., it contains $\tilde{\mu}_i$ copies of i'), each element of $\mathrm{Im}(\varphi(\nu, b))$ is a relative $R[B^+(\mathcal{Q}, P')]$-invariant of V'-content $\tilde{\mu}$. But then we are through, thanks to Lemma (2.5) below and to the fact that $d_\mu(V')(b^{(\mu_1)}\otimes\cdots\otimes b^{(\mu_r)})$ is the only canonical tableau of content $\tilde{\mu}$. □

2.5. LEMMA. *For every partition* α, *take in* $L_\alpha V'\otimes_{\mathbf{R}}\mathbf{Q}(\mathcal{Q})$ *the element*

$$C_\alpha=d_\alpha(V')(1'\wedge\cdots\wedge\alpha_1'\otimes 1'\wedge\cdots\wedge\alpha_2'\otimes\cdots\otimes 1'\wedge\cdots\wedge\alpha_l')\,, \qquad l=l(\alpha)$$

(C_α *is sometimes called the canonical tableau of* $L_\alpha V'$). *Then the relative* $R[B^+(\mathcal{Q}, P')]$-*invariant elements of* $L_\alpha V'\otimes_{\mathbf{R}}\mathbf{Q}(\mathcal{Q})$ *are spanned (over* $\mathbf{Q}(\mathcal{Q})$) *by* C_α.

PROOF. Combine $(L_\alpha V')_{\tilde{\alpha}}=R\cdot C_\alpha$ with a multiparameter version of Theorem 6.5.2 in [P-W]. □

2.6. For each $\nu\subseteq\lambda$ such that $\gamma(\lambda/\mu;\nu)\neq 0$, we wish to describe a subset of $B(\lambda/\nu)$, say $B'(\lambda/\nu)$, such that $\sharp B'(\lambda/\nu)=\gamma(\lambda/\mu;\nu)$. Let $T\in L_{\lambda/\nu}V'$ be a standard tableau, of content $\tilde{\mu}$, and such that its associated word, $\mathrm{as}(T)=(a_1,\dots,a_{|\mu|})$, is a lattice permutation. Then μ is the content of the transpose lattice permutation $(\mathrm{as}(T))^\sim$. (Explicitly, $(\mathrm{as}(T))^\sim=(\tilde{a}_1,\dots,\tilde{a}_{|\mu|})$, where $\tilde{a}_i$ is the number of times a_i occurs in as(T) in the range $(a_1,\dots,a_i)$.) Let $\tilde{T}$ be the tableau obtained from T by replacing every entry a_i of T by $\tilde{a}_i$. For each $i\in\{1,\dots,s\}$ and each $j\in\{1,\dots,r\}$, we set t_{ji} to be the number of j's occuring in the i-th row of $\tilde{T}$. We denote by $b(T)$ the element $b(t)\in B(\lambda/\nu)$, corresponding to this choice of t_{ji}'s.

2.7. Given any row-standard tableau T, we can consider the word $w(T)$ formed by writing one after the other all the rows of T, starting from the top. As all such words can be ordered lexicographically, we can say that $T<_{\mathrm{lex}}T'$ if and only if $w(T)<_{\mathrm{lex}}w(T')$. It is then easy to see that the following holds.

PROPOSITION. *If we write* $b(T)\in\Lambda_{\lambda/\nu}V'$ *as a linear combination of row-standard tableaux, then*

$$b(\mathrm{T}) = \pm\mathcal{Q}^*\mathrm{T} + \sum_k c_k \mathrm{T}_k, \qquad c_k \in \boldsymbol{Z}[\mathcal{Q}, \mathcal{Q}^{-1}],$$

where $\mathcal{Q}^*$ stands for a power of $\mathcal{Q}$, and each T_k is a row-standard tableau $<_{\mathrm{lex}} \mathrm{T}$.

Since there are exactly $\gamma(\lambda/\mu; \nu)$ tableaux $\mathrm{T} \in L_{\lambda/\nu}\mathrm{V}'$ which are standard, of content $\tilde{\mu}$, and such that as(T) is a lattice permutation, the above Proposition implies that the elements $b(\mathrm{T})$ form a subset of $B(\lambda/\nu)$ of cardinality $\gamma(\lambda/\mu; \nu)$. It is precisely this subset which we call $B'(\lambda/\nu)$.

2.8. Consider the family of elements of $L_{\lambda/\mu}\mathrm{V} \hookrightarrow L_\lambda(\mathrm{V}' \oplus \mathrm{V})$:

$$\mathscr{F} = \{\varphi(\nu, b)(x) \mid \gamma(\lambda/\mu; \nu) \neq 0,\ b \in B'(\lambda/\nu), \text{ and } d_\nu(\mathrm{V})(x) \text{ is a standard tableau}\}.$$

We claim that $\mathscr{F}$ is an R-basis of $L_{\lambda/\mu}\mathrm{V}$.

PROPOSITION. *The elements of $\mathscr{F}$ are linearly independent over R.*

PROOF. Suppose that there exist nonzero coefficients $r_{\nu,b,x} \in R$ such that $\sum_{\mathscr{F}} r_{\nu,b,x}\varphi(\nu, b)(x) = 0$, i.e., such that $\sum_{\nu,b,x} r_{\nu,b,x} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_\nu(\lambda)(x \otimes b)) = 0$ in $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$. This is the same as

$$\sum_{\nu,b} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_\nu(\lambda)(y_{\nu,b} \otimes b)) = 0, \tag{3}$$

where $y_{\nu,b} = \sum_x r_{\nu,b,x} x$. Let ν_0 be the (lexicographically) smallest ν occuring in (3). Order the set $B'(\lambda/\nu_0) = \{b(\mathrm{T}_1), \ldots, b(\mathrm{T}_p)\}$ as follows:

$$b(\mathrm{T}_i) < b(\mathrm{T}_j) \quad \text{if and only if} \quad w(\mathrm{T}_i) <_{\mathrm{lex}} w(\mathrm{T}_j).$$

Let $b_0 = b(\mathrm{T}_0)$ be the highest $b(\mathrm{T}_i) \in B'(\lambda/\nu_0)$ occuring in $\sum_{\nu,b} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_\nu(\lambda)(y_{\nu,b} \otimes b))$. Clearly, $d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes b_0))$ is not in general a linear combination of standard tableaux of $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$, with respect to the order $1 < \cdots < N < 1' < \cdots < M'$, since violations of column-standardness may occur in b_0. Apply therefore to $d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes b_0))$ the straightening law of $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$ with respect to $1 < \cdots < N < 1' < \cdots < M'$. One gets (recall Proposition (2.7)):

$\pm\mathcal{Q}^* d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes \mathrm{T}_0))$
$+$(a linear combination of standard tableaux with V-shape$> \nu_0$)
$+$(a linear combination of standard tableaux with V-shape$= \nu_0$ and V′-part$<_{\mathrm{lex}}\mathrm{T}_0$).

Because of our choice of ν_0 and b_0, (3) then implies that $d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes \mathrm{T}_0)) = 0$, i.e.,

$$\sum_x r_{\nu_0,b_0,x} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(x \otimes \mathrm{T}_0)) = 0.$$

But this is a linear combination of standard tableaux in $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$, with respect to the order $1 < \cdots < N < 1' < \cdots < M'$, so that $r_{\nu_0,b_0,x} = 0$ for each x, which contradicts our assumption on the coefficients $r_{v,b,x}$. □

2.9. COROLLARY. *$\mathscr{F}$ is a basis for $L_{\lambda/\mu}V \otimes_{\boldsymbol{R}} \boldsymbol{Q}(\mathscr{Q})$.*

PROOF. By definition of $\mathscr{F}$, $\sharp\mathscr{F} = \mathrm{rk}(L_{\lambda/\mu}V)$. By Theorem (1.10) (ii), the latter rank is constant on all rings. So Proposition (2.8) says that $\mathscr{F}$ is a basis for the vector space $L_{\lambda/\mu}V \otimes_{\boldsymbol{R}} \boldsymbol{Q}(\mathscr{Q})$. □

2.10. COROLLARY. *$\mathscr{F}$ is a basis for $L_{\lambda/\mu}V$.*

PROOF. It suffices to show that $\mathscr{F}$ is a system of generators for $L_{\lambda/\mu}V$. Let $y \in L_\lambda(V' \oplus V)$ be any tableau of type

$$\begin{array}{llllllll}
1' & \cdot & \cdot & \cdot & \mu_1' & \circ & \circ & \circ \\
1' & \cdot & \cdot & \mu_2' & \circ & \circ & \circ & \\
1' & \cdot & \mu_3' & \circ & \circ & & & \\
\cdot & \cdot & \circ & \circ & & & & \\
\cdot & \cdot & \circ & \circ & & & & \\
\circ & \circ & \circ & & & & & \\
\circ & \circ & & & & & &
\end{array},$$

where the little circles stand for basis elements of V.

Since $y \in L_{\lambda/\mu}V$, Corollary (2.9) says that in the quotient field of R, there exist (unique) coefficients $q_{v,b,x}$, such that

$$y = \sum_{\mathscr{F}} q_{v,b,x}\varphi(v, b)(x)\,. \tag{4}$$

To both sides of (4), apply the straightening law with respect to $1 < \cdots < N < 1' < \cdots < M'$. In the left-hand side, only coefficients in R occur. In the right-hand side, if v_0 denotes the smallest V-shape coupled with a nonzero $\sum_x q_{v,b,x}x$, and $b_0 = b(T_0)$ denotes the highest element of $B'(\lambda/v_0)$ (cf. ordering in the proof of Proposition (2.8)) occurring with a nonzero $\sum_x q_{v_0,b,x}x$, we find that the term $\pm\mathscr{Q}^* d_\lambda(V' \oplus V)(\varphi_{v_0}(\lambda)(\sum_x q_{v_0,b_0,x}x \otimes T_0))$ must cancel with something in the left-hand side; since each $d_{v_0}(V)(x) \in L_v V$ is standard, it follows that $q_{v_0,b_0,x} \in R$ for every x.

Write next (4) as:

$$y - \sum_x q_{v_0,b_0,x}\varphi(v_0, b_0)(x) = \sum_{(v,b) \neq (v_0,b_0)} \varphi(v, b)\left(\sum_x q_{v,b,x}x\right). \tag{4'}$$

Reasoning for (4′) as done for (4), it follows that $q_{v_1,b_1,x} \in R$, where (v_1, b_1) is the pair (v, b) coming immediately before (v_0, b_0) in the total ordering:

$(v, b) < (v', b') \Leftrightarrow$ either $v > v'$, or $v = v'$ and $b < b'$

in the ordering of $B'(\lambda/v)$ given in the proof of (2.8).

Repeating the argument as many times as necessary, the proofs is completed. □

2.11. THEOREM. *Up to a filtration,* $L_{\lambda/\mu}\mathrm{V} \simeq \sum_\nu \gamma(\lambda/\mu; \nu) L_\nu \mathrm{V}$ *as* $R[GL(\mathcal{Q}, \mathrm{P})]$-*comodules.*

PROOF. For every ν such that $\gamma(\lambda/\mu; \nu) \neq 0$, let M_ν denote the R-span (in $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$) of all elements $\varphi(\tau, b)(x)$ of $\mathcal{F}$ such that $\tau \geq \nu$. Also let $\dot{M}_\nu$ denote the R-span of all $\varphi(\tau, b)(x)$ such that $\tau > \nu$. Clearly, we have the isomorphism of free R-modules:

$$M_\nu/\dot{M}_\nu \xrightarrow{\psi_\nu} L_\nu\mathrm{V} \oplus \cdots \oplus L_\nu\mathrm{V} \qquad (\gamma(\lambda/\mu; \nu) \text{ summands}).$$

$\{M_\nu\}$ will be the required filtration, if we show that each ψ_ν is an $R[GL(\mathcal{Q}, \mathrm{P})]$-isomorphism. In order to do so, it suffices to prove that for every fixed $b_0 \in B'(\lambda/\nu)$, and for every basis element $y \in \Lambda_\nu\mathrm{V}$, $\varphi(\nu, b_0)(y) - \varphi(\nu, b_0)(\sum r_i x_i) \in \dot{M}_\nu$, where $\sum r_i d_\nu(\mathrm{V})(x_i)$ is obtained by application to the tableau $d_\nu(\mathrm{V})(y)$ of the straightening law of $L_\nu\mathrm{V}$. Notice however that $\varphi(\nu, b_0)(y) \in L_{\lambda/\mu}\mathrm{V} \subseteq L_\lambda(\mathrm{V}' \oplus \mathrm{V})$ can be written in two ways:

$$\varphi(\nu, b_0)(y) = \sum_{\mathcal{F}} r_{\tau,b,x} \varphi(\tau, b)(x)\,, \tag{5}$$

by Corollary (2.10), and

$$\varphi(\nu, b_0)(y) = \sum r_i \varphi(\nu, b_0)(x_i) + L.C.\,, \tag{6}$$

where $L.C.$ denotes a linear combination of tableaux, standard with respect to $1 < \cdots < N < 1' < \cdots < M'$, and with V-part $> \nu$. This last equality is obtained by eliminating in the V-part of $\varphi(\nu, b_0)(y)$ all violations of standardness, with respect to $1 < \cdots < N < 1' < \cdots < M'$.

Comparing (5) and (6), it follows that

$$\varphi(\nu, b_0)(y) - \varphi(\nu, b_0)(\sum r_i x_i) = \sum_{\mathcal{F}} r_{\tau,b,x} \varphi(\tau, b)(x)$$

with $r_{\tau,b,x} = 0$ whenever $\tau \leq \nu$. Hence $\varphi(\nu, b_0)(y) - \varphi(\nu, b_0)(\sum r_i x_i) \in \dot{M}_\nu$ as wished. □

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