

## A LITTLEWOOD-RICHARDSON RULE FOR FACTORIAL SCHUR FUNCTIONS

ALEXANDER I. MOLEV AND BRUCE E. SAGAN

ABSTRACT. We give a combinatorial rule for calculating the coefficients in the expansion of a product of two factorial Schur functions. It is a special case of a more general rule which also gives the coefficients in the expansion of a skew factorial Schur function. Applications to Capelli operators and quantum immanants are also given.

### 1. INTRODUCTION

As  $\lambda$  runs over all partitions with length  $l(\lambda) \leq n$ , the Schur polynomials  $s_\lambda(x)$  form a distinguished basis in the algebra of symmetric polynomials in the independent variables  $x = (x_1, \dots, x_n)$ . By definition,

$$s_\lambda(x) = \frac{\det(x_j^{\lambda_i + n - i})_{1 \leq i, j \leq n}}{\prod_{i < j} (x_i - x_j)}.$$

Equivalently, these polynomials can be defined by the combinatorial formula

$$(1) \quad s_\lambda(x) = \sum_T \prod_{\alpha \in \lambda} x_{T(\alpha)},$$

summed over semistandard tableaux  $T$  of shape  $\lambda$  with entries in the set  $\{1, \dots, n\}$ , where  $T(\alpha)$  is the entry of  $T$  in the cell  $\alpha$ .

Any product  $s_\lambda(x)s_\mu(x)$  can be expanded as a linear combination of Schur polynomials:

$$(2) \quad s_\lambda(x)s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x).$$

The classical Littlewood-Richardson rule [LR] gives a method for computing the coefficients  $c_{\lambda\mu}^\nu$ . These same coefficients appear in the expansion of a skew Schur function

$$s_{\nu/\lambda}(x) = \sum_\mu c_{\lambda\mu}^\nu s_\mu(x).$$

A number of different proofs and variations of this rule can be found in the literature; see, e.g. [M1, S1], and the references therein.

---

Received by the editors September 2, 1997 and, in revised form, January 15, 1998.

1991 *Mathematics Subject Classification*. Primary 05E05; Secondary 05E10, 17B10, 17B35, 20C30.

*Key words and phrases*. Capelli operator, factorial Schur function, Littlewood-Richardson rule, quantum immanant, Young tableau.

To state the rule, we introduce the following notation. If  $T$  is a tableau, then let  $\text{cw}(T)$  be the (reverse) *column word* of  $T$ , namely the sequence obtained by reading the entries of  $T$  from top to bottom in successive columns starting from the right-most column. We will call the associated total order on the cells of  $T$  *column order* and write  $\alpha < \beta$  if cell  $\alpha$  comes before cell  $\beta$  in this order. A word  $w = a_1 \cdots a_N$  in the symbols  $1, \dots, n$  is a *lattice permutation* if for  $1 \leq r \leq N$  and  $1 \leq i < n$  the number of occurrences of  $i$  in  $a_1 \cdots a_r$  is at least as large as the number of occurrences of  $i + 1$ .

The Littlewood-Richardson rule says that the coefficient  $c_{\lambda\mu}^\nu$  is equal to the number of semistandard tableaux  $T$  of the shape  $\nu/\mu$  and weight  $\lambda$  such that  $\text{cw}(T)$  is a lattice permutation. (One usually uses *row words* in the formulation of the rule. However, it is known that these two versions are equivalent [FG].) In particular,  $c_{\lambda\mu}^\nu$  is zero unless  $\lambda, \mu \subseteq \nu$  and  $|\nu| = |\lambda| + |\mu|$ .

We will now state an equivalent formulation of this rule [JP, Z, KR] and establish some notation to be used in Section 3. Let  $R$  denote a sequence of diagrams

$$\mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \dots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu,$$

where  $\rho \rightarrow \sigma$  means that  $\rho \subset \sigma$  with  $|\sigma/\rho| = 1$ . Let  $r_i$  denote the row number of  $\rho^{(i)}/\rho^{(i-1)}$ . Then the sequence  $r_1 \dots r_l$  is called the *Yamanouchi symbol* of  $R$ . Equivalently,  $R$  corresponds to a standard tableau  $T$  of shape  $\nu/\mu$  where  $r_i$  is the row number of the entry  $i$  in  $T$ . A semistandard tableau  $T$  *fits*  $\nu/\mu$  if  $\text{cw}(T)$  is the Yamanouchi symbol for some standard Young tableau of shape  $\nu/\mu$ . For example,

$$T = \begin{array}{ccc} 1 & 1 & 2 \\ & 2 & 3 \end{array}$$

fits  $(4, 3, 1)/(2, 1)$  since  $\text{cw}(T) = 21312 = r_1 \dots r_5$  corresponds to the standard tableau

$$\begin{array}{cc} & 2 & 4 \\ & 1 & 5 \\ 3 & & \end{array}$$

or equivalently to the shape sequence

$$R: \mu = (2, 1) \rightarrow (2, 2) \rightarrow (3, 2) \rightarrow (3, 2, 1) \rightarrow (4, 2, 1) \rightarrow (4, 3, 1) = \nu.$$

The coefficient  $c_{\lambda\mu}^\nu$  is then equal to the number of semistandard tableaux  $T$  of shape  $\lambda$  that fit  $\nu/\mu$ .

The factorial Schur function  $s_\lambda(x|a)$  is a polynomial in  $x$  and a doubly-infinite sequence of variables  $a = (a_i)$ . It can be defined as the ratio of two alternants (3) (see the beginning of Section 2) by analogy with the ordinary case. This approach goes back to Lascoux [L1]. The  $s_\lambda(x|a)$  are also a special case of the double Schubert polynomials introduced by Lascoux and Schützenberger as explained in [L2]. The combinatorial definition (4) for the particular sequence  $a$  with  $a_i = i - 1$  (again, see the beginning of Section 2) is due to Biedenharn and Louck [BL] while the case for general  $a$  is due to Macdonald [M2] and Goulden-Greene [GG]. The equivalence of (3) and (4) was established independently in [M2] and [GG].

Specializing  $a_i = 0$  for all  $i$ , the functions  $s_\lambda(x|a)$  turn into  $s_\lambda(x)$ . They form a basis in the symmetric polynomials in  $x$  over  $\mathbb{C}[a]$  so one can define the corresponding Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu(a)$ ; see (5). Our main result is Theorem 3.1 which gives a combinatorial rule for calculating a two-variable generalization  $c_{\theta\mu}^\nu(a, b)$  of these coefficients (8), where  $\theta$  is a skew diagram. In the case

$|\nu| = |\theta| + |\mu|$  the rule turns into a rule for computing the intertwining number of the skew representations of the symmetric group corresponding to the diagrams  $\theta$  and  $\nu/\mu$  [JP, Z]. Specializing further to  $\mu = \emptyset$  (respectively  $\theta = \lambda$ ), we get the classical Littlewood-Richardson rule in the first (respectively second) formulation above. A Pieri rule for multiplication of a double Schubert polynomial by a complete or elementary symmetric polynomial is given by Lascoux and Veigneau [V]. Lascoux has pointed out that the Newton interpolation formula in several variables [LS] can also be used to give an alternative proof of the factorial Littlewood-Richardson rule.

In Section 4 we consider the specialization  $a_i = i - 1$ . The corresponding coefficients  $c'_{\lambda\mu}(a)$  turn out to be the structure constants for the center of the universal enveloping algebra for the Lie algebra  $\mathfrak{gl}(n)$  and for an algebra of invariant differential operators in certain distinguished bases. We also obtain a formula which relates these coefficients to the dimensions of skew diagrams. This implies a symmetry property of these coefficients.

2. PRELIMINARIES

Let  $x = (x_1, \dots, x_n)$  be a finite sequence of variables and let  $a = (a_i), i \in \mathbb{Z}$ , be a doubly-infinite variable sequence. The *generalized factorial Schur function* for a partition  $\lambda$  of length at most  $n$  can be defined as follows [M2]. Let

$$(y|a)^k = (y - a_1) \cdots (y - a_k)$$

for each  $k \geq 0$ . Then

$$(3) \quad s_\lambda(x|a) = \frac{\det [(x_j|a)^{\lambda_i+n-i}]_{1 \leq i, j \leq n}}{\prod_{i < j} (x_i - x_j)}.$$

There is an explicit combinatorial formula for  $s_\lambda(x|a)$  analogous to (1):

$$(4) \quad s_\lambda(x|a) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)}),$$

where  $T$  runs over all semistandard tableaux of shape  $\lambda$  with entries in  $\{1, \dots, n\}$ ,  $T(\alpha)$  is the entry of  $T$  in the cell  $\alpha \in \lambda$  and  $c(\alpha) = j - i$  is the content of  $\alpha = (i, j)$ .

The highest homogeneous component of  $s_\lambda(x|a)$  in  $x$  obviously coincides with  $s_\lambda(x)$ . Therefore the polynomials  $s_\lambda(x|a)$  form a basis for  $R[x]^{S_n}$  where  $R = \mathbb{C}[a]$ , and one can define Littlewood-Richardson type coefficients  $c'_{\lambda\mu}(a)$  by

$$(5) \quad s_\lambda(x|a)s_\mu(x|a) = \sum_\nu c'_{\lambda\mu}(a)s_\nu(x|a).$$

Comparing the highest homogeneous components in  $x$  on both sides and using the Littlewood-Richardson Rule for the  $s_\lambda(x)$  we see that

$$(6) \quad c'_{\lambda\mu}(a) = \begin{cases} c'_{\lambda\mu} & \text{if } |\nu| = |\lambda| + |\mu|, \\ 0 & \text{if } |\nu| > |\lambda| + |\mu|. \end{cases}$$

Contrary to the classical case, the coefficients  $c'_{\lambda\mu}(a)$  turn out to be nonzero if  $|\nu| < |\lambda| + |\mu|$  and  $\lambda, \mu \subseteq \nu$ . This makes it possible to compute them using induction on  $|\nu/\mu|$  while keeping  $\lambda$  fixed.

The starting point of our calculation is the fact that the polynomials  $s_\lambda(x|a)$  possess some (characteristic) vanishing properties; see [S2, O1]. We use the following result from [O1]. For a partition  $\rho$  with  $l(\rho) \leq n$  define an  $n$ -tuple  $a_\rho = (a_{\rho_1+n}, \dots, a_{\rho_n+1})$ .

**Theorem 2.1** (Vanishing Theorem). *Given partitions  $\lambda, \rho$  with  $l(\lambda), l(\rho) \leq n$*

$$s_\lambda(a_\rho|a) = \begin{cases} 0 & \text{if } \lambda \not\subseteq \rho, \\ \prod_{(i,j) \in \lambda} (a_{\lambda_i+n-i+1} - a_{n-\lambda_j^t+j}) & \text{if } \lambda = \rho, \end{cases}$$

where  $\lambda^t$  is the diagram conjugate to  $\lambda$ .

In particular,  $s_\lambda(a_\lambda|a) \neq 0$  for any specialization of the sequence  $a$  such that  $a_i \neq a_j$  if  $i \neq j$ . We reproduce the proof of the Vanishing Theorem from [O1] for completeness.

*Proof.* The  $ij$ -th entry of the determinant in the numerator of the right hand side of (3) for  $x = a_\rho$  is

$$(7) \quad (a_{\rho_j+n-j+1} - a_1) \cdots (a_{\rho_j+n-j+1} - a_{\lambda_i+n-i}).$$

The condition  $\lambda \not\subseteq \rho$  implies that there exists an index  $k$  such that  $\rho_k < \lambda_k$ . Then for  $i \leq k \leq j$  we have

$$1 \leq \rho_j + n - j + 1 \leq \rho_k + n - k + 1 \leq \lambda_k + n - k \leq \lambda_i + n - i,$$

and so all the entries (7) with  $i \leq k \leq j$  are zero. Hence, the determinant is zero which proves the first part of the theorem.

Now let us set  $x = a_\lambda$  in (3). Then the  $ij$ -th entry of the determinant is

$$(a_{\lambda_j+n-j+1} - a_1) \cdots (a_{\lambda_j+n-j+1} - a_{\lambda_i+n-i}),$$

which equals zero for  $i < j$  and is nonzero for  $i = j$ . This means that the matrix is lower triangular with nonzero diagonal elements. Taking their product and dividing by

$$\prod_{i < j} (a_{\lambda_i+n-i+1} - a_{\lambda_j+n-j+1})$$

we get the desired equation. □

### 3. CALCULATING THE COEFFICIENTS

We will be able to prove more general results by introducing a second infinite sequence of variables denoted  $b = (b_i), i \in \mathbb{Z}$ . Let  $\theta$  and  $\mu$  be skew and normal (i.e., skewed by  $\emptyset$ ) diagrams, respectively. Define Littlewood-Richardson type coefficients  $c_{\theta\mu}^\nu(a, b)$  by the formula

$$(8) \quad s_\theta(x|b)s_\mu(x|a) = \sum_\nu c_{\theta\mu}^\nu(a, b)s_\nu(x|a),$$

where  $s_\theta(x|b)$  is defined as in (4) with  $\lambda$  replaced by  $\theta$  and  $a$  replaced by  $b$ .

As in Section 1, consider a sequence of diagrams

$$(9) \quad R : \mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \dots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu,$$

and let  $r_i$  be the row number of  $\rho^{(i)}/\rho^{(i-1)}$ . Construct the set  $\mathcal{T}(\theta, R)$  of semistandard  $\theta$ -tableaux  $T$  with entries from  $\{1, \dots, n = |x|\}$  such that  $T$  contains cells  $\alpha_1, \dots, \alpha_l$  with

$$\alpha_1 < \dots < \alpha_l \quad \text{and} \quad T(\alpha_i) = r_i, \quad 1 \leq i \leq l,$$

where  $<$  is column order. Let us distinguish the entries in  $\alpha_1, \dots, \alpha_l$  by barring each of them. For example, if  $n = 2$  and

$$R : (2, 1) \rightarrow (2, 2) \rightarrow (3, 2)$$

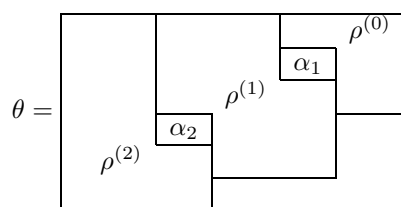
so that  $r_1 r_2 = 21$ , then for  $\theta = (3, 2)/(1)$  we have

$$\mathcal{T}(\theta, R) = \left\{ \begin{array}{c} \overline{1} \quad \overline{2} \quad 1 \\ \overline{1} \quad \overline{2} \end{array}, \begin{array}{c} \overline{1} \quad \overline{2} \\ 1 \quad 2 \end{array}, \begin{array}{c} \overline{1} \quad \overline{2} \\ 1 \quad 2 \end{array}, \begin{array}{c} \overline{1} \quad \overline{2} \\ 1 \quad 2 \end{array}, \begin{array}{c} \overline{1} \quad \overline{2} \\ 2 \quad 2 \end{array} \right\}.$$

We also let

$$\mathcal{T}(\theta, \nu/\mu) = \bigsqcup_R \mathcal{T}(\theta, R),$$

where the union is over all sequences  $R$  of the form (9). Finally, for each cell  $\alpha$  with  $\alpha_i < \alpha < \alpha_{i+1}$ ,  $0 \leq i \leq l$ , set  $\rho(\alpha) = \rho^{(i)}$ . (Inequalities involving cells with out-of-range subscripts are ignored.) For instance, if  $l = |\nu/\mu| = 2$ , then the following schematic diagram gives the layout of the  $\rho(\alpha)$



We are now in a position to state the Littlewood-Richardson rule for the  $c_{\theta\mu}^\nu(a, b)$ . The reader should compare the following formula with the combinatorial one for the  $s_\lambda(x|a)$  in (4).

**Theorem 3.1.** *The coefficient  $c_{\theta\mu}^\nu(a, b)$  is zero unless  $\mu \subseteq \nu$ . If  $\mu \subseteq \nu$ , then*

$$c_{\theta\mu}^\nu(a, b) = \sum_{T \in \mathcal{T}(\theta, \nu/\mu)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{ unbarred}}} ((a_{\rho(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}).$$

As immediate specializations of this result, note the following.

1. If  $a = b$  and  $\theta$  is normal, then this is a Littlewood-Richardson rule for the  $s_\lambda(x|a)$ .
2. If  $a = b$  and  $\mu$  is empty, then this is a rule for the expansion of a skew factorial Schur polynomial.
3. If  $|\nu| = |\theta| + |\mu|$ , then  $c_{\theta\mu}^\nu(a, b)$  is independent of  $a$  and  $b$  and equals the number of semistandard tableaux of shape  $\theta$  that fit  $\nu/\mu$ . This coincides with the number of pictures between  $\theta$  and  $\nu/\mu$  [JP, Z]. In particular,  $c_{\theta\mu}^\nu(a, b) = c_{\theta\mu}^\nu$ , an ordinary Littlewood-Richardson coefficient.
4. If  $\mu = \emptyset$  and  $\theta = \lambda$  is normal, then this is a rule for the re-expansion of a factorial Schur polynomial in terms of those for a different sequence of second variables. In particular,

$$s_\lambda(x|a) = \sum_{\nu \subseteq \lambda} g_{\lambda\nu}(a) s_\nu(x)$$

where

$$g_{\lambda\nu}(a) = (-1)^{|\lambda/\nu|} \sum_{T \in \mathcal{T}(\lambda, \nu)} \prod_{\substack{\alpha \in \lambda \\ T(\alpha) \text{ unbarred}}} a_{T(\alpha)+c(\alpha)}.$$

A different expression for  $g_{\lambda\nu}(a)$  in terms of double Schubert polynomials is provided by the Newton interpolation formula in several variables [LS].

We present the proof of Theorem 3.1 as a chain of propositions.

Note that the first claim of the theorem follows immediately from the Vanishing Theorem. Indeed, let  $\nu$  be minimal (with respect to containment) among all partitions in (8) such that  $c_{\theta\mu}^\nu(a, b) \neq 0$ . Suppose  $\nu \not\supseteq \mu$ . Then setting  $x = a_\nu$  in (8) and using the first part of the Vanishing Theorem gives

$$0 = c_{\theta\mu}^\nu(a, b)s_\nu(a_\nu|a).$$

But by the Vanishing Theorem’s second part we have  $s_\nu(a_\nu|a) \neq 0$  and so a contradiction to  $c_{\theta\mu}^\nu(a, b) \neq 0$ .

We shall assume hereafter that  $\mu \subseteq \nu$  and also write

$$|a_\rho| = a_{\rho_1+n} + \cdots + a_{\rho_n+1}.$$

**Proposition 3.2.** *If  $\mu \subseteq \nu$  with  $|\nu/\mu| = l$ , then*

$$\frac{s_\mu(a_\nu|a)}{s_\nu(a_\nu|a)} = \sum_{\mu \rightarrow \rho^{(1)} \rightarrow \cdots \rightarrow \rho^{(l-1)} \rightarrow \nu} \frac{1}{(|a_\nu| - |a_{\rho^{(0)}}|) \cdots (|a_\nu| - |a_{\rho^{(l-1)}}|)},$$

where  $\rho^{(0)} = \mu$ .

*Proof.* Setting  $x = a_\mu$  in (8) and using the Vanishing Theorem gives

$$(10) \quad c_{\theta\mu}^\mu(a, b) = s_\theta(a_\mu|b).$$

Further, for  $\theta = (1)$  and  $a = b$  relation (8) takes the form (cf. [OO, Theorem 9.1])

$$s_{(1)}(x|a)s_\mu(x|a) = s_{(1)}(a_\mu|a)s_\mu(x|a) + \sum_{\mu \rightarrow \rho} s_\rho(x|a)$$

which follows from (10), (6), and the Branching Theorem for the ordinary Schur functions.

Setting  $x = a_\nu$  in the previous equation and using the Vanishing Theorem we get

$$(11) \quad s_{(1)}(a_\nu|a)s_\mu(a_\nu|a) = s_{(1)}(a_\mu|a)s_\mu(a_\nu|a) + \sum_{\mu \rightarrow \rho \subseteq \nu} s_\rho(a_\nu|a).$$

We have

$$s_{(1)}(a_\nu|a) - s_{(1)}(a_\mu|a) = |a_\nu| - |a_\mu|$$

and so (11) gives

$$\frac{s_\mu(a_\nu|a)}{s_\nu(a_\nu|a)} = \frac{1}{|a_\nu| - |a_\mu|} \sum_{\mu \rightarrow \rho \subseteq \nu} \frac{s_\rho(a_\nu|a)}{s_\nu(a_\nu|a)}.$$

Induction on  $|\nu/\mu|$  completes the proof. □

It will be convenient to have a notation for sums like those occurring in the previous proposition. So let

$$(12) \quad H(\mu, \rho) = \sum_{\mu \rightarrow \rho^{(1)} \rightarrow \cdots \rightarrow \rho^{(r-1)} \rightarrow \rho} \frac{1}{(|a_\rho| - |a_{\rho^{(0)}}|) \cdots (|a_\rho| - |a_{\rho^{(r-1)}}|)},$$

and

$$(13) \quad H'(\rho, \nu) = \sum_{\rho \rightarrow \rho^{(r+1)} \rightarrow \cdots \rightarrow \rho^{(l-1)} \rightarrow \nu} \frac{1}{(|a_\rho| - |a_{\rho^{(r+1)}}|) \cdots (|a_\rho| - |a_{\rho^{(l)}}|)},$$

where  $\rho^{(0)} = \mu$  and  $\rho^{(l)} = \nu$ .

**Proposition 3.3.** *We have the formula*

$$c_{\theta\mu}^\nu(a, b) = \sum_{\mu \subseteq \rho \subseteq \nu} s_\theta(a_\rho|b)H(\mu, \rho)H'(\rho, \nu).$$

*Proof.* We use induction on  $|\nu/\mu|$ , noting that (10) is the base case  $|\nu/\mu| = 0$ . Set  $x = a_\nu$  in (8) and divide both sides by  $s_\nu(a_\nu|a)$ . By Proposition 3.2 we get

$$c_{\theta\mu}^\nu(a, b) = s_\theta(a_\nu|b)H(\mu, \nu) - \sum_{\sigma \subset \nu} c_{\theta\mu}^\sigma(a, b)H(\sigma, \nu).$$

By the induction hypotheses we can write this as

$$\begin{aligned} c_{\theta\mu}^\nu(a, b) &= s_\theta(a_\nu|b)H(\mu, \nu) - \sum_{\sigma \subset \nu} \sum_{\mu \subseteq \rho \subseteq \sigma} s_\theta(a_\rho|b)H(\mu, \rho)H'(\rho, \sigma)H(\sigma, \nu) \\ &= s_\theta(a_\nu|b)H(\mu, \nu) - \sum_{\mu \subseteq \rho \subset \nu} s_\theta(a_\rho|b)H(\mu, \rho) \sum_{\rho \subseteq \sigma \subset \nu} H'(\rho, \sigma)H(\sigma, \nu). \end{aligned}$$

To complete the proof we note that

$$\sum_{\rho \subseteq \sigma \subseteq \nu} H'(\rho, \sigma)H(\sigma, \nu) = 0,$$

which follows from the identity

$$\sum_{i=1}^k \frac{1}{(u_1 - u_2) \cdots (u_1 - u_i)(u_k - u_i) \cdots (u_k - u_{k-1})} = 0,$$

which holds for any variables  $u_1, \dots, u_k$  by induction on  $k > 1$ . (In the denominator an empty product is, as usual, equal to 1.) □

Note that a different expression for the  $c_{\theta\mu}^\nu(a, b)$  in terms of divided differences can be deduced from the Newton interpolation formula in several variables [LS].

**Proposition 3.4.** *We have the recurrence relation*

$$(14) \quad c_{\theta\mu}^\nu(a, b) = \frac{1}{|a_\nu| - |a_\mu|} \left( \sum_{\mu \rightarrow \mu'} c_{\theta\mu'}^\nu(a, b) - \sum_{\nu' \rightarrow \nu} c_{\theta\mu}^{\nu'}(a, b) \right).$$

*Proof.* By Proposition 3.3 it suffices to check that

$$H(\mu, \rho)H'(\rho, \nu) = \frac{1}{|a_\nu| - |a_\mu|} \left( \sum_{\mu \rightarrow \mu'} H(\mu', \rho)H'(\rho, \nu) - \sum_{\nu' \rightarrow \nu} H(\mu, \rho)H'(\rho, \nu') \right).$$

This follows from the relations

$$\sum_{\mu \rightarrow \mu'} H(\mu', \rho) = (|a_\rho| - |a_\mu|)H(\mu, \rho)$$

and

$$\sum_{\nu' \rightarrow \nu} H'(\rho, \nu') = (|a_\rho| - |a_\nu|)H'(\rho, \nu). \quad \square$$

Given a sequence

$$R : \mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \dots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu,$$

and an index  $k \in \{1, \dots, l\}$  introduce a set of  $\theta$ -tableaux  $\mathcal{T}_k(\theta, R)$  having entries from the set  $\{1, \dots, n\}$  as follows. Each tableau  $T \in \mathcal{T}_k(\theta, R)$  contains cells  $\alpha_1, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_l$  such that

$$\alpha_1 < \dots < \alpha_{k-1} < \alpha_{k+1} < \dots < \alpha_l \quad \text{and} \quad T(\alpha_i) = r_i, \quad 1 \leq i \leq l, i \neq k.$$

As usual, we distinguish the entries in the  $\alpha_i, i \neq k$ , by barring them. Now modify the  $\rho(\alpha)$  for  $R$  by defining, for cells with unbarred entries,

$$\rho^+(\alpha) = \begin{cases} \rho^{(k)} & \text{if } \alpha_{k-1} < \alpha < \alpha_{k+1}, \\ \rho(\alpha) & \text{otherwise,} \end{cases}$$

and

$$\rho^-(\alpha) = \begin{cases} \rho^{(k-1)} & \text{if } \alpha_{k-1} < \alpha < \alpha_{k+1}, \\ \rho(\alpha) & \text{otherwise.} \end{cases}$$

Also define corresponding weights

$$\begin{aligned} \mathcal{S}(R) &= \sum_{T \in \mathcal{T}(\theta, R)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{ unbarred}}} ((a_{\rho(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}), \\ \mathcal{S}_k^+(R) &= \sum_{T \in \mathcal{T}_k(\theta, R)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{ unbarred}}} ((a_{\rho^+(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}), \end{aligned}$$

and similarly for  $\mathcal{S}_k^-(R)$ . So Theorem 3.1 is equivalent to

$$(15) \quad c_{\theta\mu}^\nu(a, b) = \sum_R \mathcal{S}(R).$$

**Proposition 3.5.** *Given a sequence  $R$  we have*

$$(16) \quad \mathcal{S}(R) = \frac{1}{|a_\nu| - |a_\mu|} \sum_{k=1}^l (\mathcal{S}_k^+(R) - \mathcal{S}_k^-(R)).$$

*Proof.* It suffices to show that for each  $k$  we have

$$\mathcal{S}_k^+(R) - \mathcal{S}_k^-(R) = (|a_{\rho^{(k)}}| - |a_{\rho^{(k-1)}}|) \mathcal{S}(R).$$

Formula (16) will then follow from the relation

$$\sum_{k=1}^l (|a_{\rho^{(k)}}| - |a_{\rho^{(k-1)}}|) = |a_\nu| - |a_\mu|.$$

For a given  $T \in \mathcal{T}_k(\theta, R)$  the factors in the formulas for  $\mathcal{S}_k^+(R)$  and  $\mathcal{S}_k^-(R)$  are identical except for the case where  $\alpha_{k-1} < \alpha < \alpha_{k+1}$  and  $T(\alpha) = r_k$ . To see what happens when we divide  $\mathcal{S}_k^+(R) - \mathcal{S}_k^-(R)$  by

$$|a_{\rho^{(k)}}| - |a_{\rho^{(k-1)}}| = (a_{\rho^{(k)}})_{r_k} - (a_{\rho^{(k-1)}})_{r_k},$$

fix  $T$  and consider its contribution to the quotient. We need the following easily proved formula, where we are thinking of  $u = (a_{\rho^{(k)}})_{r_k}$ ,  $v = (a_{\rho^{(k-1)}})_{r_k}$  and  $m_i = b_{T(\alpha)+c(\alpha)}$  as  $\alpha$  runs over all cells of  $T$  with  $\alpha_{k-1} < \alpha < \alpha_{k+1}$  and  $T(\alpha) = r_k$ :

$$\frac{\prod_{i=1}^s (u - m_i) - \prod_{i=1}^s (v - m_i)}{u - v} = \sum_{j=1}^s (u - m_1) \cdots \widehat{(u - m_j)} \cdots (v - m_{j+1}) \cdots (v - m_s)$$



(a hat indicates the factor is to be omitted). The right-hand side of this expression can now be interpreted as the contribution to  $\mathcal{S}(R)$  of all tableaux gotten from  $T$  by barring one of the  $r_k$  between  $\alpha_{k-1}$  and  $\alpha_{k+1}$  in column order.  $\square$

We now prove (15) by induction on  $|\nu/\mu|$ . Equation (10) takes care of the case  $|\nu/\mu| = 0$ . By the induction hypothesis,

$$\sum_{\mu \rightarrow \mu'} c_{\theta\mu'}^\nu(a, b) = \sum_R \mathcal{S}_1^+(R) \quad \text{and} \quad \sum_{\nu' \rightarrow \nu} c_{\theta\mu}^{\nu'}(a, b) = \sum_R \mathcal{S}_l^-(R).$$

So formulas (14), (16) and the following proposition complete the proof of (15) and hence Theorem 3.1.

**Proposition 3.6.** *We have*

$$\sum_R \sum_{k=1}^{l-1} \mathcal{S}_k^-(R) = \sum_R \sum_{k=2}^l \mathcal{S}_k^+(R).$$

*Proof.* We can rewrite this formula as follows:

$$(17) \quad \sum_{R,k,T} \text{wt}^-(R, k, T) = \sum_{R',k',T'} \text{wt}^+(R', k', T'),$$

where  $T \in \mathcal{T}_k(\theta, R)$ ,  $k = 1, \dots, l-1$ , and  $T' \in \mathcal{T}_{k'}(\theta, R')$ ,  $k' = 2, \dots, l$ , with weights defined by

$$\text{wt}^-(R, k, T) = \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{ unbarred}}} ((a_{\rho^-(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)})$$

and similarly define  $\text{wt}^+(R', k', T')$ . To prove (17) we will construct a bijection  $(R, k, T) \longleftrightarrow (R', k', T')$  preserving the weights in the sense that  $\text{wt}^-(R, k, T) = \text{wt}^+(R', k', T')$ . There are three cases.

*Case 1.* Suppose that the skew diagram  $\rho^{(k+1)}/\rho^{(k-1)}$  consists of two cells in different rows and columns. Then  $R'$  is the sequence obtained from  $R$  by replacing  $\rho^{(k)}$  by the other diagram  $\rho'^{(k)}$  such that  $\rho^{(k-1)} \rightarrow \rho'^{(k)} \rightarrow \rho^{(k+1)}$  while  $k' = k + 1$  and  $T' = T$ .

*Case 2.* Let  $\rho^{(k+1)}/\rho^{(k-1)}$  have two cells in the same row. Then  $R' = R$ ,  $k' = k + 1$  and  $T' = T$ .

*Case 3.* Let  $\rho^{(k+1)}/\rho^{(k-1)}$  have two cells in the same column, say in rows  $r$  and  $r + 1$ . Let  $(i + 1, j) = (i_1, j_1)$  be the cell of  $T$  containing the corresponding  $\overline{r + 1}$ . If there is an  $r$  in cell  $(i, j)$  then it must be unbarred. In this case let  $T'$  be  $T$  with the bar moved from the  $r + 1$  to the  $r$ ,  $R' = R$ , and  $k' = k + 1$ . Weights are preserved since  $(a_{\rho^{(k-1)}})_r = (a_{\rho^{(k+1)}})_{r+1}$  and  $T(\alpha) + c(\alpha)$  is invariant under the change.

Now suppose cell  $(i, j)$  of  $T$  contains an entry less than  $r$  or  $(i, j) \notin \theta$  and let  $j' = j'_1$  be the column of the left-most  $r + 1$  in row  $i + 1$ . Since this subcase is more complicated than the others, the reader may wish to follow along with the example given after the end of this proof. Let  $s$  be the maximum integer such that for  $1 \leq t \leq s$  we have

1. there is an  $\overline{r + t}$  in cell  $(i + t, j_t)$  for some  $j_t$  corresponding to a cell in the same column as those of  $\rho^{(k+1)}/\rho^{(k-1)}$ ,

- 2. if  $(i + t, j'_t)$  contains the left-most  $r + t$  in row  $i + t$ , then  $(i + t, j_t)$  is between  $(i + t - 1, j_{t-1})$  and  $(i + t - 1, j'_{t-1})$  in column order. (Assume this is true vacuously when  $t = 1$ .)

Note that the condition on cell  $(i, j)$  implies that none of the  $r + t$ 's to the left of the one in  $(i + t, j_t)$  can be barred.

We now form  $T'$  by moving the bar in cell  $(i + t, j_t)$  to cell  $(i + t, j'_t)$  and replacing the  $r + t$ 's in cells  $(i + t, j'_t), (i + t, j'_t + 1), \dots, (i + t, j_t)$  by  $r + t - 1$ 's. Note that the result will still be a semistandard tableau because of the assumption about  $(i, j)$  and the choice of elements to decrease. Since the elements from the given column of  $\rho^{(k+1)}/\rho^{(k-1)}$  are still added in the correct order in  $T'$ , it determines a valid  $R'$ , complete except for the step where a cell is added in row  $r + s$  of that column which should be done immediately following the addition of  $r + s - 1$ . Then  $k'$  is the position of this  $r + s$ . Invariance of weights follows from considerations like those in the first subcase, noting that the contribution to  $\text{wt}^-$  of each entry decreased in  $T$  is the same as that of the element on its right to  $\text{wt}^+$  in  $T'$ .

The inverse of this construction is similar and left to the reader. This completes the proof of the Theorem 3.1. □

As an example of the last subcase, suppose we have the  $R$  sequence

$$(3, 2, 2, 2) \rightarrow (3, 3, 2, 2) \rightarrow (3, 3, 3, 2) \rightarrow (4, 3, 3, 2) \rightarrow (4, 3, 3, 2, 1) \rightarrow (4, 3, 3, 3, 1)$$

with Yamanouchi symbol  $r_1 \dots r_5 = 23154$ . Let  $k = 1$  so  $r = 2$  and consider

$$T = \begin{matrix} 1 & 1 & 1 & 1 & \bar{1} & 1 \\ 2 & 3 & 3 & 3 & 3 & \bar{3} \\ 4 & 4 & \bar{4} & 4 & \bar{5} \end{matrix} .$$

Then  $(i + 1, j) = (2, 6)$  and  $s = 2$  with  $r + 1, r + 2 = 3, 4$  so

$$T' = \begin{matrix} 1 & 1 & 1 & 1 & \bar{1} & 1 \\ 2 & \bar{2} & 2 & 2 & 2 & 2 \\ \bar{3} & 3 & 3 & 4 & \bar{5} \end{matrix} .$$

Column reading the barred elements of  $T'$  and inserting  $r + 2 = 4$  after  $r + 1 = 3$  gives the Yamanouchi symbol 15234 corresponding to the  $R'$  sequence

$$(3, 2, 2, 2) \rightarrow (4, 2, 2, 2) \rightarrow (4, 2, 2, 2, 1) \rightarrow (4, 3, 2, 2, 1) \rightarrow (4, 3, 3, 2, 1) \rightarrow (4, 3, 3, 3, 1)$$

and  $k' = 5$ .

#### 4. MULTIPLICATION RULES FOR CAPELLI OPERATORS AND QUANTUM IMMANANTS

Let  $E_{ij}, i, j = 1, \dots, n$ , denote the standard basis of the general linear Lie algebra  $\mathfrak{gl}(n)$ . Denote by  $Z(\mathfrak{gl}(n))$  the center of the universal enveloping algebra  $U(\mathfrak{gl}(n))$ . Given  $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{C}^n$  consider a  $\mathfrak{gl}(n)$ -module  $L(\kappa)$  of highest weight  $\kappa$ . That is,  $L(\kappa)$  is generated by a nonzero vector  $v$  such that

$$\begin{aligned} E_{ii} \cdot v &= \kappa_i v, & \text{for } i = 1, \dots, n; \\ E_{ij} \cdot v &= 0 & \text{for } 1 \leq i < j \leq n. \end{aligned}$$

Any element  $z \in Z(\mathfrak{gl}(n))$  acts as a scalar  $\omega(z) = \omega_\kappa(z)$  in  $L(\kappa)$  and this scalar is independent of the choice of the highest weight module  $L(\kappa)$ . Moreover,  $\omega(z)$  is

a symmetric polynomial in the shifted variables  $x_1, \dots, x_n$  where  $x_i = \kappa_i + n - i$ . The mapping  $z \mapsto \omega(z)$  defines an algebra isomorphism

$$\omega : Z(\mathfrak{gl}(n)) \rightarrow \mathbb{C}[x]^{S_n}$$

called the *Harish-Chandra isomorphism*; see e.g. Dixmier [D, Section 7.4].

For any positive integer  $m$  consider the natural action of the complex Lie group  $GL(n)$  in the algebra  $\mathcal{P}$  of polynomials on the vector space  $\mathbb{C}^n \otimes \mathbb{C}^m$ . The corresponding Lie algebra  $\mathfrak{gl}(n)$  then acts by differential operators

$$\pi(E_{ij}) = \sum_{a=1}^m x_{ia} \partial_{ja},$$

where the  $x_{ia}$  are the coordinates on  $\mathbb{C}^n \otimes \mathbb{C}^m$  and the  $\partial_{ia}$  are the corresponding partial derivatives. This representation is uniquely extended to an algebra homomorphism

$$\pi : U(\mathfrak{gl}(n)) \rightarrow \mathcal{PD}$$

where  $\mathcal{PD}$  is the algebra of polynomial coefficient differential operators in the  $x_{ia}$ . The image of  $Z(\mathfrak{gl}(n))$  under  $\pi$  is contained in the subalgebra  $\mathcal{PD}^G$  of differential operators invariant with respect to the action of the group  $G = GL(n) \times GL(m)$ . Moreover, if  $m \geq n$ , then this restriction is an algebra isomorphism which can be called the *Capelli isomorphism*; see [H, HU] for further details. So if  $m \geq n$ , we have the triple isomorphism

$$\mathbb{C}[x]^{S_n} \xleftarrow{\omega} Z(\mathfrak{gl}(n)) \xrightarrow{\pi} \mathcal{PD}^G.$$

Distinguished bases in the three algebras which correspond to each other under these isomorphisms were constructed in [O1] (see also [N, O2, M3]).

In the algebra  $\mathbb{C}[x]^{S_n}$  the basis is formed by the polynomials  $s_\lambda(x|a)$  with  $l(\lambda) \leq n$  and the sequence  $a$  specialized to  $a_i = i - 1$  for all  $i \in \mathbb{Z}$ . We shall denote these polynomials by  $s_\lambda^*(x)$ . Explicitly [BL],

$$s_\lambda^*(x) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - T(\alpha) - c(\alpha) + 1),$$

where  $T$  runs over all semistandard tableaux of shape  $\lambda$  with entries in  $\{1, \dots, n\}$ . We shall denote by  $f_{\lambda\mu}^\nu$  the coefficient  $c_{\lambda\mu}^\nu(a)$  in this specialization of  $a$ . In other words, the  $f_{\lambda\mu}^\nu$  can be defined by the formula

$$s_\lambda^*(x) s_\mu^*(x) = \sum_\nu f_{\lambda\mu}^\nu s_\nu^*(x).$$

To describe the basis in  $Z(\mathfrak{gl}(n))$  we introduce some more notation. Given  $k$  matrices  $A, B, \dots, C$  of size  $p \times q$  with entries from an algebra  $\mathcal{A}$  we regard their tensor product  $A \otimes B \otimes \dots \otimes C$  as an element

$$\sum A_{a_1 i_1} B_{a_2 i_2} \dots C_{a_k i_k} \otimes e_{a_1 i_1} \otimes e_{a_2 i_2} \otimes \dots \otimes e_{a_k i_k} \in \mathcal{A} \otimes (\text{Mat}_{pq})^{\otimes k},$$

where  $\text{Mat}_{pq}$  denotes the space of complex  $p \times q$ -matrices and the  $e_{ai}$  are the standard matrix units. The symmetric group  $S_k$  acts in a natural way in the tensor space  $(\mathbb{C}^n)^{\otimes k}$ , so that we can identify permutations from  $S_k$  with elements of the algebra  $(\text{Mat}_{nn})^{\otimes k}$ .

For a diagram  $\lambda$  with  $l(\lambda) \leq n$  denote by  $T_0$  the  $\lambda$ -tableau obtained by filling in the cells by the numbers  $1, \dots, k = |\lambda|$  from left to right in successive rows starting from the first row. We let  $R_\lambda$  and  $C_\lambda$  denote the row symmetrizer and column

antisymmetrizer of  $T_0$  respectively. By  $c_\lambda(r)$  we denote the content of the cell occupied by  $r$ . Introduce the matrix  $E = (E_{ij})$  whose  $ij$ -th entry is the generator  $E_{ij}$  and set

$$\mathbb{S}_\lambda = \frac{1}{h(\lambda)} \operatorname{tr}(E - c_\lambda(1)) \otimes \cdots \otimes (E - c_\lambda(k)) \cdot R_\lambda C_\lambda,$$

where the trace is taken over all the tensor factors  $\operatorname{Mat}_{nn}$ , and  $h(\lambda)$  is the product of the hook-lengths of the cells of  $\lambda$ :

$$h(\lambda) = \prod_{\alpha \in \lambda} h_\alpha.$$

The elements  $\mathbb{S}_\lambda$  with  $l(\lambda) \leq n$  form a basis in the algebra  $Z(\mathfrak{gl}(n))$ . In [O1] they were called the *quantum immanants*.

Let us now describe the basis in the algebra  $\mathcal{PD}^G$ . The representation  $\pi$  can be written in a matrix form as follows:

$$\pi : E \mapsto XD^t,$$

where  $X$  and  $D$  are the  $n \times m$  matrices formed by the coordinates  $x_{ia}$  and the derivatives  $\partial_{ia}$ , respectively, while  $D^t$  is the matrix transposed to  $D$ . We introduce the following differential operators:

$$\Delta_\lambda = \frac{1}{k!} \operatorname{tr} X^{\otimes k} \cdot (D^t)^{\otimes k} \cdot \chi^\lambda,$$

where  $\chi^\lambda$  is the irreducible character of  $S_k$  corresponding to  $\lambda$ . Explicitly,

$$\Delta_\lambda = \frac{1}{k!} \sum_{i_1, \dots, i_k} \sum_{a_1, \dots, a_k} \sum_{s \in S_k} \chi^\lambda(s) x_{i_1 a_1} \cdots x_{i_k a_k} \partial_{i_{s(1)} a_1} \cdots \partial_{i_{s(k)} a_k}.$$

The operators  $\Delta_\lambda$  with  $l(\lambda) \leq n$  form a basis in  $\mathcal{PD}^G$ . They are called the *higher Capelli operators*.

The following identities were proved in [O1] (for other proofs see [N, O2, M3]):

$$\omega(\mathbb{S}_\lambda) = s_\lambda^*(x) \quad \text{and} \quad \pi(\mathbb{S}_\lambda) = \Delta_\lambda.$$

Using Theorem 3.1 we obtain the following multiplication rules for the elements  $\mathbb{S}_\lambda$  and the operators  $\Delta_\lambda$ .

**Theorem 4.1.** *We have*

$$\mathbb{S}_\lambda \mathbb{S}_\mu = \sum_\nu f_{\lambda\mu}^\nu \mathbb{S}_\nu$$

and

$$\Delta_\lambda \Delta_\mu = \sum_\nu f_{\lambda\mu}^\nu \Delta_\nu$$

where the coefficients  $f_{\lambda\mu}^\nu$  are given by

$$(18) \quad f_{\lambda\mu}^\nu = \sum_{T \in \mathcal{T}(\lambda, \nu/\mu)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{ unbarred}}} (\rho(\alpha)_{T(\alpha)} + n - 2T(\alpha) - c(\alpha) + 1)$$

with  $R$ ,  $T(\lambda, \nu/\mu)$ , and  $\rho(\alpha)$  defined in Theorem 3.1. □

Proposition 3.3 enables us to obtain another formula for  $f_{\lambda\mu}^\nu$ . For a skew diagram  $\nu/\mu$  let

$$h(\nu/\mu) = \frac{|\nu/\mu|!}{\dim \nu/\mu},$$

where  $\dim \nu/\mu$  is the number of standard  $\nu/\mu$ -tableaux. In particular, if  $\mu$  is empty;  $h(\nu)$  is the product of the hook-lengths of the cells of  $\nu$ .

In the specialization of the sequence  $a$  under consideration we obtain from (12) and (13) that

$$H(\mu, \rho) = \frac{1}{h(\rho/\mu)} \quad \text{and} \quad H'(\rho, \nu) = \frac{(-1)^{|\nu/\rho|}}{h(\nu/\rho)}.$$

Moreover, by Proposition 3.2,

$$\frac{s_\lambda(a_\rho|a)}{s_\rho(a_\rho|a)} = \frac{1}{h(\rho/\lambda)}$$

and by the Vanishing Theorem

$$s_\rho(a_\rho|a) = h(\rho).$$

Thus Proposition 3.3 becomes the following:

**Proposition 4.2.** *One has the formula*

$$(19) \quad f_{\lambda\mu}^\nu = \sum_{\lambda, \mu \subseteq \rho \subseteq \nu} (-1)^{|\nu/\rho|} \frac{h(\rho)}{h(\nu/\rho)h(\rho/\lambda)h(\rho/\mu)}. \quad \square$$

Formula (19) implies the following symmetry property of the coefficients  $f_{\lambda\mu}^\nu$ .

**Corollary 4.3.** *If  $l(\lambda^t), l(\mu^t), l(\nu^t) \leq n$ , then*

$$f_{\lambda^t\mu^t}^{\nu^t} = f_{\lambda\mu}^\nu.$$

*Proof.* This follows immediately from the relation  $h(\nu^t/\mu^t) = h(\nu/\mu)$ . □

*Remarks.* 1. It follows from (18) that the coefficients  $f_{\lambda\mu}^\nu$  are integers while the summands on the right hand side of (19) need not be. In fact the numbers  $h(\nu/\mu)$  need not be integers either, e.g.,  $h((3, 2)/(1)) = 24/5$ .

2. Note that since in the case of  $|\nu| = |\lambda| + |\mu|$  the  $f_{\lambda\mu}^\nu$  coincide with the classical Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ , the latter can be computed using (19) as well, but this does not appear to be very useful for practical purposes. For example, consider  $\lambda = \mu = (1^n)$  and  $\nu = (2^r 1^{n-r})$ , then (19) gives

$$f_{\lambda\mu}^\nu = \sum_{k=0}^r (-1)^{r-k} \frac{(n+1)!}{k!(r-k)!(n-k+1)}$$

while, directly from (18), we get

$$(20) \quad f_{\lambda\mu}^\nu = (n-r)!$$

As a final example, take  $m = n$  in the definition of  $\Delta_\lambda$ . Then for  $\lambda = (1^n)$  we get the classical Capelli operator [C]:

$$\Delta_{(1^n)} = \det X \det D.$$

We find from (18) that the coefficients  $f_{(1^n)(1^n)}^\nu$  are zero except for  $\nu = (2^r 1^{n-r})$ ,  $r = 0, 1, \dots, n$ . So by (20) the square of the Capelli operator is given by

$$(\det X \det D)^2 = \sum_{r=0}^n (n-r)! \Delta_{(2^r 1^{n-r})}.$$

#### ACKNOWLEDGMENT

We would like to thank G. D. James, A. N. Kirillov, A. Lascoux and S. Sahi for valuable discussions.

#### REFERENCES

- [BL] L. Biedenharn and J. Louck, A new class of symmetric polynomials defined in terms of tableaux, *Advances in Appl. Math.* **10** (1989), 396–438. MR **91c**:05189
- [C] A. Capelli, Sur les opérations dans la théorie des formes algébriques, *Math. Ann.* **37** (1890), 1–37.
- [D] J. Dixmier, “Algèbres enveloppantes,” Gauthier-Villars, Paris, 1974. MR **58**:16803a
- [FG] S. Fomin and C. Greene, A Littlewood–Richardson miscellany, *European J. Combin.* **14** (1993), 191–212. MR **94b**:05213
- [GG] I. Goulden and C. Greene, A new tableau representation for supersymmetric Schur functions, *J. Algebra* **170** (1994), 687–703. MR **96f**:05187
- [H] R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–570. MR **90h**:22015a
- [HU] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, *Math. Ann.* **290** (1991), 569–619. MR **92j**:17004
- [JP] G. D. James and M. H. Peel, Specht series for skew representations of symmetric groups, *J. Algebra* **56** (1979), 343–364. MR **80h**:20021
- [KR] A. N. Kirillov and N. Yu. Reshetikhin, The Bethe ansatz and the combinatorics of Young tableaux, *J. Soviet Math.* **41** (1988), 925–955.
- [L1] A. Lascoux, Puissances extérieures, déterminants et cycles de Schubert, *Bull. Soc. Math. France* **102** (1974), 161–179. MR **51**:529
- [L2] A. Lascoux, “Interpolation,” Lectures at Tianjin University, June 1996.
- [LS] A. Lascoux and M.-P. Schützenberger, Interpolation de Newton à plusieurs variables, in “Séminaire d’Algèbre P. Dubreil et M.-P. Malliavin, 36ème année (Paris, 1983–1984),” Lecture Notes in Math., Vol. 1146, Springer-Verlag, New York, NY, 1985, 161–175. MR **88h**:05020
- [LR] D. E. Littlewood and A. R. Richardson, Group characters and algebra, *Philos. Trans. Roy. Soc. London Ser. A* **233** (1934), 49–141.
- [M1] I. G. Macdonald, “Symmetric functions and Hall polynomials,” 2nd edition, Oxford University Press, Oxford, 1995. MR **96h**:05207
- [M2] I. G. Macdonald, Schur functions: theme and variations, in “Actes 28-e Séminaire Lotharingien,” I.R.M.A., Strasbourg, 1992, 5–39. MR **95m**:05245
- [M3] A. Molev, Factorial supersymmetric Schur functions and super Capelli identities, in “A. A. Kirillov Seminar on Representation Theory,” S. Gindikin, ed., Amer. Math. Soc. Transl., Amer. Math. Soc., Providence, 1998, 109–137. CMP 98:12
- [N] M. Nazarov, Yangians and Capelli identities, in “A. A. Kirillov Seminar on Representation Theory,” S. Gindikin, ed., Amer. Math. Soc. Transl., Amer. Math. Soc., Providence, 1998, 139–163. CMP 98:2
- [O1] A. Okounkov, Quantum immanants and higher Capelli identities, *Transformation Groups* **1** (1996), 99–126. MR **97j**:17010
- [O2] A. Okounkov, Young basis, Wick formula, and higher Capelli identities, *Int. Math. Research Notes* (1996), 817–839. MR **98b**:17009
- [OO] A. Okounkov and G. Olshanski, Shifted Schur functions, *St. Petersburg Math. J.* **9** (1997), no. 2; q-alg/9605042.
- [S1] B. E. Sagan, “The symmetric group: representations, combinatorial algorithms, and symmetric functions,” 2nd edition, Springer-Verlag, New York, to appear.

- [S2] S. Sahi, The spectrum of certain invariant differential operators associated to a Hermitian symmetric space, in “Lie Theory and Geometry,” J.-L. Brylinski, R. Brylinski, V. Guillemin, V. Kac eds., Progress in Math., Vol. 123, Birkhäuser, Boston, 1994, 569–576. MR **94d**:43013
- [V] S. Veigneau, “Calcul symbolique et calcul distribué en combinatoire algébrique, Ph.D. thesis, Université de Marne-la-Vallée, Marne-la-Vallée, 1996.
- [Z] A. V. Zelevinsky, A generalization of the Littlewood–Richardson rule and the Robinson–Schensted–Knuth correspondence, *J. Algebra* **69** (1981), 82–94. MR **82j**:20028

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, SYDNEY, NSW 2006,  
AUSTRALIA

*E-mail address:* `alexm@maths.usyd.edu.au`

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN  
48824-1027

*E-mail address:* `sagan@math.msu.edu`