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A LOCAL LAGRANGE INTERPOLATION METHOD BASED ON C^1 CUBIC SPLINES ON FREUDENTHAL PARTITIONS

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ABSTRACT. A trivariate Lagrange interpolation method based on C^1 cubic splines is described. The splines are defined over a special refinement of the Freudenthal partition of a cube partition. The interpolating splines are uniquely determined by data values, but no derivatives are needed. The interpolation method is local and stable, provides optimal order approximation, and has linear complexity.

§1. INTRODUCTION

Let $\mathcal{V} := \{\eta_i\}_{i=1}^n$ be a set of points in $\mathbb{R}^3$. In this paper we are interested in the following problem.

Problem 1. Find a tetrahedral partition $\triangle$ whose set of vertices includes $\mathcal{V}$, an N-dimensional space $\mathcal{S}$ of trivariate splines defined on $\triangle$ with a prescribed smoothness r, and a set of additional points $\{\eta_i\}_{i=n+1}^N$ such that for every choice of the data $\{z_i\}_{i=1}^N$, there is a unique spline $s \in \mathcal{S}$ satisfying

$$s(\eta_i) = z_i, \qquad i = 1, \ldots, N. \tag{1.1}$$

We call $P := \{\eta_i\}_{i=1}^N$ and $\mathcal{S}$ a *Lagrange interpolation pair*.

It is easy to solve this problem using C^0 splines; see Remark 1. In this case no additional interpolation points are needed. However, the situation is much more complicated if we want to use C^r splines with $r \geq 1$. For $r = 1$ the problem was first solved in [29] for the special case where the points in $\mathcal{V}$ are the vertices of a cube partition. The method uses quintic splines, and provides full approximation power six for sufficiently smooth functions. In [23] we recently described two C^1 methods which solve the problem for arbitrary initial point sets $\mathcal{V}$. The first method is based on quadratic splines, and requires that each tetrahedron be split into 24 subtetrahedra. The second method is based on cubic splines, and requires that each tetrahedron be split into 12 subtetrahedra. Both methods provide approximation order three for sufficiently smooth functions. For the quadratic spline case this is the optimal approximation order, but for the cubic case it is suboptimal.

The aim of this paper is to describe a new method based on cubic C^1 splines which yields *optimal approximation order four*. To achieve this, we restrict ourselves

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to the case where the initial points $\mathcal{V}$ are the vertices of a cube partition; see Section 2 and Remark 2. The construction involves the following steps: First we create the Freudenthal tetrahedral partition associated with the cube partition. Next we decompose the cube partition into five classes of cubes, and color the tetrahedra of the Freudenthal partition black or white in such a way that tetrahedra with a common face have different colors. Based on this decomposition, we refine some of the tetrahedra using so-called partial Worsey-Farin splits. Then we choose the set of interpolation points P to include $\mathcal{V}$ as well as certain points on the edges and faces of the Freudenthal partition. We show that our interpolation method has the following properties:

1) The method is local in the sense that the value of a spline interpolant s at a point η depends only data values associated with points near η.
2) The method is stable in the sense that small changes in the data values result in small changes in s.
3) The method yields optimal approximation order four.
4) The computational complexity of the method is linear in the number of points in $\mathcal{V}$.

The paper is organized as follows. In Section 2 we classify the cubes, define Freudenthal partitions, describe a coloring of the tetrahedra, and introduce partial Worsey-Farin splits. In Section 3 we give an algorithm for constructing our Lagrange interpolation pair and state our main theorem. We devote Section 4 to a review of the basic Bernstein–Bézier theory which is essential for our analysis. Two useful lemmas about bivariate splines are proved in Section 5, while in Section 6 we discuss interpolation with spaces of C^1 cubic splines defined on partial Worsey-Farin splits. We give a proof of the main theorem in Section 7, and establish error bounds for the corresponding interpolation operator in Section 8. In Section 9 we give a formula for the dimension of our spline space and compare it to other related spline spaces. We conclude the paper with remarks in Section 10.

§2. Freudenthal partitions

Let n be an odd integer, and let

$$Q_{ijk} := [ih, (i+1)h] \times [jh, (j+1)h] \times [kh, (k+1)h],$$

for each $i, j, k = 0, \dots, n-1$, where $h = 1/n$. Let

$$\diamondsuit = \bigcup_{i,j,k=0}^{n-1} \{Q_{ijk}\}. \tag{2.1}$$

We call $\diamondsuit$ the *uniform cube partition* of the unit cube $\Omega = [0,1]^3$.

Let $\mathcal{V} := \{v_{ijk} := (ih, jh, kh)\}_{i,j,k=0}^{n}$ be the set of $(n+1)^3$ vertices of $\diamondsuit$. Our aim in this paper is to construct a Lagrange interpolating pair P and $\mathcal{S}$ based on this initial point set $\mathcal{V}$. We will choose $\mathcal{S}$ to be a space of C^1 cubic splines defined over an appropriate tetrahedral partition $\triangle$ associated with $\diamondsuit$. As a first step towards defining $\triangle$, we describe a special tetrahedral partition associated with a uniform cube partition.

Definition 2.1. Let $\triangle_{\mathcal{F}}$ be the tetrahedral partition obtained from $\diamondsuit$ as follows: for each $0 \le i, j, k \le n-1$, intersect the cube Q_{ijk} with the three planes

$$y - x = (j-i)h, \qquad z - x = (k-i)h, \qquad z - y = (k-j)h.$$

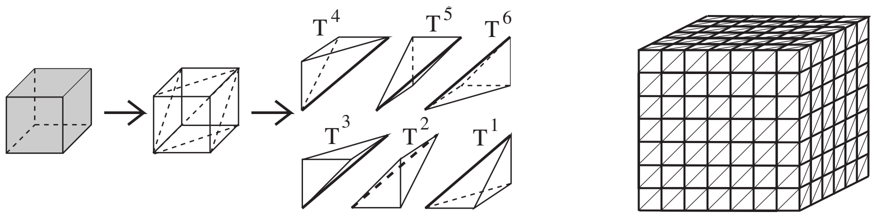
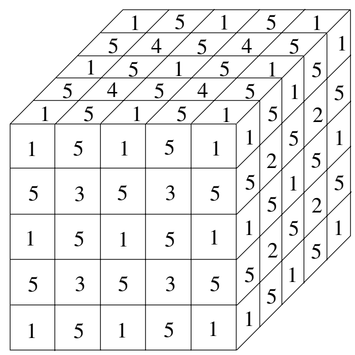


FIGURE 1. The Freudenthal partition is obtained by splitting each cube in a uniform cube partition into six tetrahedra. It can be regarded as a generalization of the classical bivariate three-direction mesh.



class	i	j	k
$\mathcal{K}_1$	even	even	even
$\mathcal{K}_2$	even	odd	odd
$\mathcal{K}_3$	odd	even	odd
$\mathcal{K}_4$	odd	odd	even

FIGURE 2. Classifying cubes.

We call $\triangle_{\mathcal{F}}$ the *Freudenthal partition of* Ω.

It is clear from this definition that each cube Q_{ijk} in $\diamondsuit$ is split into six tetrahedra. For each $0 \le i, j, k \le n-1$, we write

$$T^1_{ijk} := \langle v_{ijk}, v_{i+1,j,k}, v_{i+1,j+1,k}, v_{i+1,j+1,k+1} \rangle$$

for the tetrahedron with the listed vertices. We number the remaining tetrahedra as $T^2_{ijk}, \ldots, T^6_{ijk}$ in clockwise order as shown in Figure 1. Note that all six of these tetrahedra share the edge e_{ijk} connecting v_{ijk} to $v_{i+1,j+1,k+1}$.

It turns out (cf. the dimension result in [13]) that $\triangle_{\mathcal{F}}$ is not a sufficiently fine triangulation to allow the construction of a Lagrange interpolating pair using C^1 cubic splines defined over $\triangle_{\mathcal{F}}$, at least not with the desirable properties listed in the Introduction. Instead, we define $\mathcal{S}$ over a refined partition $\triangle$ obtained by splitting certain tetrahedra of $\triangle_{\mathcal{F}}$ into subtetrahedra. As a first step towards defining $\triangle$, we begin by sorting the cubes in $\diamondsuit$ into five different classes. This step is motivated

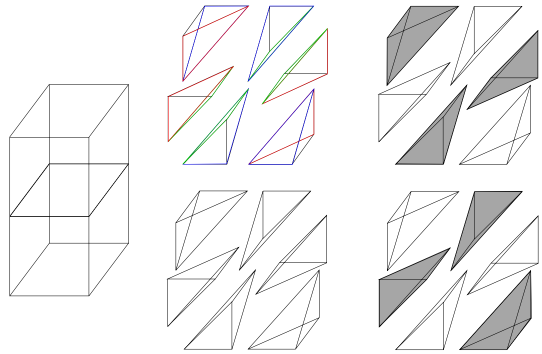


FIGURE 3. Black and white coloring of the tetrahedra of $\triangle_{\mathcal{F}}$.

by our earlier work on Lagrange interpolation with both bivariate and trivariate splines [18]–[19],[21]–[26].

Given a cube $Q_{ijk} \in \Diamond$, we classify it according to the parity of its subscripts. In particular, if the triple i, j, k fits one of the cases in the table in Figure 2, then we assign Q_{ijk} to the corresponding class $\mathcal{K}_1, \ldots, \mathcal{K}_4$. All cubes not covered by the table are assigned to the class $\mathcal{K}_5$. These are precisely the cubes in $\Diamond$ with an odd number of odd subscripts. Figure 2 shows the result of this classification process for the case $n = 5$. It is easy to check that

1) for each $i = 1, \ldots, 4$, any two cubes in the same class $\mathcal{K}_i$ are disjoint;
2) cubes in different classes $\mathcal{K}_1, \ldots, \mathcal{K}_4$ can touch each other only along an edge of $\Diamond$;
3) cubes in class $\mathcal{K}_5$ can touch each other only along an edge of $\Diamond$.

Our next step is to separate the tetrahedra of $\triangle_{\mathcal{F}}$ into two disjoint classes. As a means of visualizing where tetrahedra are located within the cubes, we assign two different colors to them. For all i, j, k, ℓ such that $i+j+k+\ell$ is odd, we define T^{ℓ}_{ijk} to be black. All other tetrahedra are defined to be white. Note that in each cube Q_{ijk}, three of the tetrahedra are black, and three are white, see Figure 3, where the cubes on the top and bottom correspond to $i + j + k$ odd and even, respectively. Moreover, black tetrahedra share triangular faces only with white tetrahedra, and vice versa. This is an immediate consequence of König's theorem (see e.g. [14]) since the number of tetrahedra in $\triangle_{\mathcal{F}}$ touching each other at an interior vertex v_{ijk} of $\Diamond$ is always even. It will be essential for our interpolation method that two intersecting black (white) tetrahedra only touch each other along a common edge or at a common vertex.

We now introduce some schemes for splitting individual tetrahedra in $\triangle_{\mathcal{F}}$.

Definition 2.2. Let T be a tetrahedron, and let v_T be its barycenter. Given an integer $0 \le m \le 4$, let $F_1, \ldots, F_m$ be distinct faces of T, and for each $i = 1, \ldots, m$, let v_{F_i} be a point in the interior of F_i. Then we define the *m-th order partial Worsey-Farin split $\triangle^m_{WF}$ of T* to be the tetrahedral partition obtained by the following steps:

1) connect v_T to each of the four vertices of T;
2) connect v_T to the points v_{F_i} for $i = 1, \ldots, m$;
3) connect v_{F_i} to the three vertices of F_i for $i = 1, \ldots, m$.

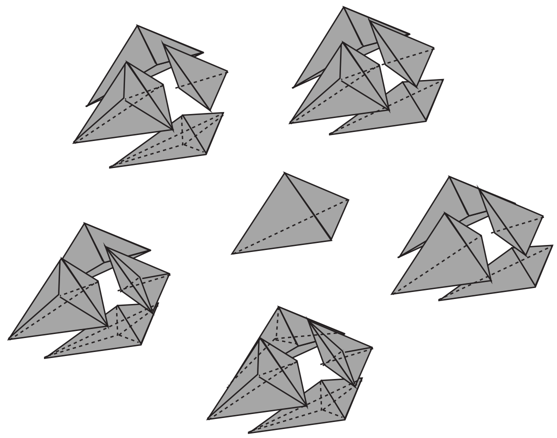


FIGURE 4. Partial Worsey-Farin splits subdivide a given tetrahedron into $4, 6, 8, 10$, or 12 subtetrahedra.

It is easy to see that the m-th order partial Worsey-Farin split of a tetrahedron results in $4 + 2m$ subtetrahedra. When $m = 0$, T is split into four subtetrahedra. This split was introduced in [1], and following [17], we refer to it as the *Alfeld split* of T. When $m = 4$, T is split into twelve tetrahedra. This split was introduced in [34], and following [17], we refer to it as the *Worsey-Farin split* of T. We illustrate the five different partial Worsey-Farin splits in Figure 4, where Alfeld's split is on the right, and m-th order Worsey-Farin splits are presented in counterclockwise order for $m = 1, \ldots, 4$.

§3. Constructing a Lagrange interpolating pair

Let $\diamondsuit$ be a cube partition with vertices $\mathcal{V}$, and let $\triangle_{\mathcal{F}}$ be the Freudenthal partition associated with $\diamondsuit$. Suppose the cubes in $\diamondsuit$ have been separated into classes $\mathcal{K}_1, \ldots, \mathcal{K}_5$, and that the tetrahedra of $\triangle_{\mathcal{F}}$ are colored as described in Section 2. We now give an algorithm to construct an associated Lagrange interpolation pair. The algorithm will involve applying partial Worsey-Farin splits to some tetrahedra of $\triangle_{\mathcal{F}}$. To uniquely define such splits (cf. Definition 2.2), we have to specify points v_F on the faces to be split. We adopt the following principle for tetrahedra which are to be split:

1) if F is a face of $\triangle_{\mathcal{F}}$ that is shared by two tetrahedra T and $\widetilde{T}$ in $\triangle_{\mathcal{F}}$, choose v_F to be the intersection of F with the line connecting the barycenters v_T and $v_{\tilde{T}}$ of T and $\widetilde{T}$;
2) otherwise, choose v_F to be the barycenter of F.

Algorithm 3.1. 1) *Define all edges of $\triangle_{\mathcal{F}}$ to be "unmarked", put the points of $\mathcal{V}$ in P, and set $\triangle := \triangle_{\mathcal{F}}$ and $\mathcal{T} := \emptyset$.*
2) *For each edge $e := \langle u, v\rangle$ of a cube $Q_{ijk} \in \mathcal{K}_1$, add the points $(2u + v)/3$ and $(u + 2v)/3$ to P.*
3) *For $\ell = 1, 2, 3, 4$,*
For each cube $Q_{ijk} \in \mathcal{K}_\ell$,
For $m = 1, 3, 5$,

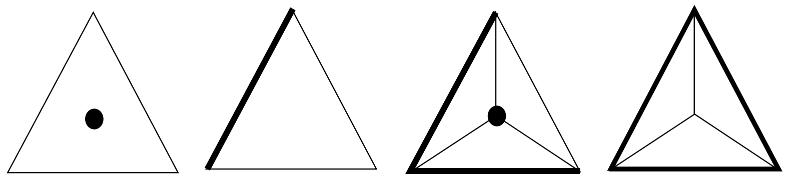


FIGURE 5. Algorithm 3.1 inserts new interpolation points in the interior of the faces of tetrahedra in two cases only. Thick lines indicate edges which have already been marked at the time step 3b) of the algorithm is carried out.

a) *If $T := T^m_{ijk}$ has faces with either two or three marked edges, then create a partial Worsey-Farin split of T based on splitting these faces, and replace T in $\triangle$ by the subtetrahedra resulting from the split of T.*
b) *Add v_F to P for each face F of T that has exactly two marked edges or does not contain any marked edges.*
c) *Mark all edges of T and add it to $\mathcal{T}$.*

4) *For each cube $Q_{ijk} \in \mathcal{K}_5$ and for each $m = 2, 4, 6$, carry out* 3a) − 3c).
5) *For each white tetrahedron $T \in \triangle_{\mathcal{F}}$, create a partial Worsey-Farin split of T based on the faces of T that are shared with black tetrahedra in $\triangle_{\mathcal{F}}$ that have been split in the previous steps. If T has an unmarked (boundary) edge e, increase the order of the partial Worsey-Farin split of T by splitting the adjacent faces of e and add v_F to P for these two faces. Add T to $\mathcal{T}$, and update $\triangle$ by replacing T by the new subtetrahedra.*
6) *Set*

$$\mathcal{S} := \{s \in C^1(\Omega) : s|_T \in \mathcal{P}_3 \text{ for all } T \in \triangle\}, \tag{3.1}$$

where $\mathcal{P}_3$ is the space of trivariate polynomials of degree 3.

The most important output of this algorithm is the point set P and the spline space $\mathcal{S}$ which are going to form our Lagrange interpolation pair. However, the algorithm also produces a refinement $\triangle$ of the Freudenthal partition $\triangle_{\mathcal{F}}$, and an ordering $\mathcal{T}$ of the tetrahedra in $\triangle_{\mathcal{F}}$. The refinement is constructed in such a way as to make the final interpolation method local. The ordering plays a crucial order in the proof that P and $\mathcal{S}$ are a Lagrange interpolation pair.

The marking of edges in Algorithm 3.1 plays two roles. First, it helps us decide when to insert new interpolation points into the set P, as illustrated in Figure 5. The marked edges are also used in deciding which faces (if any) of a given tetrahedron should be used to create a partial Worsey-Farin split. It turns out that the algorithm makes use of all five types of partial Worsey-Farin splits; see Remark 4. We are ready to state the main result of this paper.

Theorem 3.2. *P and $\mathcal{S}$ form a Lagrange interpolation pair.*

We give the proof of this theorem in Section 7 below. In Section 8 we show that the corresponding interpolation operator is stable, and develop error bounds for how well it approximates smooth functions. We give a formula for the dimension of $\mathcal{S}$ in Section 9.

§4. Bernstein–Bézier techniques

We recall some standard Bernstein–Bézier notation. For more details, see [17]. Given a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ and an integer d, we define the associated domain points to be $\mathcal{D}_{T,d} := \{\xi^{T,d}_{ijkl} := \frac{(iv_1+jv_2+kv_3+lv_4)}{d}\}_{i+j+k+l=d}$. We say that the domain point $\xi^{T,d}_{ijkl}$ is at a *distance* $d-i$ *from the vertex* v_1, with similar definitions for the other vertices. We say that $\xi^{T,d}_{ijkl}$ is at a *distance* $i+j$ *from the edge* $e := \langle v_3, v_4\rangle$, with similar definitions for the other edges of T. If $\triangle$ is a tetrahedral partition of a polygonal domain Ω in $\mathbb{R}^3$, we write $\mathcal{D}_{\triangle,d}$ for the collection of all domain points associated with tetrahedra in $\triangle$, where common points in neighboring tetrahedra are not repeated. Given $\rho > 0$, we refer to the set $D_\rho(v)$ of all domain points in $\mathcal{D}_{\triangle,d}$ which are within a distance ρ of v as the *ball of radius* ρ *around* v. Similarly, we refer to the set $R_\rho(v)$ of all domain points which are at a distance ρ from v as the *shell of radius* ρ *around* v. If e is an edge of $\triangle$, we define the *tube of radius* ρ *around* e to be the set $t_\rho(e)$ of domain points whose distance to e is at most ρ.

Given $0 \le r < d$, let

$$\mathcal{S}^r_d(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d \text{ all } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree d. Now suppose $s \in \mathcal{S}^0_d(\triangle)$. Then for each triangle $T \in \triangle$, $s|_T$ can be written in the *B-form*

$$s|_T := \sum_{i+j+k+\ell=d} c^T_{ijk\ell}\, B^{T,d}_{ijk\ell}, \tag{4.1}$$

where $B^{T,d}_{ijk\ell}$ are the *Bernstein basis polynomials* of degree d associated with T. Conversely, given a collection of such coefficients associated with each triangle, there is a unique associated spline in $\mathcal{S}^0_d(\triangle)$ provided that the coefficients associated with domain points on a face between any two neighboring tetrahedra have common values. Thus, every spline $s \in \mathcal{S}^0_d(\triangle)$ is uniquely determined by the set of coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{\triangle,d}}$.

Since we are interested in C^1 splines, we recall the conditions on the coefficients of a spline $s \in \mathcal{S}^0_d(\triangle)$ that ensure C^1 smoothness. Suppose that $T := \langle v_1, v_2, v_3, v_4\rangle$ and $\widetilde{T} := \langle v_1, v_2, v_3, v_5\rangle$ are two tetrahedra in $\triangle$ that share the *oriented triangular face* $F := \langle v_1, v_2, v_3\rangle$, and suppose $\{c^T_{ijkl}\}_{i+j+k+l=d}$ and $\{c^{\widetilde{T}}_{ijkl}\}_{i+j+k+l=d}$ are the corresponding B-coefficients. Then the polynomials $s|_T$ and $s|_{\widetilde{T}}$ join with C^1 smoothness across the face F if and only if

$$c^T_{i,j,k,1} = \lambda_1\, c^{\widetilde{T}}_{i+1,j,k,0} + \lambda_2\, c^{\widetilde{T}}_{i,j+1,k,0} + \lambda_3\, c^{\widetilde{T}}_{i,j,k+1,0} + \lambda_4\, c^{\widetilde{T}}_{i,j,k,1}, \tag{4.2}$$

for all $i + j + k = d - 1$, where $\lambda_1, \ldots, \lambda_4$ are the barycentric coordinates of v_5 relative to T.

The smoothness condition (4.2) can be used to calculate the B-coefficient on the left provided the B-coefficients on the right are given. Clearly this calculation is stable since the weights $\{\lambda_\nu\}^4_{\nu=1}$ depend only on the smallest angle in the two tetrahedra $T, \widetilde{T}$. Note that if exactly one of the λ_ν vanishes at v_5, then the smoothness conditions in (4.2) degenerate to smoothness conditions of bivariate type. Furthermore, if two of the λ_ν vanish at v_5, then the smoothness conditions in (4.2) degenerate to familiar smoothness conditions for univariate splines.

An important tool for dealing with splines on triangulations and tetrahedral partitions is the concept of determining set; see [17] and references therein. We recall that a subset $\mathcal{M}$ of $\mathcal{D}_{\triangle,d}$ is said to be a *determining set* for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ provided that setting the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of $s \in \mathcal{S}$ determines all other coefficients, or, equivalently if $s \in \mathcal{S}$ and its coefficients satisfy $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $s \equiv 0$. It is known that if $\mathcal{M}$ is a determining set for a spline space $\mathcal{S}$, then $\dim \mathcal{S} \leq \#\mathcal{M}$. A determining set $\mathcal{M}$ is called a *minimal determining set (MDS) for* $\mathcal{S}$ provided that there is no smaller determining set. In this case $\dim \mathcal{S} = \#\mathcal{M}$. A determining set $\mathcal{M}$ for a spline space $\mathcal{S}$ is an MDS if and only if setting the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ determines all coefficients of $s \in \mathcal{S}$ in a consistent way, i.e., such that all smoothness conditions are satisfied.

§5. Two bivariate lemmas

Suppose F is a triangle, and that $\{B_{ijk}^3\}_{i+j+k=3}$ are the bivariate Bernstein basis polynomials of degree three associated with F; see [17]. Then every bivariate polynomial p of degree three can be written uniquely in the B-form

$$p = \sum_{i+j+k=3} c_{ijk} B_{ijk}^3.$$

Lemma 5.1. *Suppose that we are given all of the coefficients c_{ijk} of a bivariate cubic polynomial p except for c_{111}. Then for any given real number z and any point v_F in the interior of F, there exists a unique c_{111} so that $p(v_F) = z$.*

Proof. The interpolation condition gives

$$B_{111}^3(v_F)\, c_{111} = z - {\sum}'_{i+j+k=3} c_{ijk} B_{ijk}^3(v_F),$$

where the prime on the sum indicates that the term $i = j = k = 1$ is to be skipped. This equation can be uniquely solved for c_{111} since $B_{111}^3(v_F) \neq 0$. If we take v_F to be the barycenter of F, then $B_{111}^3(v_F) = 2/9$ regardless of the size and shape of F, and the computation of the coefficient c_{111} is a stable process. □

Given a triangle $F := \langle u_1, u_2, u_3\rangle$, let v_F be a point in its interior. Then the well-known *Clough-Tocher split F_{CT} of F* is the triangulation obtained from F by connecting v_F to the three vertices of F. It consists of three subtriangles $F_i := \langle v_F, u_i, u_{i+1}\rangle$, $i = 1, 2, 3$, where we identify $u_4 = u_1$. Our next result concerns interpolation using a C^1 bivariate cubic spline defined on the Clough-Tocher split of a triangle F. Suppose s is such a spline, and that $\{c_{ijk}^1\}_{i+j+k=3}$ are its B-coefficients corresponding to the domain points associated with the subtriangle F_1.

Lemma 5.2. *Suppose that we are given all of the coefficients of $s \in \mathcal{S}_3^1(F_{CT})$ except for $c_{300}^1, c_{210}^1, c_{201}^1, c_{111}^1$. Then for any given real number z, there exists a unique choice of these coefficients so that $s(v_F) = z$.*

Proof. We set $c_{300}^1 = z$ which ensures that $s(v_F) = s(\xi_{300}^1) = z$. Suppose r, s, t are the barycentric coordinates of v_F relative to F. Then taking account of the C^1 smoothness conditions across the interior edges of F_{CT}, the remaining three unknown coefficients must satisfy the linear system

$$\begin{pmatrix} 1 & -s & 0 \\ 0 & -r & 1 \\ r & 0 & s \end{pmatrix} \begin{pmatrix} c_{210}^1 \\ c_{111}^1 \\ c_{201}^1 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix},$$

where the R_i are combinations of known coefficients. This system has a unique solution since the determinant of the matrix is $-2rs \neq 0$. The stability of this computation depends on the size of r and s, which in turn depends on the location of v_F within F. $\square$

§6. Interpolation by C^1 cubic splines on Partial Worsey-Farin splits

Throughout this section we suppose that $0 \le m \le 4$, and that $\triangle_{WF}^m$ is the m-th order Worsey-Farin split of a tetrahedron $T := \langle v_1, v_2, v_3, v_4 \rangle$ based on the split point v_T. We begin with a useful technical lemma.

Lemma 6.1. *For all $0 \le m \le 4$, $\mathcal{S}_2^1(\triangle_{WF}^m) \equiv \mathcal{P}_2$.*

Proof. Let $\mathcal{M}$ be the set of ten domain points lying on the edges of T. We claim that $\mathcal{M}$ is a determining set for $\mathcal{S}_2^1(\triangle_{WF}^m)$. To see this, suppose s is a spline in $\mathcal{S}_2^1(\triangle_{WF}^m)$ whose coefficients corresponding to the domain points $\xi \in \mathcal{M}$ are set to zero. Then using the C^1 smoothness conditions, it is easy to see that all other coefficients of s must be zero. Now the fact that $\mathcal{M}$ is a determining set implies $\dim \mathcal{S}_2^1(\triangle_{WF}^m) \le \#\mathcal{M} = 10$. But since $\mathcal{P}_2$ is a ten dimensional subspace of $\mathcal{S}_2^1(\triangle_{WF}^m)$, we conclude that $\mathcal{S}_2^1(\triangle_{WF}^m) \equiv \mathcal{P}_2$. $\square$

Corollary 6.2. *Suppose $s \in \mathcal{S}_3^1(\triangle_{WF}^m)$. Then $s \in C^2(v_T)$, i.e., all of the polynomial pieces on subtetrahedra sharing the vertex v_T have common derivatives at v_T up to order 2.*

Proof. We can regard the coefficients of s associated with domain points in the ball $D_2(v_T)$ as coefficients of a spline $g \in \mathcal{S}_2^1(\triangle_{WF}^m)$. But Lemma 6.1 implies $g \in \mathcal{P}_2$, and so $g \in C^2(v_T)$. Since derivatives of s and g up to order 2 at v_T match, the claim follows. $\square$

Theorem 6.3. *Fix $0 \le m \le 4$, and let $\mathcal{D}$ be the set of domain points associated with cubic splines defined on $\triangle_{WF}^m$. Let $\mathcal{M}$ be the union of the following sets of points in $\mathcal{D}$:*

1) *for each $i = 1, \ldots, 4$, $D_1(v_i) \cap T_i$ for some tetrahedron $T_i \in \triangle_{WF}^m$ containing v_i;*
2) *for each face F of T that is not split, the point ξ_{111}^F;*
3) *for each face F of T that is split, the points $\{\xi_{111}^{F_i}\}_{i=1}^3$, where F_1, F_2, F_3 are the subfaces of F.*

Then $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_3^1(\triangle_{WF}^m)$.

Proof. We have to show that if $s \in \mathcal{S}_3^1(\triangle_{WF}^m)$, then setting its coefficients corresponding to domain points in $\mathcal{M}$ consistently determines all remaining coefficients, i.e., in such a way that all smoothness conditions are satisfied. First, for each $1 \le i \le 4$, set the coefficients corresponding to $D_1(v_i) \cap T_i$. Then by the C^1 smoothness, all coefficients corresponding to domain points in the balls $D_1(v_i)$ are determined. Now if F is a face of T that is not split, then setting the coefficient c_{111}^F determines all coefficients of s associated with domain points in F. Similarly, if F is a split face of T, then setting the three coefficients associated with the domain points $\{\xi_{111}^{F_i}\}_{i=1}^3$ uniquely determines all other coefficients of s corresponding to domain points in the face F since this corresponds to the standard bivariate Clough-Tocher macro-element; see [10], [16], [17].

At this point we have determined all coefficients of s corresponding to domain points on the shell $R_3(v_T)$. But then the C^1 smoothness conditions across the interior faces of $\triangle_{WF}^m$ can be used to compute all coefficients of s corresponding to domain points on the edges of the shell $R_2(v_T)$. Now the fact that $s \in C^2(v_T)$ coupled with the proof of Lemma 6.1 shows that all remaining coefficients of s are consistently determined. □

Since the set $\mathcal{M}$ in Theorem 6.3 is a minimal determining set, it follows that $\dim \mathcal{S}_3^1(\triangle_{WF}^m) = \#\mathcal{M} = 20 + 2m$.

§7. Proof of Theorem 3.2

Let $\{z_\xi\}_{\xi \in P}$ be a given set of real numbers. We show how to compute the B-coefficients of a spline $s \in \mathcal{S}$ so that $s(\xi) = z_\xi$ for all $\xi \in P$. We begin with coefficients corresponding to domain points located at the vertices of $\triangle_{\mathcal{F}}$ which are just the points of $\mathcal{V}$. For each $\xi \in \mathcal{V}$, we choose the B-coefficient c_ξ of s to be z_ξ. This ensures $s(\xi) = z_\xi$ for all $\xi \in \mathcal{V}$. Next for each edge e of a cube in class $\mathcal{K}_1$, we determine the coefficients corresponding to the two domain points ξ, η in the interior of e by using the fact that s reduces to a univariate cubic polynomial on e. Since the set P contains these two points, we get two equations $s(\xi) = z_\xi$ and $s(\eta) = z_\eta$ for determining the coefficients c_ξ and c_η. Since the points ξ and η are equally spaced on e, we get the same 2×2 nonsingular system for every edge. Now for each vertex of v of $\triangle_{\mathcal{F}}$, we use the C^1 continuity at v to compute all remaining coefficients of s corresponding to domain points in the ball $D_1(v)$. At this point we have determined all coefficients corresponding to domain points on the edges of $\triangle_{\mathcal{F}}$.

To compute the remaining coefficients of s, we work through the tetrahedra of $\triangle_{\mathcal{F}}$ in the order prescribed by the list $\mathcal{T}$ created in Algorithm 3.1. By the nature of Algorithm 3.1, all black tetrahedra appear in the list $\mathcal{T}$ before any white tetrahedra. Recall that two black tetrahedra cannot share a face. Suppose T_ℓ is a tetrahedron in $\mathcal{T}$, and that we have already computed the coefficients of s on all tetrahedra $T_1, \ldots, T_{\ell-1}$. We now show how to compute the coefficients of s corresponding to domain points in T_ℓ. Note that we already know the coefficients corresponding to domain points on the edges of T_ℓ. Moreover, for each edge e that is shared with a tetrahedron in the list $T_1, \ldots, T_{\ell-1}$, we can use the C^1 smoothness around e to compute coefficients of s corresponding to domain points in T_ℓ that lie in the tube $t_1(e)$.

Case 1: T_ℓ is a black tetrahedron not split by Algorithm 3.1. In this case we have to compute the four coefficients corresponding to domain points at the barycenters of the faces of T_ℓ. Let F be a face of T_ℓ. If F shares an edge e with one of the tetrahedra $T_1, \ldots, T_{\ell-1}$, then we already know the coefficient c_{111}^F since it corresponds to the domain point ξ_{111}^F in the tube $t_1(e)$. If F does not share an edge with any of the tetrahedra $T_0, \ldots, T_{\ell-1}$, then P contains a point $v_{_F}$ in the interior of F; see Figure 5. In this case we use Lemma 5.1 to determine the coefficient c_{111}^F from the interpolation condition at $v_{_F}$.

Case 2: T_ℓ is a black tetrahedron that was subjected to a partial Worsey-Farin split. Let $v_{_{T_\ell}}$ be its barycenter. We first compute the coefficients of s corresponding to domain points on the shell $R_3(v_{_{T_\ell}})$, i.e., on the outer faces of T_ℓ. Let F be such a face. If none of the edges of F are shared with $T_1, \ldots, T_{\ell-1}$, then F is not split and P contains a point $v_{_F}$ in the interior of F; see Figure 5. In this case we can use Lemma 5.1 to determine the coefficient c^F_{111}. If exactly one of the edges of F is shared, then again F is not split, and the coefficient c^F_{111} is already determined by C^1 smoothness around that edge. If exactly two of the edges of F are shared with tetrahedra in the list $T_1, \ldots, T_{\ell-1}$, then the face F was subjected to a Clough-Tocher split as described in Section 5, and P contains the split point $v_{_F}$ see Figure 5. In this case we can use Lemma 5.2 to compute the remaining coefficients of s associated with domain points on F. Finally, if all three edges of F are shared with tetrahedra in the list $T_1, \ldots, T_{\ell-1}$, then we already know the coefficients associated with the barycenters of the three subtriangles making up the Clough-Tocher split of F (since they lie in tubes of radius 1 around shared edges). But this corresponds to the classical Clough-Tocher macro-element, see [17], and all coefficients associated with domain points on F are consistently determined. We have now computed all coefficients on the shell $R_3(v_{_{T_\ell}})$. By the argument in the proof of Theorem 6.3, all remaining coefficients of s corresponding to domain points in the ball $D_2(v_{_{T_\ell}})$ are consistently determined.

Case 3: T_ℓ is a white tetrahedron. As in Case 2, to determine the coefficients of s corresponding to domain points in T_ℓ, it suffices to do so for the coefficients on the outer faces. Suppose F is such a face. If F is shared with a black tetrahedron, then the coefficients of s corresponding to domain points on F are already determined by C^0 continuity. Now suppose F is not shared with a black tetrahedron. In this case we can argue exactly as in Case 2.

To complete the proof we must verify that all C^1 smoothness conditions of $\mathcal{S}$ are satisfied. The only case where this is not immediately clear involves smoothness conditions connecting coefficients on two sides of a face F shared by a black tetrahedron T and a white tetrahedron $\widetilde{T}$. If the face F is not split, it is clear that all C^1 smoothness conditions are satisfied since they either come from C^1 smoothness at a vertex or in a tube around an edge. If the face F is split, the required C^1 smoothness follows from the fact that the point $v_{_F}$ used to create the Clough-Tocher split of F lies on the line segment joining the barycenters v_T and $v_{_{\tilde T}}$ of the two tetrahedra T and $\widetilde{T}$ sharing the face F; see [34]. □

It may be helpful to explain the procedure in the above proof in a little more detail. The first tetrahedron T_1 in the list $\mathcal{T}$ is of the form T^1_{ijk} with $Q_{ijk} \in \mathcal{K}_1$. We already know the coefficients of s corresponding to domain points on the edges of T_1. For each face F of T_1 we have inserted an interpolation point at the barycenter of F, and Lemma 5.1 can be used to compute the corresponding coefficient c^F_{111}; see Figure 6. This gives all the coefficients of $s|_{T_1}$.

The second tetrahedron T_2 in the list $\mathcal{T}$ is of the form T^3_{ijk} with $Q_{ijk} \in \mathcal{K}_1$. We already know the coefficients of s corresponding to domain points on the edges of T_2, and we know that T_2 shares an edge e with T_1. Thus, we not only know the coefficients of $s|_{T_2}$ corresponding to domain points on this edge, but also in the tube $t_1(e)$. This leaves just two undetermined coefficients of $s|_{T_2}$ corresponding to domain points at the barycenters of the two faces that do not contain e. But P contains these two points (see Figure 6), and we can again use Lemma 5.1 to compute the associated coefficients.

The third tetrahedron T_3 in the list $\mathcal{T}$ is of the form T^5_{ijk} with $Q_{ijk} \in \mathcal{K}_1$. We already know the coefficients of s corresponding to domain points on the edges of T_3, and we know that T_3 shares an edge e with T_1. Thus, we know all the coefficients of $s|_{T_3}$ except for the two corresponding to domain points at the barycenters of the

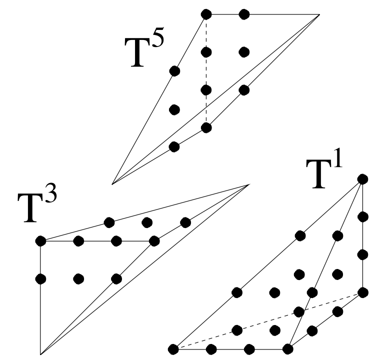


FIGURE 6. Interpolation points (black dots) in the black tetrahedra T^m_{ijk}, $m = 1, 3, 5$, contained in $Q_{ijk} \in \mathcal{K}_1$.

two faces that do not contain e. But P contains these two points (see Figure 6), and we can again use Lemma 5.1 to compute the associated coefficients.

We can repeat these steps to deal with all of the black tetrahedra lying in cubes of class $\mathcal{K}_1$. This is done simultaneously for all cubes in $\mathcal{K}_1$, which is possible since these cubes do not have any vertices in common, and thus there is no interference of the corresponding coefficients with each other; see the proof of Lemma 8.1. Then the next tetrahedron in the list $\mathcal{T}$ has the form $T := T^1_{ijk}$, where the corresponding cube Q_{ijk} is of class $\mathcal{K}_2$. This tetrahedron shares one edge e with a previously processed tetrahedron contained in a cube of class $\mathcal{K}_1$, and so besides knowing the coefficients corresponding to domain points on the edges of T, we also know them for domain points in the tube $t_1(e)$. This determines the coefficients associated with the barycenters of the two faces sharing that edge. The coefficients corresponding to the barycenters of the other two faces of T are determined by Lemma 5.1. The next tetrahedron in the list $\mathcal{T}$ has the form $T := T^3_{ijk}$ with Q_{ijk} of class $\mathcal{K}_2$. This tetrahedron shares one edge with a previously processed tetrahedra, and is treated similarly. The next tetrahedron in the list $\mathcal{T}$ has the form $T := T^5_{ijk}$ with Q_{ijk} of class $\mathcal{K}_2$. This tetrahedron shares exactly two edges with previously processed tetrahedra, and by the algorithm, the face F containing these edges will have been split about a split point that has been inserted in P. In this case Lemma 5.2 can be used to determine all coefficients of s corresponding to domain points lying on F. This process can now be repeated for each of the cubes of class $\mathcal{K}_2$.

The procedure for processing the remaining tetrahedra in the list $\mathcal{T}$ is analogous, but some care is needed to determine exactly which edges are shared with previously processed tetrahedra. We have included Figure 7 as an aid to visualizing which edges these are. Suppose we have completed all of the cubes in class $\mathcal{K}_1$, and now examine a cube Q in class $\mathcal{K}_2$. Then the edges of Q that belong to black tetrahedra in cubes of class $\mathcal{K}_1$ are marked with dark lines in the cube on the left in Figure 7. Similarly, suppose we have completed all of the cubes in class $\mathcal{K}_1$ and $\mathcal{K}_2$, and now examine a cube Q in class $\mathcal{K}_3$. Then the edges of Q that belong to black tetrahedra in $\mathcal{K}_1 \cup \mathcal{K}_2$ are marked with dark lines in the second cube from the left in the figure. The figure also shows the situation for cubes in the classes $\mathcal{K}_4$ and $\mathcal{K}_5$. Note that if a cube Q has any edges that are on the outside faces of the domain Ω, then they may not be shared with edges of black tetrahedra in lower classes. This can happen

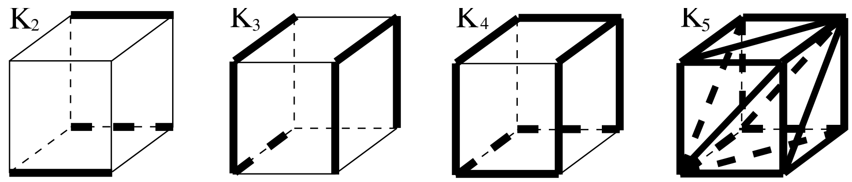


FIGURE 7. Edges shared by a cube Q in class $\mathcal{K}_\nu$ with edges of black tetrahedra in cubes of classes $\mathcal{K}_1, \ldots, \mathcal{K}_{\nu-1}$.

for cubes in the classes $\mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5$. Figure 7 shows only the generic case where none of the edges of Q are on the outside faces of Ω.

§8. Bounds on the error of the interpolant

Let P and $\mathcal{S}$ be the Lagrange interpolation pair constructed in the previous sections. Then for every $f \in C(\Omega)$, there exists a unique spline $\mathcal{I}f \in \mathcal{S}$ such that

$$\mathcal{I}f(\xi) = f(\xi), \qquad \xi \in P.$$

This defines a linear projector $\mathcal{I}$ mapping $C(\Omega)$ onto $\mathcal{S}$. In order to give an error bound for $f - \mathcal{I}f$, we first show that $\mathcal{I}$ is local and stable.

Lemma 8.1. *Fix $f \in C(\Omega)$, and let T be a tetrahedron in $\triangle$. Suppose T lies in the cube Q in the cube partition, and let Ω_T be the union of the (at most 27) cubes that have at least one point in common with Q. Then $\mathcal{I}f|_T$ depends only on values of f in Ω_T, and*

$$\|\mathcal{I}f\|_T \le K \; \|f\|_{\Omega_T}, \tag{8.1}$$

where K is an absolute constant.

Proof. Let $s = \mathcal{I}f$. We first show that the B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{T,3}}$ depend only on values of f in Ω_T. To see this, we review the way in which coefficients are determined. If $\xi \in \mathcal{V}$ is a vertex of the original cube partition, then the associated coefficient is just equal to $f(\xi)$. Coefficients corresponding to domain points lying on edges of cubes in $\mathcal{K}_1$ are determined from data at points on the same edges. These in turn determine coefficients corresponding to domain points in the disks $D_1(v)$ around the vertices of $\triangle_{\mathcal{F}}$. We conclude that any coefficient corresponding to a domain point on an edge of $\triangle_{\mathcal{F}}$ can depend only on data values at points in Ω_T.

Now consider coefficients on the faces of black tetrahedra. They are computed in three ways: from a C^1 smoothness condition around an edge, or using either Lemma 5.1 or 5.2. Thus, a coefficient in one black tetrahedron can influence the value of a coefficient in a neighboring one, leading to a certain propagation. However, due to the ordering imposed on the tetrahedra by Algorithm 3.1, and the fact that black tetrahedra do not share faces, there can be no propagation from a black tetrahedron in a cube in class $\mathcal{K}_i$ to another black tetrahedron in a different cube in the same class. Thus, any propagation from a cube $\mathcal{K}_i$ is limited to a sequence of neighboring cubes, each in a different class. There can also be propagation into a white tetrahedron, but no propagation from one white tetrahedron to another. It follows from the way in which the classes of cubes are defined that for every $\xi \in T$, c_ξ depends only on values of f in Ω_T, see also Figure 1.

We now turn to stability. In view of the fact that the Bernstein basis polynomials associated with T are nonnegative and form a partition of unity, it suffices to show that the coefficients c_ξ of $s|_T$ satisfy

$$|c_\xi| \le K \|f\|_{\Omega_T}, \tag{8.2}$$

for all domain points $\xi \in \mathcal{D}_{T,3}$. If $\xi \in \mathcal{V}$, then (8.2) clearly holds with $K = 1$. Now consider $\xi \notin \mathcal{V}$. Suppose $\xi \in \mathcal{D}_{T,3}$ lies on the shell $R_1(v)$ for some vertex v of $\triangle_{\mathcal{F}}$. Then c_ξ can be computed from the gradient of s at v. This is a stable computation since the angles in the partition $\triangle_{\mathcal{F}}$ are bounded away from zero by an absolute constant independent of n. On the other hand, the gradient of s at v can be stably computed from B-coefficients of s corresponding to domain points on the edges of $\diamondsuit$ attached to v. Next, suppose c_ξ is a coefficient that is computed from Lemma 5.1. In this case c_ξ is stably determined from the coefficients associated with the other domain points in T (which have already been determined) coupled with the value $f(v_F)$, where v_F is the barycenter of a face F of a tetrahedron T in $\triangle_{\mathcal{F}}$. Finally, suppose c_ξ is a coefficient that is computed from Lemma 5.2. Then, ξ lies on a face of a tetrahedron in $\triangle_{\mathcal{F}}$ which has been subjected to a Clough-Tocher split. Since $\triangle_{\mathcal{F}}$ is a Freudenthal partition, it is easy to see (cf. [13]) that the common vertices of any two tetrahedra $T := \langle v_1, v_2, v_3, v_4\rangle$ and $\widetilde{T} := \langle v_1, v_2, v_3, v_5\rangle$ in $\triangle_{\mathcal{F}}$ can be arranged such that the relation $v_5 = v_1 + v_2 - v_4$ is satisfied. Hence, the point v_F where the line segment through the barycenters $v_T = (v_1 + v_2 + v_3 + v_4)/4$ and $v_{\tilde{T}} = (v_1 + v_2 + v_3 + v_5)/4$ intersects the triangular face $F = \langle v_1, v_2, v_3\rangle$ is

$$v_F = (v_T + v_{\tilde{T}})/2 = (v_1 + v_2 + v_3)/4 + (v_4 + v_5)/8 = (3v_1 + 3v_2 + 2v_3)/8.$$

It follows that the split point v_F has barycentric coordinates $(3/8, 2/8, 3/8)$ in all cases, and so the inverse of the matrix appearing in the proof of Lemma 5.2 is bounded by an absolute constant. The remaining coefficients of s corresponding to domain points in $\mathcal{D}_T$ are determined from those that we have aleady computed by applying smoothness conditions, and thus satisfy (8.2) with a constant depending only on the smallest angle in the faces of the tetrahedra in $\triangle$. Since this angle does not depend on n, K is an absolute constant. $\square$

Given an integer $m \ge 1$ and any compact subset B of Ω, let $W^m_\infty(B)$ be the usual Sobolev space defined on B with seminorm

$$|f|_{m,\infty,B} := \sum_{|\alpha|=m} \|D^\alpha f\|_B. \tag{8.3}$$

Here α is a multi-index of length $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$, and $\|.\|_B$ denotes the infinity norm on B.

We now give an error bound for the interpolation operator $\mathcal{I}$ which shows in particular that the C^1 cubic spline space $\mathcal{S}$ has full approximation power. Let $|\triangle|$ be the *mesh size* of $\triangle$, i.e., the maximum diameter of the tetrahedra in $\triangle$.

Theorem 8.2. *Suppose $f \in W^{m+1}_\infty(\Omega)$ for some $0 \le m \le 3$. Then there exists an absolute constant K such that*

$$\|D^\alpha (f - \mathcal{I}f)\|_\Omega \le K\ |\triangle|^{m+1-|\alpha|} |f|_{m+1,\infty,\Omega}, \tag{8.4}$$

for all multi-indices α with $0 \le |\alpha| \le m$.

Proof. The proof is similar to the proof of Theorem 6.3 in [29]; see also Theorem 9 in [23]. Fix m, and let $f \in W_\infty^{m+1}(\Omega)$. Fix $T \in \triangle$, and let Ω_T be as in Lemma 8.1. We first show that

$$\|D^\alpha(f - \mathcal{I}f)\|_T \leq K_1 |\Omega_T|^{m+1-|\alpha|} |f|_{m+1,\infty,\Omega_T}, \tag{8.5}$$

where α is a multi-index with $0 \leq |\alpha| \leq m$. By Lemma 4.3.8 of [9], there exists a cubic polynomial p such that

$$\|D^\beta(f - p)\|_{\Omega_T} \leq K_2\, |\Omega_T|^{m+1-|\beta|} |f|_{m+1,\infty,\Omega_T}, \tag{8.6}$$

for all $0 \leq |\beta| \leq m$, where $|\Omega_T|$ is the diameter of Ω_T. Since $\mathcal{I}p = p$, it follows that

$$\|D^\alpha(f - \mathcal{I}f)\|_T \leq \|D^\alpha(f - p)\|_T + \|D^\alpha \mathcal{I}(f - p)\|_T.$$

In view of (8.6) with $\beta = \alpha$, it suffices to estimate the second term. By the Markov inequality [36] and (8.1),

$$\|D^\alpha \mathcal{I}(f - p)\|_T \leq K_3\, |T|^{-\alpha} \|\mathcal{I}(f - p)\|_T \leq K_4\, |T|^{-\alpha} \|f - p\|_{\Omega_T},$$

where $|T|$ is the diameter of T. Because of the geometry of the partition, $|\Omega_T| \leq K_5 |T|$ and $|T| \leq K_6 |\triangle|$ for some absolute constants K_5 and K_6. Now inserting (8.6) for $\beta = 0$, and combining the above, we get (8.5). Taking the maximum over all tetrahedra in $\triangle$ leads to (8.4). $\square$

§9. Dimension of the space $\mathcal{S}$

Suppose P and $\mathcal{S}$ are the interpolation pair constructed above. For comparison with other spline interpolation methods, we now count the number of points in P, which by Theorem 3.2 is also the dimension of $\mathcal{S}$.

Theorem 9.1. *For all odd integers n,*

$$\dim \mathcal{S} = \#P = 8n^3 + 18n^2 + 15n + 5.$$

Proof. To count the number of points in P, we count the number of cubes in each of the classes $\mathcal{K}_\nu$ of the cube partition $\diamondsuit$, and then count the number of points of P in a given cube, but not in any cube already counted. Unfortunately, this is nontrivial since cubes on the boundary of Ω are special. We summarize our counts in Table 1. The first column describes the type of cube. Here we have introduced the notation L,R,F,B,T, and b to stand for left, right, front, back, top, and bottom, respectively. We use the letter O to stand for "other". With this notation, cubes in the class 1Bb are those of class $\mathcal{K}_1$ that lie along the edge along the back and bottom of Ω. The classes in this table are disjoint, so for example a cube in 5F is not in any of the classes 5LF, 5RF, 5FT, or 5Fb.

There are six classes of cubes where some white tetrahedron has an edge that is on the boundary of Ω and is not shared with any edge of a black tetrahedra. These are 1Bb, 1LT, 1RF, 5LB, 5Rb, and 5FT. In each of these cases Algorithm 3.1 inserts two extra points in the set P. We emphasize the counting of these extra points by writing +2 in the column labelled n_i.

For each class we give the number of such cubes n_c, the number of points n_i in each cube of this class (but not in any cube higher in the table), and the total number of points $n_t := n_c \times n_i$ in cubes of this class. Adding the numbers in the second column gives the total number of cubes n^3. Adding the numbers in the last column of Table 1 gives us $\#P$. $\square$

TABLE 1. The number of cubes n_c contained in each class, the number n_i of points in each cube of that class, and the total number $n_t := n_c \times n_i$ of points in all cubes of the given class.

class	n_c	n_i	n_t
1Bb	$(n+1)/2$	40+2	$21(n+1)$
1LT	$(n+1)/2$	40+2	$21(n+1)$
1RF	$(n+1)/2$	40+2	$21(n+1)$
1O	$(n^3+3n^2-9n-11)/8$	40	$5n^3+15n^2-45n-55$
2	$(n^3-n^2-n+1)/8$	6	$3(n^3-n^2-n+1)/4$
3F	$(n^2-2n+1)/4$	6	$3(n^2-2n+1)/2$
3B	$(n^2-2n+1)/4$	8	$2(n^2-2n+1)$
3O	$(n^3-5n^2+7n-3)/8$	6	$3(n^3-5n^2+7n-3)/4$
4T	$(n^2-2n+1)/4$	4	n^2-2n+1
4b	$(n^2-2n+1)/4$	6	$3(n^2-2n+1)/2$
4O	$(n^3-5n^2+7n-3)/8$	4	$(n^3-5n^2+7n-3)/2$
5LF	$(n-1)/2$	4	$2(n-1)$
5LB	$(n-1)/2$	6+2	$4(n-1)$
5LT	$(n-1)/2$	4	$2(n-1)$
5Lb	$(n-1)/2$	6	$3(n-1)$
5RF	$(n-1)/2$	2	$n-1$
5RB	$(n-1)/2$	4	$2(n-1)$
5RT	$(n-1)/2$	4	$2(n-1)$
5Rb	$(n-1)/2$	4+2	$3(n-1)$
5FT	$(n-1)/2$	4+2	$3(n-1)$
5Fb	$(n-1)/2$	4	$2(n-1)$
5BT	$(n-1)/2$	6	$3(n-1)$
5Bb	$(n-1)/2$	4	$2(n-1)$
5L	$(n^2-4n+3)/2$	4	$2(n^2-4n+3)$
5R	$(n^2-4n+3)/2$	2	n^2-4n+3
5F	$(n^2-4n+3)/2$	2	n^2-4n+3
5B	$(n^2-4n+3)/2$	4	$2(n^2-4n+3)$
5b	$(n^2-4n+3)/2$	4	$2(n^2-4n+3)$
5T	$(n^2-4n+3)/2$	4	$2(n^2-4n+3)$
5O	$(n^3-6n^2+12n-7)/2$	2	$n^3-6n^2+12n-7$

The result of Theorem 9.1 can be compared with the dimensions of C^1 cubic splines on other tetrahedral partitions of the uniform cube partition $\diamondsuit$. It was shown in [13] that $\dim \mathcal{S}_3^1(\triangle_{\mathcal{F}}) = 12n^2 + 18n + 4$. Furthermore, for type-6 tetrahedral partitions $\triangle_6$ (each cube Q is subdivided in 24 congruent tetrahedra which have a common vertex at the center of Q), it is known [12] that $\dim \mathcal{S}_3^1(\triangle_6) = 6n^3+24n^2+18n^2+4$. Hence, compared with these two spaces, $\mathcal{S}$ has more degrees of freedom when $n > 1$. On the other hand, if we apply the full Worsey-Farin split to each tetrahedron of $\triangle_{\mathcal{F}}$, then it follows from [34] and some elementary computations that the dimension of the space of C^1 cubic splines on the resulting partition is equal to $18n^3 + 30n^2 + 18n + 4$. Our space $\mathcal{S}$ has a much lower dimension while still providing full approximation power.

§10. Remarks

Remark 1. Suppose $\mathcal{V}$ is an arbitrary set of points in $\mathbb{R}^3$, and that $\triangle$ is some tetrahedral partition with vertices $\mathcal{V}$. Set $P := \mathcal{V}$, and let $\mathcal{S} := \mathcal{S}_1^0(\triangle)$ be the space of continuous linear splines. Then clearly P and $\mathcal{S}$ form a Lagrange interpolation pair. It is also straightforward to create a Lagrange interpolation pair using C^0 splines of higher degree provided we add an appropriate set of additional interpolation points.

Remark 2. The results in this paper can be extended to more general domains Ω consisting of the union of a set of uniform boxes, although the description of a Lagrange interpolation pair becomes more technical since there are many special cases when tetrahedra have edges on the boundary of Ω. Even the case of a uniform cube partition of the unit cube with n an even integer is more complicated than the odd case presented here. Bivariate results in [21], [25] can help guide the construction.

Remark 3. For an early reference to Freudenthal partitions, see [11].

Remark 4. Algorithm 3.1 makes use of all five types of partial Worsey-Farin splits. For example, a first order split is applied to the tetrahedra T_{ijk}^5 for $Q_{ijk} \in \mathcal{K}_2 \cup \mathcal{K}_3$. The second order split is used on certain tetrahedra T_{ijk}^3 for $Q_{ijk} \in \mathcal{K}_3 \cup \mathcal{K}_4$. The third order split is used on certain tetrahedra T_{ijk}^2 for $Q_{ijk} \in \mathcal{K}_5$. The fourth order split is always applied to the tetrahedra T_{ijk}^4 and T_{ijk}^6, where $Q_{ijk} \in \mathcal{K}_5$. There are also some tetrahedra which are not split at all, for instance the tetrahedra $T_{ijk}^1, T_{ijk}^3, T_{ijk}^5$ when $Q_{ijk} \in \mathcal{K}_1$. For others that are not split, see the table in the following remark.

Remark 5. As an indication of the complexity of the refined partition $\triangle$, we tabulate the number of split faces (nsf) of black tetrahedra in classes $\mathcal{K}_1, \dots, \mathcal{K}_4$. The top row of the table shows the classes of cubes to which the tetrahedra belong. Note that we have separated class 3 into 3F (those in front), and 3O (the remainder). Similarly, we have separated class 4 into 4b (those on the bottom), 4T (those on the top), and 4O (the remainder). The first row shows the indices m of the tetrahedra T_{ijk}^m in each class which have no split faces. The second and third rows show the indices of those that have one or two split faces, respectively.

nsf	1	2	3O	3F	4O	4b	4T
0	1,3,5	1,3	1	1	1	1	1
1		5	5	3,5		5	3,5
2			3		3,5	3	

Remark 6. It was observed in [2],[3] that degrees of freedom can be removed from spline spaces by forcing them to satisfy certain isolated smoothness conditions. This idea can be used to remove the interpolation condition in Lemma 5.2 by imposing one extra C^2-super smoothness condition. This leads to a Lagrange interpolation pair where the point set P has fewer points than the one constructed here, and the corresponding spline space $\mathcal{S}$ has lower dimension. However, the associated interpolation method has almost the same properties as the one presented above.

Remark 7. For the bivariate case, the problem of constructing a Lagrange interpolating pair utilizing smooth splines defined on triangulations has been studied in a number of recent papers [18]–[19], [21]–[22], [24]–[26]. Our most recent paper [19] (see also [18]) deals with the general case of arbitrary smoothness and given initial triangulations. On the other hand, much less is known for the trivariate case; see [23], [29].

Remark 8. Spline spaces where the restriction of a spline to a (refined) tetrahedron T is determined from Hermite data at points in T are called *macro-element spaces*. Trivariate macro-elements have been constructed in [1], [4]–[6], [15], [29]–[35]. Except for the spaces discussed in [29], all of these constructions require Hermite data, and cannot be modified to work with Lagrange data. Recently, some local quasi-interpolation methods that are based on trivariate splines and require only Lagrange data have been constructed; see [20], [27], [33]. These methods differ from the method discussed here in that no data is interpolated. On the other hand, a common feature is that the methods are based on piecewise polynomials of low polynomial degree, which is important for applications such as the contouring three-dimensional data, also called iso-surfacing.

Remark 9. The fact that C^1 cubic splines on a partial Worsey-Farin split automatically have supersmoothness C^2 at the split point v_T (see Corollary 6.2) generalizes immediately to C^1 spaces of splines of arbitrary degree $q \geq 3$.

Remark 10. It can be shown using the same kind of arguments as in Lemma 6.1 that for the Alfeld split $\triangle^0_{WF}$ of a tetrahedron, $\mathcal{S}^1_3(\triangle^0_{WF}) = \mathcal{P}_3$. Following the arguments of Lemma 6.1, it follows that if $s \in \mathcal{S}^1_3(\triangle^0_{WF})$, then s is automatically in $C^3(v_T)$, where v_T is the split point. This fact also holds for all C^1 spaces of arbitrary degree $q \geq 3$ on the Alfeld split.

Remark 11. Trivariate spline spaces are much more difficult to analyze than bivariate spline spaces, where questions of dimension, stable local bases, and approximation power are already very challenging; see [17]. For results on dimension and the construction of bases, see [7], [8], [12], [13], [17]. For a survey on interpolation using bivariate splines, see [24].

Remark 12. It is known that constructing well-behaved tetrahedral partitions of given points in $\mathbb{R}^3$ is nontrivial. Currently available methods in computational geometry are able to solve this problem with algorithmic complexity $\mathcal{O}(n^2 \log(n))$, where n is the number of given data points. In this regard, using uniform type partitions as proposed in this paper has obvious advantages from a computational point of view, since it leads to an interpolation method with linear algorithmic complexity. In order to find Lagrange interpolation pairs starting with arbitrary point sets $\mathcal{V}$, we will have to allow more complicated tetrahedral partitions. We are currently investigating this more general problem.

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