J. DIFFERENTIAL GEOMETRY
54 (2000) 139-176

A LOCAL PROOF OF PETRI'S CONJECTURE AT THE GENERAL CURVE

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Abstract

A proof of Petri's general conjecture on the unobstructedness of linear systems on a general curve is given, using only the local properties of the deformation space of the pair (curve, line bundle).

1. Introduction

Let L_0 denote a holomorphic line bundle of degree d over a compact Riemann surface C_0. The Petri conjecture stated that, if C_0 is a curve of general moduli, the mapping

$$\mu_0 : H^0\left(L_0\right) \otimes H^0\left(\omega_{C_0} \otimes L_0^{\vee}\right) \rightarrow H^0\left(\omega_{C_0}\right)$$

is injective. Later, this assertion was given a more modern interpretation making it a central question in the study of curves and their linear series—what is now called Brill-Noether theory.

To recap the modern formulation we proceed as in [1]. Let $C_0^{(d)}$ denote the d-th symmetric product of C_0 and let $\Delta \subseteq C_0^{(d)} \times C_0$ denote the tautological divisor. Let

$$\mathbb{P}^r = \mathbb{P}\left(H^0\left(L_0\right)\right).$$

For the projection

$$p_* : C_0^{(d)} \times C_0 \rightarrow C_0^{(d)}$$

Received February 4, 2000.

and exact sequence

$$0 \to O_{C_0^{(d)} \times C_0} \to O_{C_0^{(d)} \times C_0} (\Delta) \to O_{C_0^{(d)} \times C_0} (\Delta)\Big|_{\Delta} \to 0,$$

one has that

$$T_{C_0^{(d)}} = p_* \left(O_{C_0^{(d)} \times C_0} (\Delta)\Big|_{\Delta} \right).$$

Applying the derived functor $Rp_* \circ \mathcal{O}_{\mathbb{P}^r \times C_0}$ to the above exact sequence as in (2.6) of [1], one obtains an exact sequence

$$0 \to N_{\mathbb{P}^r \backslash C_0^{(d)}} \to O_{\mathbb{P}^r} \otimes H^1 (O_{C_0}) \to O_{\mathbb{P}^r} (1) \otimes H^1 (L_0) \to 0,$$

where $N_{A \backslash B}$ denotes the normal bundle of A in B. So the dual of the kernel of μ_0 above is exactly

$$H^1 \left(N_{\mathbb{P}^r \backslash C_0^{(d)}} \Big|_{\mathbb{P}^r} \right).$$

Via the standard short exact sequence of normal bundles, Petri's conjecture becomes the assertion

$$H^1 \left(N_{\mathbb{P}^r \backslash C_0^{(d)}} \right) = 0,$$

that is, the deformation theory of linear series is unobstructed at a curve of general moduli.

There are several proofs of Petri's conjecture, proofs via degeneration by Gieseker [7] and Eisenbud-Harris [6] and a proof via specialization to the locus of curves on a general $K3$-surface due to Lazarsfeld [11] (see also [12]). However the only proof based on properties of the infinitesimal deformation of the general curve, as opposed to some specialization of it, is a proof for $r \leq 2$ by Arbarello and Cornalba in [1]. In conversations concerning his joint work with Cornalba, Arbarello explained to the author the viewpoint of [2] that there should exist a generalization to higher order of the following result (which appears both in [2] and [1]):

Let

$$\mathfrak{D}_n (L_0)$$

denote the sheaf of holomorphic differential operators of order $\leq n$ on sections of the line bundle L_0. (If

$$L_0 = \mathcal{O}_{C_0}$$

we denote this sheaf simply as $\mathfrak{D}_n$.) The first-order deformations the pair (L_0, C_0) are in natural one-to-one correspondence with the elements

$$\psi \in H^1\left(\mathfrak{D}_1\left(L_0\right)\right)$$

in such a way that a section s_0 of L_0 deforms to first order with the deformation ψ if and only if the element

$$\psi\left(s_0\right) \in H^1\left(L_0\right)$$

is zero.

Furthermore he pointed out that an appropriate higher-order generalization of this fact and a simple Wronskian argument would immediately yield a "local" proof of Petri's general conjecture at the general curve (see §4 below). The purpose of this paper is to carry out that generalization.

The general idea of the proof is to use the Kuranishi theory of (curvilinear) C^∞-trivializations of deformations of complex manifolds as it applies to the total space the dual line bundle $L_0^\vee$. Roughly speaking, if we denote the t-disk as Δ and are given a C^∞-trivialization

$$F_\sigma = (\sigma, \pi) : M \to M_0 \times \Delta$$

of a deformation M/Δ of a complex manifold M_0, Kuranishi associated to this situation a power series

$$\xi = \xi_1 t + \xi_2 t^2 + \dots ,$$

where each ξ_j is a $(0,1)$-form with coefficients in (a subsheaf of) the tangent bundle of M_0. F_σ is not allowed to be an arbitrary C^∞-isomorphism over Δ. The relevant restriction is that trajectory of each point on M_0 must be holomorphic, that is,

$$\sigma^{-1}\left(x_0\right) \subseteq M$$

must be a holomorphic disk for each $x_0 \in M_0$. This is of course just a restriction on the choice of trivialization; it implies no restriction on the deformation M/Δ. For such a trivialization, the holomorphic functions f on M have a very nice form; namely we can write power-series expansions

$$f \circ F_\sigma^{-1} = f_0 + f_1 t + f_2 t^2 + \dots$$

such that the holomorphicity condition

$$\overline{\partial}_M f = 0$$

becomes just

$$\left(\overline{\partial}_{M_0} - \xi\right)\left(f_0 + f_1 t + f_2 t^2 + \ldots\right) = 0.$$

Although later on we will actually need to consider a slightly more general case in the body of this paper, it is perhaps helpful as an introduction to give the line of reasoning of the paper in the case in which M_0 happens to be the total space of a holomorphic line bundle

$$q_0 : L_0^\vee \to C_0$$

over a compact Riemann surface C_0. One easily sees that the deformation is a deformation of holomorphic line bundles if and only if the Kuranishi data ξ^L are invariant under the action of the $\mathbb{C}^*$-action on $L_0^\vee$. In fact, if χ denotes the $(1,0)$ Euler vector field on $L_0^\vee$ associated with the natural $\mathbb{C}^*$-action on the line bundle, this is just the condition

$$\left[\chi, \xi_j^L\right] = 0$$

for all j, that is, that the ξ_j^L can be written everywhere locally in the form

$$q_0^*(\alpha) \cdot \chi + q_0^*(\beta) \cdot \tau_L \tag{1}$$

where α and β are $(0,1)$-forms on C_0 and τ_L is a lifting of a $(1,0)$-vector-field τ_C on C_0 such that

$$[\chi, \tau_L] = 0.$$

(The "associated" or "compatible" Kuranishi data for the deformation of C_0 is just given by $\xi_j^C = \beta \cdot \tau_C$.) Sections s of L are just functions f on $L_0^\vee$ for which

$$L_\chi(f) = f,$$

where L_χ denotes Lie differentiation with respect to the vector field χ.

Suppose now we have a line-bundle deformation $(L/\Delta, C/\Delta)$ of (L_0, C_0) with compatible trivializations

$$\begin{aligned} \sigma &: C \to C_0, \\ \lambda &: L^\vee \to L_0^\vee, \end{aligned}$$

and a section s of L whose zeros are given by

$$\sigma^{-1}\left(zeros\left(s_0\right)\right).$$

Rescaling λ in the fiber direction we arrive at a trivialization of the deformation $L^{\vee}$ of $L_0^{\vee}$ for which s is constant, that is,

$$s = s_0 \circ \lambda.$$

We call such compatible trivializations of C_0 and $L_0^{\vee}$ "adapted" to the section s.

Of course we have twisted the almost complex structure on C_0 and $L_0^{\vee}$ to achieve this trivialization. To keep track of this twisting, we consider only "Schiffer-type" deformations C of C_0, for which the twist in almost complex structure is given almost everywhere by a gauge transformation, that is, by a power series

$$\beta = \beta_1 t + \beta_2 t^2 + \dots ,$$

where the β_j are C^{∞}-vector-fields of type $(1,0)$ on $C_0 - \{p\}$ and meromorphic in a small analytic neighborhood of p. Then we take

$$\xi^C = \frac{e^{[\beta,\ ]} - 1}{[\beta,\ ]}\left(\overline{\partial}_{C_0}\beta\right)$$

(see [5]) and get a compatible trivialization of $L^{\vee}/\Delta$ by lifting the β_j to vector fields $\tilde{\beta}_j$ on $L_0^{\vee}$ with

$$\left[\tilde{\beta}_j, \chi\right] = 0$$

with the same meromorphic property near $q_0^{-1}(p)$. Holomorphicity of a section s becomes the condition

$$\left(\overline{\partial}_{L_0^{\vee}}\left(e^{L_{-\beta}}(f)\right)\right) = 0$$

on the power series

$$f = f_0 + f_1 t + f_2 t^2 + \dots$$

representing s as a function on $L_0^{\vee} \times \Delta$. That is, the condition is simply that the pull-back of f via the gauge transformation is a power series whose coefficients are meromorphic sections of L_0.

If we have a holomorphic section s of L whose restriction to s_0 has simple zeros D_0, and if $\tilde{\beta}$ is zero in a small analytic neighborhood of

$$D = zero\,(s) \subset C$$

then there is a C^{∞}-automorphism

$$\Phi : C_0 \times \Delta \to C_0 \times \Delta$$

defined over Δ such that:

1. Φ is holomorphic in a small analytic neighborhood of $D \cup \{p\}$.
2. $$\Phi\left(\{x_0\} \times \Delta\right)$$ is a holomorphic disk for each $x_0 \in C_0$.
3. $$\Phi \circ F_{\sigma}\left(D\right) = D_0 \times \Delta.$$

The rough (imprecise) idea is that trivialization $\Phi \circ F_{\sigma}$ can also be considered to be of Schiffer type for some vector field

$$\gamma = \gamma_1 t + \gamma_2 t^2 + \ldots.$$

γ lifts to a vector field $\tilde{\gamma}$ associated to a Schiffer-type trivialization of the deformation $L^{\vee}/\Delta$ of $L_0^{\vee}$ which is adapted to the section s. Since by construction s corresponds to the "constant" power series

$$f_0 + 0 \cdot t + 0 \cdot t^2 + \ldots,$$

we have the equation

$$\left(\overline{\partial}_{L_0^{\vee}}\left(e^{L-\tilde{\gamma}}\left(f_0\right)\right)\right) = 0,$$

that is,

$$\left[\overline{\partial}_{L_0^{\vee}}, e^{L-\tilde{\gamma}}\right]\left(f_0\right) = 0. \tag{2}$$

It is in this way that we produce elements of $H^1\left(\mathfrak{D}_{n+1}\left(L_0\right)\right)$ for all $n \geq 0$ which must annihilate sections s_0 of L_0 which extend to sections of L. (The difficulty is of course that the elements of $H^1\left(\mathfrak{D}_{n+1}\left(L_0\right)\right)$

depend on the choice of s_0. To remedy this we will eventually have to replace the deformation C/Δ of C_0 with the deformation

$$\mathbb{P}/\Delta = \mathbb{P}\left(H^0\left(L/\Delta\right)\right)$$

of $\mathbb{P}\left(H^0\left(L_0\right)\right)$ and replace L with $\mathcal{O}\left(1\right)$.)

As one of the simplest concrete examples, let

$$C_0 = \frac{\mathbb{C}}{\mathbb{Z}+\mathbb{Z}\sqrt{-1}}$$

with linear holomorphic coordinate z on $\mathbb{C}$. For a C^{∞}-function ρ supported on $\{z : |z| \leq 1/8\}$ and identically 1 on $\{z : |z| \leq 1/16\}$, let

$$\begin{aligned}\beta_1 &= \frac{\rho}{z}\cdot\frac{\partial}{\partial z},\\ \beta_j &= 0,\ j>1.\end{aligned}$$

This is a non-trivial deformation since, to first order it is given by the generator

$$\overline{\partial}_{C_0}\left(\frac{\rho}{z}\cdot\frac{\partial}{\partial z}\right) \in H^1\left(T_{C_0}\right).$$

For L_0 we can take the line bundle of degree 2 given by the divisor

$$D_0 = \left\{\frac{1+\sqrt{-1}}{4}\right\} + \left\{\frac{3+3\sqrt{-1}}{4}\right\}$$

with corresponding section s_0. Let s be some extension of the section s_0. For a trivialization

$$F_\sigma : C \to C_0 \times \Delta$$

associated to the above Kuranishi data, the zero set $D = D' + D''$ of the section s is given by two power series

$$\begin{aligned} z &= a\left(t\right) = \frac{1+\sqrt{-1}}{4} + a_1 t + \ldots\ ,\\ z &= b\left(t\right) = \frac{3+3\sqrt{-1}}{4} + b_1 t + \ldots\ ,\end{aligned}$$

since the deformation of (almost) complex structure is zero near D_0. So near D' we recursively solve for

$$\Phi\left(z,t\right) = \left(a'\left(z,t\right),t\right)$$

such that

$$a'\left(a\left(t\right),t\right)\equiv\frac{1+\sqrt{-1}}{4},$$

and similarly near D'' for

$$\Phi\left(z,t\right)=\left(b'\left(z,t\right),t\right)$$

such that

$$b'\left(b\left(t\right),t\right)\equiv\frac{3+3\sqrt{-1}}{4}.$$

Near $\{0\}\times\Delta$ we take

$$\Phi\left(z,t\right)=\left(z,t\right)$$

and then extend Φ to a family of diffeomorphism on all of C_0 by a C^{∞} patching argument. For the new trivialization

$$\Phi\circ F_{\sigma}:C\to C_0\times\Delta$$

the divisor D giving the line bundle L is "constant" so that the pull-back of s_0 via the product structure gives rise to a compatible trivialization of L.

The Petri proof will follow from doing this process (for a line-bundle deformation of L_0 for which all sections extend) for every Schiffer-type variation of a generic curve C_0. We show that the set of equations (2) we obtain implies that the higher μ-maps

$$\mu_{n+1}:\ker\left(\mu_n\right)\to H^0\left(\omega_{C_0}^{n+2}\right)=H^1\left(T_{C_0}^{n+1}\right)$$

are all zero. As Arbarello-Cornalba-Griffiths-Harris showed twenty years ago, this implies Petri's conjecture.

We shall use Dolbeault cohomology throughout this paper. In particular, the sheaf $\mathfrak{D}_n\left(L_0\right)$ has both a left and a right $\mathcal{O}_{C_0}$-module structure and we define

$$A^{0,i}\left(\mathfrak{D}_n\left(L_0\right)\right):=A_{C_0}^{0,i}\otimes_{\mathcal{O}_{C_0}}\mathfrak{D}_n\left(L_0\right)$$

where $A^{0,i}$ is the sheaf of C^{∞}-$(0,i)$-forms. Also the context will hopefully eliminate any confusion between two standard notation used in this paper, namely the notation L and L_0 for line bundles and the notation

$$L_{\tau}^{k}=\underbrace{L_{\tau}\circ\ldots\circ L_{\tau}}_{k-times}$$

where L_τ denotes Lie differentiation with respect to a vector field τ.

The author wishes to thank E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris for the original concept and general framework of this paper, and E. Arbarello and M. Cornalba in particular for many helpful conversations without which this work could not have been completed. Also he wishes to thank the referee and R. Miranda for ferreting out an elusive mistake in a previous version of this paper, E. Casini and C. Hacon for help with the rewrite (especially for pointing me toward Lemma 2.6), and the Scuola Normale Superiore, Pisa, Italia, for its hospitality and support during part of the period of this research.

2. Deformations of manifolds and differential operators

2.1 Review of formal Kuranishi theory

We begin with a brief review of the Newlander-Nirenberg-Kuranishi theory of deformations of complex structures (see [10], [9], II.1 of [8], or [4]). Let

$$M \xrightarrow{\pi} \Delta = \{t \in \mathbb{C} : |t| < 1\} \tag{3}$$

be a deformation of a complex manifold M_0 of dimension m. Since we are doing formal deformation theory, all calculations will actually take place over the formal neighborhood of 0 in Δ. However, convergence will not be an issue in anything that we do since we will always be working from a situation in which we are *given* a geometric deformation and deriving consequences in the category of formal deformations.

Definition 2.1. A C^∞-diffeomorphism

$$F_\sigma = (\sigma, \pi) : M \to M_0 \times \Delta$$

will be called a trivialization of the deformation M/Δ if

$$\sigma|_{M_0} = identity$$

and

$$\sigma^{-1}(x_0)$$

is an analytic disk for each $x_0 \in M_0$.

The next four lemmas are standard from formal Kuranishi theory:

Lemma 2.1. *Let*

$$T^*_{M_0}$$

denote the complexification of the real cotangent bundle of M_0. Given any trivialization F_σ, the holomophic cotangent bundle of M_t under the C^∞-isomorphisms

$$M_t \cong M_0$$

induced by F_σ corresponds to a subbundle

$$T_t^{1,0} \subseteq T^*_{M_0}.$$

If

$$\pi^{1,0} + \pi^{0,1} : T^*_{M_0} \to T^{1,0}_{M_0} \oplus T^{0,1}_{M_0}$$

are the two projections, the retriction

$$\pi^{1,0} : T_t^{1,0} \to T^{1,0}_{M_0}$$

is an isomorphism for small t so that the composition

$$T^{1,0}_{M_0} \overset{\left(\pi^{1,0}\right)^{-1}}{\longrightarrow} T_t^{1,0} \overset{\pi^{0,1}}{\longrightarrow} T^{0,1}_{M_0},$$

gives a C^∞-mapping

$$\xi\left(t\right) : T^{1,0}_{M_0} \to T^{0,1}_{M_0}$$

which determines the deformation of (almost) complex structure.

Thus, at least formally, we can write

$$\xi\left(t\right) = \sum\nolimits_{i>0} \xi_i t^i$$

with each $\xi_i \in A^{0,1}_{M_0}\left(T_{1,0}\right)$, that is, each ξ_i is a $(0,1)$-form with coefficients in the holomorphic tangent bundle $T_{1,0}$ of M_0.

Lemma 2.2. *Every relative complex-valued C^∞ -differential form ω on M/Δ of type $(0,q)$ corresponds on a (formal) neighborhood of M_0 to a form*

$$\pi^{0,q}\left(\left(F_\sigma^{-1}\right)^*\left(\omega\right)\right) = \sum\nolimits_{i,j=0}^{\infty} \omega_{i,j} t^i \bar{t}^j$$

on

$$M_0 \times \Delta$$

and so, working modulo $\bar{t}$, gives a holomorphic family

$$\omega_\sigma := \sum\nolimits_{i=0}^{\infty} \omega_{i,0} t^i.$$

of C^{∞}-forms. This correspondence is a formal isomorphism

$$(\ )_{\sigma}^{q} : \frac{A_{M/\Delta}^{0,q}}{\{\bar{t}\}} \to A_{M_0}^{0,q} \otimes \mathbb{C}[[t]] .$$

If we have two different trivializations σ and σ', we have a formal isomorphism

$$G_{\sigma'\sigma}^{q} = (\ )_{\sigma'}^{q} \circ ((\ )_{\sigma}^{q})^{-1} .$$

Lemma 2.3. *For any C^{∞}-function f on M write*

$$f \circ F_{\sigma}^{-1} = \sum\nolimits_{i,j=0}^{\infty} f_{i,j} t^{i} \bar{t}^{j}$$

and define as above

$$f_{\sigma} = \sum\nolimits_{i,j=0}^{\infty} f_{i,0} t^{i}.$$

Further define

$$\begin{aligned} \bar{D}_{\sigma}(f_{\sigma}) &:= \left(\bar{\partial}_{M_0} - \sum\nolimits_{j=1}^{\infty} t^{j}\xi_{j}\right)(f_{\sigma}) \\ &= \sum\nolimits_{i=0}^{\infty} \bar{\partial}_{M_0} f_{i,\sigma} t^{i} - \sum\nolimits_{i=0,j=1}^{\infty,\infty} \xi_{j}(f_{i,\sigma}) t^{i+j}. \end{aligned}$$

Then

$$(\ )_{\sigma}^{1} \circ \bar{\partial}_{M} = \bar{D}_{\sigma} \circ (\ )_{\sigma}^{0} ,$$

and

$$\bar{D}_{\sigma} = G_{\sigma\sigma'}^{1} \circ \bar{D}_{\sigma'} \circ G_{\sigma'\sigma}^{0}.$$

Also

$$f_{\sigma} \circ F_{\sigma}$$

is holomorphic on M if and only if

$$\bar{D}_{\sigma}(f_{\sigma}) = 0.$$

We next ask which sequences $\xi_j \in A^{0,1}(T_{1,0})$ come from a trivialization of a deformation (3). Before answering this question, we need to make precise the various actions of an element $\xi \in A^{0,k}(T_{1,0})$ on $\sum A^{p,q}(M_0)$. For any ξ we write the action via contraction as

$$\langle \xi \mid \ \rangle ,$$

and "Lie differentiation" as

$$L_{\xi} := \langle \xi \mid \rangle \circ d + (-1)^{k} d \circ \langle \xi \mid \rangle .$$

The sign is so chosen that, writing any element of $A^{0,k}(T_{1,0})$ locally as a sum of terms

$$\xi = \bar{\eta} \otimes \chi$$

for some closed $(0,k)$-form $\bar{\eta}$ and $\chi \in A^{0,0}(T_{1,0})$. Then

$$L_{\xi} = \bar{\eta} \otimes L_{\chi}.$$

(Warning: Since, as an operator on $A^{0,q}(M_0)$, $L_{f\xi} = f L_{\xi}$, one has

$$\left[\overline{\partial}, L_{\xi}\right] = L_{\overline{\partial}\xi} : A^{0,q}(M_0) \rightarrow A^{0,q+k+1}(M_0)$$

however the identity does not hold as an operator on $A^{p,q}(M_0)$ for $p > 0$.)

Also we compute

$$\begin{aligned} L_{\xi} L_{\xi'} - (-1)^{\deg \bar{\eta} \cdot \deg \bar{\eta}'} & L_{\xi'} L_{\xi} \\ &= (\bar{\eta} \otimes L_{\chi}) \left(\bar{\eta}' \otimes L_{\chi'}\right) - (-1)^{\deg \bar{\eta} \cdot \deg \bar{\eta}'} \left(\bar{\eta}' \otimes L_{\chi'}\right) (\bar{\eta} \otimes L_{\chi}) \\ &= \bar{\eta}\bar{\eta}' \left(L_{\chi} L_{\chi'} - L_{\chi'} L_{\chi}\right) \\ &= \bar{\eta}\bar{\eta}' L_{[\chi,\chi']}. \end{aligned}$$

So, using this local presentation for

$$\xi \in A^{0,j}(T_{1,0}), \xi' \in A^{0,k}(T_{1,0}),$$

we can define

$$\left[\xi, \xi'\right] = \bar{\eta}\bar{\eta}' \left[\chi, \chi'\right] \in A^{0,j+k}(T_{1,0}).$$

Lemma 2.4. *The almost complex structures given on a coordinate neighborhood W_0 in M_0 by the the $(0,1)$-tangent distributions*

$$\left(\frac{\partial}{\partial \overline{v_{W_0}^{k}}} - \sum\nolimits_{i=1}^{\infty} \sum\nolimits_{l} h_{i,k}^{l} t^{i} \frac{\partial}{\partial v_{W_0}^{l}} \right)$$

are integrable, that is, come from a deformation/trivialization of M_0 as in Definition (2.1), *if and only if, for*

$$\xi = \sum\nolimits_{i=1}^{\infty} \sum\nolimits_{k,l} d\overline{v_{W_0}^{k}} \wedge h_{i,k}^{l} t^{i} \frac{\partial}{\partial v_{W_0}^{l}},$$

we have

$$\bar{\partial}\xi = \frac{1}{2} [\xi, \xi] .$$

Proposition 2.5. *Two trivializations F_σ and $F_{\sigma'}$ of the same deformation* (3) *are related by a holomorphic automorphism φ of M/Δ, that is, there is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M \\ \downarrow \sigma & & \downarrow \sigma' \\ M_0 & = & M_0 \end{array},$$

if and only if

$$\bar{D}_\sigma = \bar{D}_{\sigma'}.$$

Proof. One implication is immediate from the definitions of $\bar{D}_\sigma$ and $\bar{D}_{\sigma'}$. For the other, the equality

$$\xi_\sigma = \xi_{\sigma'}$$

implies that the differential of the C^∞-automorphism

$$\varphi := (\sigma, \pi)^{-1} \circ (\sigma, \pi) : M \to M$$

preserves the $(1,0)$-subspace of the (complexified) tangent space and therefore φ is holomorphic. q.e.d.

2.2 Gauge transformations

We begin now with a deformation

$$M/\Delta$$

of M_0 and let

$$F_\sigma : M \overset{(\sigma,\pi)}{\longrightarrow} M_0 \times \Delta$$

be a trivialization with associated Kuranishi data

$$\xi_\sigma.$$

Suppose that we have a one-real-parameter group of diffeomorphisms

$$\Phi_s : M_0 \times \Delta \to M_0 \times \Delta$$

defined over Δ such that

$$F_s := \Phi_s \circ F_\sigma : M \overset{(\sigma_s,\pi)}{\longrightarrow} M_0 \times \Delta$$

is a trivialization for each (sufficiently small) s and, for each $x_0 \in M_0$,

$$\Phi_s|_{\{x_0\}\times\Delta}$$

is a real-analytic family of complex-analytic embeddings of Δ in $M_0 \times \Delta$. Then, as for example in §2 of [4], there is then associated a vector field

$$\beta + \overline{\beta}$$

where

$$\beta = \sum\nolimits_{j>0} \beta_j t^j$$

and each β_j is a C^∞-vector field of type $(1,0)$ on M_0, such that, for

$$g = g_0 + g_1 t + \ldots,$$

on $M_0 \times \Delta$ we have

$$g \circ \Phi_s = e^{L_{s\beta+\overline{s\beta}}}(g). \tag{4}$$

We let

$$F_\beta := F_1 = \Phi_1 \circ F_\sigma : M \to M_0 \times \Delta.$$

Then by (4) we have for any C^∞-function g on M that

$$g_\beta = e^{L_\beta}(g_\sigma). \tag{5}$$

If ξ_s denotes the Kuranishi data for the trivialization F_s, then by direct computation

$$\frac{\partial \xi_s}{\partial s} = \left[\overline{\partial}, \beta\right] + [\beta, \xi_s].$$

(See for example Lemma 2.10 of [4].) On the other hand, if we define

$$\varsigma_\beta := \frac{e^{[\beta,\ ]} - 1}{[\beta,\ ]}\left(\left[\overline{\partial}, \beta\right]\right) \tag{6}$$

and the action

$$\xi_\beta := \beta \cdot (\xi) = e^{[\beta,\ ]}(\xi) + \varsigma_\beta, \tag{7}$$

one also has by direct computation that

$$\frac{\partial \xi_{s\beta}}{\partial s} = \left[\overline{\partial}, \beta\right] + [\beta, \xi_{s\beta}].$$

(See for example §3 of [4]. Compare with §3 of [5].) The conclusion is that $\xi_{s\beta}$ is the Kuranishi data for the trivialization F_s for all s and so, in particular

$$\xi_\beta$$

is the Kuranishi data for the trivialization $F_1 = F_\beta$.

So the group of vector fields β acts on the Kuranishi data associated to the deformation M/Δ. This action corresponds to the change of the given trivialization by a C^∞-automorphism

$$\Phi_\beta : M_0 \times \Delta \to M_0 \times \Delta \tag{8}$$

defined over Δ.

Lemma 2.6. *i)*

$$\left(e^{L_\beta}\right)\left(\overline{\partial} - \xi\right)\left(e^{-L_\beta}\right) = \overline{\partial} - \xi_\beta.$$

ii) Given a function

$$f_\beta = \sum\nolimits_i f_{\beta,i} t^i$$

on $M_0 \times \Delta$, the function

$$f_\beta \circ F_\beta$$

is holomorphic on M if and only if

$$\left(\overline{\partial} - \xi\right)\left(e^{-L_\beta}\left(f_\beta\right)\right) = 0.$$

Proof. i) This assertion is implicit in (5) but, as a check, we will do it by direct comptation.

$$\left(e^{L_\beta}\right)\left(\overline{\partial} - \xi\right)\left(e^{-L_\beta}\right) = \overline{\partial} + \left[e^{L_\beta}, \overline{\partial}\right]\left(e^{-L_\beta}\right) - e^{[\beta,\ ]}\left(\xi\right).$$

If we can show the identity

$$\left[\overline{\partial}, e^{L_\beta}\right] = \varsigma_\beta \circ e^{L_\beta}, \tag{9}$$

the lemma will follow from Lemma 2.3 since, by definition,

$$\xi_\beta = e^{[\beta,\ ]}\left(\xi\right) + \varsigma_\beta.$$

To see (9) we prove by induction that

$$\left[\overline{\partial}, \beta^{n+1}\right] = \sum\nolimits_{i=0}^{n} \binom{n+1}{i} \left([\beta,\ ]\right)^{n-i}\left[\overline{\partial}, \beta\right]\beta^i.$$

Inductively

$$\begin{aligned}\left[\overline{\partial},\beta^{n+1}\right] &= \left[\overline{\partial},\beta\right]\cdot\beta^{n}+\beta\cdot\left[\overline{\partial},\beta^{n}\right]\\ &= \left[\overline{\partial},\beta\right]\cdot\beta^{n}+\beta\Bigg(\left(\begin{array}{c}n\\0\end{array}\right)[\beta,\ ]^{n-1}\cdot\left[\overline{\partial},\beta\right]\\ &\qquad\qquad +\ldots+\left(\begin{array}{c}n\\n-1\end{array}\right)\left[\overline{\partial},\beta\right]\cdot\beta^{n-1}\Bigg)\\ &=\Bigg(\left(\begin{array}{c}n\\n\end{array}\right)\left[\overline{\partial},\beta\right]\cdot\beta^{n}\\ &\quad+\left(\begin{array}{c}\left(\begin{array}{c}n\\0\end{array}\right)\left(([\beta,\ ])^{n}\left[\overline{\partial},\beta\right]+([\beta,\ ])^{n-1}\left[\overline{\partial},\beta\right]\beta\right)\\ +\ldots+\\ \left(\begin{array}{c}n\\n-1\end{array}\right)\left(([\beta,\ ])\left[\overline{\partial},\beta\right]\beta^{n-1}+\left[\overline{\partial},\beta\right]\beta^{n}\right)\end{array}\right)\Bigg).\end{aligned}$$

Now use the identity

$$\left(\left(\begin{array}{c}n-1\\r\end{array}\right)+\left(\begin{array}{c}n-1\\r-1\end{array}\right)\right)=\left(\begin{array}{c}n\\r\end{array}\right).$$

Thus

$$\begin{aligned}\left[\overline{\partial},\left(\sum\nolimits_{n=0}^{\infty}\frac{\beta^{n}}{n!}\right)\right] &=\sum\nolimits_{n=1}^{\infty}\frac{1}{n!}\sum\nolimits_{i=0}^{n-1}\left(\begin{array}{c}n\\i\end{array}\right)([\beta,\ ])^{n-1-i}\left[\overline{\partial},\beta\right]\beta^{i}\\ &=\sum\nolimits_{k=0,i=0}^{\infty,\infty}\frac{1}{(k+1)!}([\beta,\ ])^{k}\left[\overline{\partial},\beta\right]\frac{\beta^{i}}{i!}\\ &=\xi_{\beta}\circ\left(\sum\nolimits_{i=0}^{\infty}\frac{\beta^{i}}{i!}\right).\end{aligned}$$

ii)

$$\begin{aligned}0 &= \left[\overline{\partial},e^{L_{\beta}}\circ e^{-L_{\beta}}\right]\\ &= \left[\overline{\partial},e^{L_{\beta}}\right]\circ e^{-L_{\beta}}+e^{L_{\beta}}\circ\left[\overline{\partial},e^{-L_{\beta}}\right]\\ &= \varsigma_{\beta}+e^{L_{\beta}}\circ\varsigma_{-\beta}\circ e^{-L_{\beta}},\end{aligned}$$

so that

$$e^{L_{\beta}}\circ(\varsigma_{-\beta}-\xi)\circ e^{-L_{\beta}}=-\left(\varsigma_{\beta}+\left(e^{L_{\beta}}\circ\xi\circ e^{-L_{\beta}}\right)\right).$$

q.e.d.

Suppose now that we have two trivializations

$$\begin{aligned} F_\sigma &: M \to M_0 \times \Delta, \\ F_{\sigma'} &: M \to M_0 \times \Delta \end{aligned}$$

of a given deformation

$$M/\Delta.$$

Then

$$F_{\sigma'} \circ F_\sigma^{-1}$$

is a C^∞-diffeomorphism of $M_0 \times \Delta$ and so can be realized as the value at $s = 1$ of a one-parameter group of diffeomorphisms which restrict to an analytic family of analytic embbeddings of $\{x_0\} \times \Delta$ for each $x_0 \in M_0$. Thus referring to the notation of Lemma 2.3 above we have that there is a C^∞-vector field κ of type $(1,0)$ such that

$$\begin{aligned} g_\sigma &= e^{L_{-\kappa}}\left(g_{\sigma'}\right) \\ \overline{D}_{\sigma'} &= e^{L_\kappa} \circ \overline{D}_\sigma \circ e^{L_{-\kappa}}. \end{aligned} \tag{10}$$

2.3 Schiffer-type deformations

We now consider a special class of deformations of M_0, those for which the change of complex structure can be localized at a union A_0 of codimension-one subvarieties on M_0. We let

$$\beta \in A^{0,0}_{M_0}\left(T^{1,0}_{M_0}\right) \otimes t\mathbb{C}[[t]] \tag{11}$$

be a vector field which is

i) meromorphic in an analytic neighborhood $(U_0 \times \Delta)$ of the set $(A_0 \times \Delta)$ on $(M_0 \times \Delta)$,

ii) C^∞ on $(M_0 - A_0) \times \Delta$.

Using Lemma 2.6 for the case in which we first take

$$F_\sigma : ((M_0 - A_0) \times \Delta) \to ((M_0 - A_0) \times \Delta) \tag{12}$$

in 2.2 as the identity map, we define a deformation M_β/Δ of M_0 by the integrable Kuranishi data

$$\xi_\beta := \varsigma_\beta. \tag{13}$$

Notice that $\xi_\beta = 0$ in a neighborhood of $A_0 \times \Delta$ so ξ_β corresponds to a trivialization

$$F_\beta : M_\beta \overset{(\sigma_\beta,\pi)}{\longrightarrow} M_0 \times \Delta$$

with
$$F_\beta : (\sigma_\beta)^{-1}(U_0) \to U_0 \times \Delta$$
an analytic isomorphism. Denote

$$\overline{D}_\beta := \overline{D}_{\sigma_\beta} = \overline{\partial} - \varsigma_\beta. \tag{14}$$

We call A_0 the *center* of the Schiffer-type deformation.

Let
$$A_\beta := (\sigma_\beta)^{-1}(A_0) \subseteq M_\beta.$$
From 2.2, Lemma 2.6 and the above we conclude:

Lemma 2.7.
$$f_\beta \circ F_\beta$$
is analytic on M_β if and only if
$$\overline{\partial}_{M_0}\left(e^{-L_\beta}(f_\beta)\right) = 0.$$

In fact, for any divisor B_0 supported on A_0, B_0 has a unique extension to a divisor
$$B_\beta$$
on M_β which is supported on A_β. We denote by
$$C_{B_0}$$
the vector space of functions f_0 which are C^∞ on $(M_0 - A_0)$ and meromorphic on U_0 and for which
$$B_0 + div\,(f_0)$$
is effective on U_0. Then:

Lemma 2.8. *i) A meromorphic function f on M_β with*
$$B_\beta + div\,(f)$$
effective is a formal sum
$$f_\beta := f_{\sigma_\beta} = \sum\nolimits_{i=0}^{\infty} f_{\beta,i}t^i$$
such that each $f_{\beta,i} \in C_{B_0}$ and
$$\left(\overline{\partial} - \sum\nolimits_{j=1}^{\infty} \xi_{\beta,j}t^j\right)\left(\sum\nolimits_{i=0}^{\infty} f_{\beta,i}t^i\right) = 0.$$

ii) The meromorphic functions on M_β *with*

$$B_\beta + div\,(f)$$

effective are given by the kernel of the mapping

$$e^{L_\beta} : H^0\left(\mathcal{O}_{M_0}\left(\infty \cdot A_0\right)\right) \otimes \mathbb{C}[[t]] \to H^0\left(\frac{\mathcal{O}_{M_0}\left(\infty \cdot A_0\right)}{\mathcal{O}_{M_0}\left(B_0\right)}\right) \otimes \mathbb{C}[[t]].$$

iii) If

$$i : A_0 \to M_0$$

is the inclusion map and R *denote the image of the map*

$$\left(i^{-1}\mathcal{O}_{M_0}\left(B_0\right) \otimes t\mathbb{C}[[t]]\right) \overset{\overline{\partial}\circ e^{-L_\beta}}{\longrightarrow} \left(H^1\left(\mathcal{O}_{M_0}\left(B_0\right)\right) \otimes \mathbb{C}[[t]]\right),$$

then $f_0 \in H^0\left(\mathcal{O}_{M_0}\left(B_0\right)\right)$ *extends to a global section of* $\mathcal{O}_{M_\beta}\left(B_\beta\right)$ *if and only if*

$$\left[\overline{\partial}, e^{-L_\beta}\right]\left(f_0\right) \in R.$$

Proof. i) The assertion is immediate from Lemma 2.3.

ii) Again by Lemma 2.6i) occurs exactly when f_β lies in

$$C_{B_0} \otimes \mathbb{C}[[t]] \cap image\left(H^0\left(\mathcal{O}_{M_0}\left(\infty \cdot A_0\right)\right) \otimes \mathbb{C}[[t]] \overset{e^{L_\beta}}{\longrightarrow} C_{\infty \cdot A_0} \otimes \mathbb{C}[[t]]\right).$$

iii) follows from Lemma 2.7 and the cohomology exact sequence associated to the short exact sequence

$$0 \to \mathcal{O}_{M_0}\left(B_0\right) \to \mathcal{O}_{M_0}\left(\infty \cdot A_0\right) \to \frac{\mathcal{O}_{M_0}\left(\infty \cdot A_0\right)}{\mathcal{O}_{M_0}\left(B_0\right)} \to 0.$$

q.e.d.

2.4 Gauge transformation on Schiffer-type trivializations

Next suppose we wish to change our trivialization

$$F_\beta : M_\beta \overset{(\sigma_\beta,\pi)}{\longrightarrow} M_0 \times \Delta$$

by an allowable C^∞-automorphism

$$\begin{array}{ccc} M_\beta & \overset{F_\beta}{\longrightarrow} & M_0 \times \Delta \\ = & & \downarrow \Phi \\ M_\beta & \overset{G}{\longrightarrow} & M_0 \times \Delta \end{array}$$

defined over Δ. That is

1. such that Φ preserves $A_0 \times \Delta$ as a set
2. is holomorphic on $U_0 \times \Delta$.
3. Φ restricts to an analytic embedding of each disk $\{x_0\} \times \Delta$.

To calculate the Kuranishi data for G, we proceed as in 2.2. We can assume that $\Phi = \Phi_1$ for a family Φ_s as in 2.2. We can further assume that $\Phi_s|_{U_0 \times \Delta}$ is a real analytic family of complex analytic maps. Let $\kappa = \sum_{j=1}^{\infty} \kappa_j t^j$ denote the C^∞ -vector field of type $(1,0)$ such that the family Φ_s is associated to
$$s\left(\kappa + \overline{\kappa}\right).$$
Then by (10) for $F_\sigma = F_\beta$ and $F_{\sigma'} = G$ we have
$$\begin{array}{l} g_\beta = e^{L_{-\kappa}}\left(g_{\sigma'}\right) \\ \overline{D}_{\sigma'} = e^{L_\kappa} \circ \overline{D}_\beta \circ e^{L_{-\kappa}}. \end{array}$$

Computing, using (7) and (9),
$$\begin{aligned} e^{L_\kappa} \circ \overline{D}_\beta \circ e^{L_{-\kappa}} &= e^{L_\kappa} \circ \left(\overline{\partial} - \varsigma_\beta\right) \circ e^{L_{-\kappa}} \\ &= e^{L_\kappa} \circ \left(\overline{\partial} - \left[\overline{\partial}, e^{L_\beta}\right] \circ e^{L_{-\beta}}\right) \circ e^{L_{-\kappa}} \\ &= e^{L_\kappa} \circ \left(e^{L_\beta} \circ \overline{\partial} \circ e^{L_{-\beta}}\right) \circ e^{L_{-\kappa}}. \end{aligned}$$
Thus we conclude that $g_{\sigma'}$ is holomorphic if and only if
$$\overline{\partial}_{M_0}\left(e^{L_{-\beta}} \circ e^{L_{-\kappa}}\left(g_{\sigma'}\right)\right) = 0.$$

Lemma 2.9. *For a power series*
$$g = \sum_{i=0}^{\infty} g_i t^i$$
on $M_0 \times \Delta$, $g \circ G$ is holomorphic on M_β if and only if
$$\overline{\partial}_{M_0}\left(e^{L_{-\beta}} \circ e^{L_{-\kappa}}(g)\right) = 0.$$

3. Deformations of line bundles and differential operators

3.1 The μ-maps

Let X_0 be a complex manifold and let L_0 be a holomorphic line bundle on X_0. Let

$$\mathfrak{D}(L_0), \mathfrak{D}_n(L_0)$$

denote the sheaf of (holomorphic) differential operators, respectively the sheaf of differential operators of order $\leq n$, on (sections of) the line bundle L_0. Whenever

$$H^2(\mathfrak{D}_n(L_0)) = 0$$

we have a natural exact sequence

$$H^1(\mathfrak{D}_n(L_0)) \to H^1(\mathfrak{D}_{n+1}(L_0)) \to H^1\left(S^{n+1}T_{X_0}\right) \to 0,$$

where the second last map is induced by the symbol map on differential operators. So there exists natural mappings

$$\tilde{\mu}^n : H^1(\mathfrak{D}_n(L_0)) \to Hom\left(H^0(L_0), H^1(L_0)\right) \tag{15}$$

and

$$\mu^{n+1} : H^1\left(S^{n+1}T_{X_0}\right) \to \frac{Hom\left(H^0(L_0), H^1(L_0)\right)}{image\ \tilde{\mu}^n}. \tag{16}$$

(In the next chapter we will establish Petri's conjecture on generic curve C_0 by establishing that the mappings (16) are zero for $n \geq 0$ and $X_0 = C_0$.)

Suppose now that we are given a deformation

$$L \xrightarrow{p} X \xrightarrow{\pi} \Delta \tag{17}$$

of the pair (L_0, X_0). We consider C^∞-sections of L as C^∞-functions on the dual line bundle $L^\vee$. These functions f are characterized by the properties

$$\begin{aligned} \chi(f) &= f, \\ \bar{\chi}(f) &= 0, \end{aligned} \tag{18}$$

where χ is the (holomorphic) Euler vector-field associated with the $\mathbb{C}^*$-action on $L^\vee$.

3.2 Trivializations of deformations of line bundles

We next claim that, given a trivialization σ of the deformation X/Δ and given a line bundle L/X we can make compatible trivializations

$$
\begin{array}{ccc}
L^{\vee} & \overset{F_{\lambda}=(\lambda,\pi\circ q)}{\longrightarrow} & L_0^{\vee}\times\Delta \\
\downarrow q & & \downarrow (q_0, id.) \\
X & \overset{F_{\sigma}=(\sigma,\pi)}{\longrightarrow} & X_0\times\Delta \\
\downarrow \pi & & \downarrow \\
\Delta & = & \Delta
\end{array} \tag{19}
$$

of the deformation $L^{\vee}/X$ of $L_0^{\vee}/X_0$ as in Lemma 2.1 but with the additional property that each fiber of the trivialization respects the structure of holomorphic line bundles, that is, if we denote by $\tau=\tau_{\sigma}$ the lifting of $\frac{\partial}{\partial t}$ induced by the trivialization of X/Δ, then $\tau=\tau_{\lambda}$ for the deformation $L^{\vee}$ of $L_0^{\vee}$ is obtained as a lifting of τ_{σ} such that

$$
[\tau_{\lambda},\chi]=0. \tag{20}
$$

To see that this is always possible, let $\{W\}$ be a covering of X by coordinate disks and $\{W_0\}$ the restriction of this covering to X_0. We construct a C^{∞} partition-of-unity $\{\rho_{W_0}\}$ subordinate to the induced covering of X_0. Recall that L is given with respect to the trivialization σ by holomorphic local patching data

$$
\begin{aligned}
g^{WW'}(x) &= \sum g_i^{WW'}(x_0)\,t^i \\
&= g^{W_0W_0'}(x_0)\exp\left(\sum\nolimits_{j>0} a_j^{WW'}(x_0)\,t^j\right),
\end{aligned}
$$

where $x_0=\sigma(x)$ and

$$
\sum\nolimits_{j>0} a_j^{WW'}(x_0)\,t^j=\log\frac{g^{WW'}(x)}{g^{W_0W_0'}(x_0)}.
$$

Notice that, if $V, W,$ and W' are three open sets of the cover which have non-empty intersection, then, for all $j>0$,

$$
a_j^{VW}+a_j^{WW'}=a_j^{VW'}.
$$

Define the mapping

$$
L\rightarrow L_0
$$

over $W_0 \times \Delta$ by

$$(x, v) \mapsto \left(x_0, \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0)\left(\sum\nolimits_{j>0} a_j^{WW'}(x_0)\, t^j\right)\right) \cdot v\right). \tag{21}$$

This map is well defined since, over $V \cap W$ we have

$$g^{VW}(x) = g^{V_0 W_0}(x_0) \exp\left(\sum\nolimits_{j>0} a_j^{VW}(x_0)\, t^j\right),$$

and so

$$\begin{aligned}
g^{VW}(x) \cdot \exp&\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0)\left(\sum\nolimits_{j>0} a_j^{WW'}(x_0)\, t^j\right)\right) \\
&= g^{V_0W_0}(x_0) \exp\left(\sum\nolimits_{j>0} a_j^{VW}(x_0)\, t^j\right) \\
&\qquad \cdot \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0)\left(\sum\nolimits_{j>0} a_j^{WW'}(x_0)\, t^j\right)\right) \\
&= g^{V_0W_0}(x_0) \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0) \sum\nolimits_{j>0}\left(a_j^{VW} + a_j^{WW'}\right)(x_0)\, t^j\right) \\
&= g^{V_0W_0}(x_0) \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0) \sum\nolimits_{j>0} a_j^{VW'}(x_0)\, t^j\right).
\end{aligned}$$

Referring to Lemma 2.3 our deformation/trivialization (19) is given by

$$\xi_j \in A^{0,1}\left(T_{L_0^\vee}\right)$$

for which

$$L_\chi \xi_j = L_{\bar{\chi}} \xi_j = 0. \tag{22}$$

We call a trivialization satisfying (19)-(22) a *trivialization of line bundles.* We say that the trivializations λ of $L^\vee/\Delta$ and σ of X/Δ are *compatible* if they make the diagram (19) commutative. By an elementary computation in local coordinates, sections

$$\xi_i \in A^{0,1}_{L_0^\vee} \otimes T_{L_0^\vee}$$

associated to a trivialization of line bundles lie in a subspace

$$A \subseteq A^{0,1}_{L_0^\vee} \otimes T_{L_0^\vee}$$

comprising the the middle term of an exact sequence

$$0 \to q_0^{-1}\left(A^{0,1}_{X_0}\right) \otimes_{\mathbb{C}} \mathbb{C}\chi \to A \to q_0^{-1}\left(A^{0,1}_{X_0} \otimes T_{X_0}\right) \to 0, \tag{23}$$

that is,

$$A = A^{0,1}_{X_0}\left(\mathfrak{D}_1\left(L_0\right)\right). \tag{24}$$

Notice that the first form

$$\xi_1 \in A^{0,1}_{X_0}\left(\mathfrak{D}_1\left(L_0\right)\right)$$

must be $\overline{\partial}$-closed by the integrability conditions in Lemma 2.4. Its cohomology class in

$$H^1\left(\mathfrak{D}_1\left(L_0\right)\right)$$

is the first-order deformation of the pair (X_0, L_0) given by (17) (see [1]). Its symbol is just the element of $H^1\left(T_{X_0}\right)$ giving the Kodaira-Spencer class for the compatible first-order deformation of the manifold X_0.

Lemma 3.1. *i) If X_0 is a Riemann surface C_0, the space of all (formal) deformation/trivializations of the pair (curve, line bundle) taken modulo holomorphic isomorphisms over Δ, is naturally the space of power series in t with coefficients $\xi_i \in A^{0,1}_{C_0}\left(\mathfrak{D}_1\left(L_0\right)\right)$.*

ii) In general, a (formal) holomorphic section of L is a power series

$$s = \sum_i t^i s_i$$

with coefficients s_i which are C^∞-sections of L_0 such that

$$\sum_{i=0}^{\infty}\left(\bar{\partial} s_i\right) t^i - \sum_{i=0,j=1}^{\infty,\infty} \xi_j\left(s_i\right) t^{i+j} = 0.$$

iii) Suppose

$$f \in H^0\left(L\right)$$

has divisor D such that

$$D_0 = D \cdot X_0$$

is smooth and reduced. Then there is a trivialization

$$F_\sigma : X \to X_0 \times \Delta$$

such that

$$\sigma^{-1}\left(D_0\right) = D,$$

and a unique σ-compatible trivialization

$$F_\lambda : L^\vee \to L_0^\vee \times \Delta$$

such that

$$f = f_0 \circ \lambda$$

where

$$f_0 = f|_{X_0}.$$

We call the trivialization F_λ adapted to the section f.

Proof. i) By (23) and Lemma 2.4 all integrability conditions vanish automatically.

ii) is immediate from Lemma 2.3.

iii) Let N be a tubular neighborhood of D_0 in X. On N use a partition-of-unity argument as in §5 of [3] to construct a C^∞ -retraction

$$\upsilon : N \to N \cap D_0$$

such that each fiber is an analytic polydisk. Cover N as above by coordinate disks $\{W\}$. For each $W_0 = W \cap X_0$ which meets D_0 construct a *holomorphic* projection

$$\upsilon^{-1}(W_0 \cap D_0) \to W_0,$$

which takes

$$(W \cap D) \to (W_0 \cap D_0).$$

Again as in §5 of [3], use a C$^\infty$-partition-of-unity argument to "average" these local projections to obtain a projection

$$\varkappa : N \to N \cap X_0$$

such that

$$\upsilon \circ \varkappa = \upsilon$$

and such that, for each $x_0 \in D_0$,

$$\varkappa|_{\upsilon^{-1}(x_0)}$$

is holomorphic. $\varkappa$ gives a projection σ in some neighborhood D such that

$$D = \sigma^{-1}(D_0).$$

Extend by a partition of unity argument to obtain $\sigma : X \to X_0$ and the corresponding trivialization $F_\sigma = (\sigma, \pi)$.

Now let

$$L_0 = \mathcal{O}_{X_0}(D_0),$$

and suppose D is given by local defining functions. Then, on each slice

$$v^{-1}(x_0),$$

$x_0 \in D_0$, the invertible holomorphic functions

$$\frac{z_W}{z_{W_0} \circ \sigma}$$

fit together to give an invertible C^∞-function on $W \subset N$ so that

$$h_W := \frac{z_W \circ F_\sigma^{-1}}{z_{W_0}}$$

is an invertible C^∞-function on $W_0 \times \Delta$. If $W \nsubseteq N$ put

$$h_W = 1.$$

So for patching data

$$\begin{aligned} g^{WW'}(x) &= \sum g_i^{WW'}(x_0)\, t^i \\ &= g^{W_0 W_0'}(x_0) \exp\left(\sum\nolimits_{j>0} a_j^{WW'}(x_0)\, t^j\right) \end{aligned}$$

we have

$$\sum\nolimits_{j>0} a_j^{WW'}(x_0)\, t^j = \log h_{W'} - \log h_W.$$

The σ-compatible trivialization F_λ constructed in (21) is given in this case by

$$(x, v) \mapsto \left(x_0, \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0)\left(\log h_{W'} - \log h_W\right)\right) \cdot v\right).$$

So, under this trivialization, z_W corresponds to the section of $L_0^\vee \times \Delta$ given over $(x_0, t) \in W_0 \times \Delta$ by (v, t) where

$$\begin{aligned} v &= \frac{z_{W_0}}{z_W \circ F_\sigma^{-1}} \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0)\left(\log h_{W'}\right)\right) \cdot \left(z_W \circ F_\sigma^{-1}\right) \\ &= \exp\left(\sum\nolimits_{W'} \rho_{W_0'}(x_0)\left(\log h_{W'}\right)\right) \cdot z_{W_0}. \end{aligned}$$

Now replace λ with

$$\frac{\lambda}{\exp\left(\sum_{W'} \rho_{W_0'}(x_0)\left(\log h_{W'}\right)\right) \cdot \lambda}.$$

q.e.d.

3.3 Schiffer-type deformations of line bundles

We next wish to consider a very special type of line bundle deformation. Our aim is to be able to apply Lemmas 2.7-2.8 to a case in which $M_0 = L_0^\vee$ is the total space of a line bundle and the holomorphic functions under consideration are the holomorphic sections of L_0. Let X_β/Δ be a Schiffer-type deformation as in 2.2. That is, referring to (5), suppose that X_β/Δ is given by Kuranishi data

$$\xi_{X_0}^{\beta} = \frac{e^{L_\beta} - 1}{L_\beta}\left(\left[\overline{\partial}, L_\beta\right]\right)$$

for some divisor

$$A_0 \subseteq X_0.$$

Let

$$A_\beta$$

denote the extension of A_0 to a divisor on X_β.

Let L/X_β be a deformation of L_0/X_0. By Lemma 2.9 and (21) there are compatible trivializations

$$\begin{aligned} F_\beta &: X_\beta \to X_0 \times \Delta, \\ F_\lambda &: L^\vee \to L_0^\vee \times \Delta. \end{aligned}$$

We need that $F_\lambda = F_{\tilde{\beta}}$ for some lifting $\tilde{\beta}$ of β to a vector field on $L_0^\vee \times \Delta$ for which

$$\left[\tilde{\beta}, \chi\right] = 0.$$

Lemma 3.2. *i) Suppose that L_0 is trivial over a neighborhood of A_0 and that the mapping*

$$H^0\left(\frac{\mathcal{O}_{X_0}(\infty \cdot A_0)}{\mathcal{O}_{X_0}}\right) \to H^1\left(\mathcal{O}_{X_0}\right)$$

induced by the exact sequence

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}(\infty \cdot A_0) \to \frac{\mathcal{O}_{X_0}(\infty \cdot A_0)}{\mathcal{O}_{X_0}} \to 0$$

is surjective. Then there is a lifting

$$\tilde{\beta}$$

of β *to a vector field on* $L_0^\vee \times \Delta$ *which is meromorphic above*

$$A_0$$

and otherwise C^∞ *such that* $F_{\tilde{\beta}}$ *is a trivialization of* $L^\vee$*.*

ii) Referring to i), suppose that

$$L = \mathcal{O}_{X_\beta}(D)$$

and

$$\Phi \circ F_\beta(D) = D_0 \times \Delta,$$

where D_0 *is the zero-scheme associated to a holomorphic section*

$$f_0 : L_0^\vee \to \mathbb{C}.$$

Suppose further that Φ *is holomorphic in a neighborhood of* $A_0 \times \Delta$*. Then there is a lifting* $\tilde{\Phi}$ *of* Φ *so that the section*

$$f_0 \circ \tilde{\Phi} \circ F_{\tilde{\beta}}$$

is a holomorphic section of L*.*

Proof. i) Since L is trivial near A_0, we can lift β to a vector field $\tilde{\beta}$ commuting with χ and meromorphic near A_0 by a patching argument as in 3.2. Any two liftings differ by a vector field

$$a\chi = \sum_{j>0} a_j \chi t^j,$$

where the a_j are fuctions on X_0 which are meromorphic near A_0 and C^∞ elsewhere. Given that modulo t^n

$$L^\vee = L_{\tilde{\beta}}^\vee \tag{25}$$

we use the surjectivity hypothesis in the statement of the lemma to choose a_{n+1} and achieve (25) modulo t^{n+1}.

ii) The deformation X_β is trivial in a neighborhood of $A_0 \times \Delta$, so we can choose a lifting $\tilde{\Phi}'$ of Φ which is holomorphic near $A_0 \times \Delta$ and extend by a partition-of-unity argument. Referring to Lemma 3.1iii), $\tilde{\Phi}' \circ F_{\tilde{\beta}} = (\sigma', \pi)$ and the adapted trivialization (σ, π) are related by

$$\sigma = e^b \sigma'$$

for some C^∞-function b on $X_0 \times \Delta$. Now set

$$\tilde{\Phi} = e^b \tilde{\Phi}'.$$

q.e.d.

3.4 Differential operators and basepoint-free systems

Suppose now that

$$H^0(L_0)$$

is basepoint-free. Let $\mathbb{P}_0 = \mathbb{P}\left(H^0(L_0)\right)$ and let

$$\begin{aligned} \nu &: \mathbb{P}_0 \times X_0 \to \mathbb{P}_0, \\ \rho &: \mathbb{P}_0 \times X_0 \to X_0 \end{aligned}$$

be the two projections. Let

$$\tilde{L}_0(1) = \nu^* \mathcal{O}_{\mathbb{P}_0}(1) \otimes \rho^* L_0.$$

Then by the Leray spectral sequence there are natural isomorphisms

$$\begin{aligned} \rho_* \tilde{L}_0(1) &= L_0 \otimes H^0(L_0)^\vee, \\ H^k\left(\tilde{L}_0(1)\right) &= H^k(L_0) \otimes H^0(L_0)^\vee. \end{aligned} \tag{26}$$

There is a tautological section

$$\tilde{f}_0 \in H^0\left(\tilde{L}_0(1)\right) = H^0(L_0) \otimes H^0(L_0)^\vee = End\left(H^0(L_0)\right) \tag{27}$$

given by the identity map on $H^0(L_0)$. Furthermore

$$\rho_*\left(\tilde{f}_0\right) \tag{28}$$

is given by the tautological homomorphism

$$H^0(L_0) \otimes \mathcal{O}_{X_0} \to L_0.$$

Also one easily shows by induction using the Euler sequence that

$$H^i\left(\mathfrak{D}_n\left(\mathcal{O}_{\mathbb{P}_0}(1)\right)\right) = 0$$

for all $i > 0$, so also

$$R^i \rho_* \mathfrak{D}_n\left(\tilde{L}_0(1)\right) = 0$$

and

$$H^1\left(\mathfrak{D}_n\left(\tilde{L}_0(1)\right)\right) = H^1\left(\rho_* \mathfrak{D}_n\left(\tilde{L}_0(1)\right)\right).$$

There is a natural map

$$h : \rho_* \mathfrak{D}_n\left(\tilde{L}_0(1)\right) \to \mathfrak{D}_n\left(\rho_* \tilde{L}_0(1)\right)$$

and

$$\begin{aligned}\mathfrak{D}_n\left(\rho_*\tilde{L}_0(1)\right) &= \mathfrak{D}_n\left(L_0\otimes H^0(L_0)^\vee\right)\\ &= \mathfrak{D}_n(L_0)\otimes End\left(H^0(L_0)\right).\end{aligned}$$

Now via the trace map we have a canonical splitting

$$End\left(H^0(L_0)\right)=\mathbb{C}\cdot 1\oplus End^0\left(H^0(L_0)\right),$$

where End^0 denotes trace-zero endomorphisms. Notice that

$$\mathfrak{D}'_n := h\left(\rho_*\mathfrak{D}_n\left(\tilde{L}_0(1)\right)\right)=\mathfrak{D}_n(L_0)\otimes 1\oplus\mathfrak{D}_{n-1}(L_0)\otimes End^0\left(H^0(L_0)\right)$$

so we have that

$$\mathfrak{D}'_0=\mathcal{O}_{X_0}$$

and we have the exact sequence

$$\begin{aligned}&0\to\mathfrak{D}'_n\to\mathfrak{D}'_{n+1}\\ &\xrightarrow{symbol}\left(S^{n+1}(T_{X_0})\otimes 1\right)\oplus\left(S^n(T_{X_0})\otimes End^0\left(H^0(L_0)\right)\right)\to 0\end{aligned}\tag{29}$$

is exact.

3.5 Extendable linear systems

Suppose now that the assumptions of 3.4 and Lemma 3.2 continue to hold and that, for some line bundle extension

$$L/X_\beta$$

of L_0/X_0,

$$\pi_*(L)$$

is locally free over Δ. Let

$$\mathbb{P}_{\gneqq}=\mathbb{P}_{\Delta}(\pi_*(L)).$$

Let

$$\begin{aligned}D_0&\subseteq\mathbb{P}_0\times X_0,\\ D&\subseteq\mathbb{P}_\Delta\times_\Delta X_\beta\end{aligned}$$

be the incidence divisors for the respective linear systems. The section $\tilde{f}_0$ of $\tilde{L}_0(1)$ defined in (27) has divisor D_0. For each holomorphic trivialization

$$
\begin{aligned}
T &: \mathbb{P}_\Delta \to \mathbb{P}_0 \times \Delta, \\
\tilde{T} &: \mathcal{O}_{\mathbb{P}_\Delta}(-1) \to \mathcal{O}_{\mathbb{P}_0}(-1) \times \Delta,
\end{aligned}
\tag{30}
$$

let

$$
\begin{aligned}
F_T &: \mathbb{P}\left(\tilde{L}^\vee(-1)\right) \to \mathbb{P}\left(\tilde{L}_0^\vee(-1)\right) \times \Delta, \\
\tilde{F}_T &: \tilde{L}^\vee(-1) \to \tilde{L}_0^\vee(-1) \times \Delta
\end{aligned}
$$

be the trivializations induced by $\tilde{T}$ and the trivializations

$$
\begin{aligned}
F_\beta &: X_\beta \to X_0 \times \Delta, \\
F_{\tilde{\beta}} &: L^\vee \to L_0^\vee \times \Delta
\end{aligned}
$$

defined in Lemma 3.2. We denote the infinitesimal automorphism of

$$\tilde{L}_0^\vee(-1) \times \Delta = \left(\rho^* L_0^\vee \otimes \nu^* \mathcal{O}_{\mathbb{P}_0}(-1)\right) \times \Delta$$

induced by $\tilde{\beta}$ on $\rho^* L_0^\vee$ and the identity on $\nu^* \mathcal{O}_{\mathbb{P}_0}(-1)$ as

$$\tilde{\beta}_T.$$

If we change $\tilde{T}$ in (30) to $\tilde{T}'$, then

$$e^{-L_{\tilde{\beta}_{T'}}} = e^{-L_{\tilde{\beta}_T}} \left(\tilde{T}' \circ \tilde{T}^{-1}\right)^*. \tag{31}$$

Next, since

$$F_T(D) \cdot (\mathbb{P}_0 \times \{x_0\})$$

is a hyperplane for each $x_0 \in X_0$ by basepoint-freeness, we can build a C^∞-diffeomorphism

$$\Phi_T : (\mathbb{P}_0 \times X_0) \times \Delta \to (\mathbb{P}_0 \times X_0) \times \Delta$$

such that

$$\Phi_T(F_T(D)) = D_0 \times \Delta,$$

and such that the diagram

$$
\begin{array}{ccc}
\mathbb{P}_\Delta \times_\Delta X_\beta & \xrightarrow{\Phi_T \circ F_T} & (\mathbb{P}_0 \times X_0) \times \Delta \\
\downarrow & & \downarrow \\
X_\beta & \xrightarrow{F_\beta} & X_0 \times \Delta
\end{array}
$$

is commutative, where the vertical maps are the standard projections and such that the restriction of $\Phi_T \circ F_T$ to each fiber of the left-hand projection is a linear automorphism of projective spaces. Further we can suppose that Φ_T is holomorphic over $F_{\beta}^{-1}(U_0 \times \Delta)$. Let

$$G_T = \Phi_T \circ F_T.$$

Then, by Lemma 3.1iii), there is a G_T-compatible trivialization

$$\tilde{G}_T = \tilde{\Phi}_T \circ \tilde{F}_T : \tilde{L}^{\vee}(-1) \to \tilde{L}_0^{\vee}(-1) \times \Delta$$

such that

$$\tilde{f} = \tilde{f}_0 \circ \tilde{G}$$

is a holomorphic section of $\tilde{L}(1)$ with divisor D. Since the restriction of $\tilde{\Phi}_T$ to fibers of ρ are assumed to be a holomorphic automorphism of $\mathcal{O}(-1)$ on the corresponding projective spaces, the infinitesimal automorphism of $\tilde{L}_0^{\vee}(-1) \times \Delta$ is given by

$$\tilde{\gamma}_T \in \rho^{-1} A^0_{X_0}\left(\mathfrak{gl}\left(\left(L_0^{\vee}\right)^{\oplus(r+1)}\right)\right) \otimes \mathbb{C}[[t]].$$

Thus, by Lemma 2.9,

$$\left[\overline{\partial}, e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]\left(\tilde{f}_0\right) = 0. \tag{32}$$

Notice that, if

$$e^{\tilde{A}} : \mathcal{O}_{\mathbb{P}_0}(-1) \times \Delta \to \mathcal{O}_{\mathbb{P}_0}(-1) \times \Delta$$

is any holomorphic automorphism, we also have

$$\left[\overline{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]\left(\tilde{f}_0\right) = 0. \tag{34}$$

Next, using (29) and (34), we need to analyze the elements

$$\begin{aligned}\left[\overline{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right] &\in \sum\nolimits_{n>0} H^1\left(\mathfrak{D}'_n\right) t^n \\ &= \sum\nolimits_{n>0} H^1\left(\mathfrak{D}_n\left(L_0\right)\right) \otimes End\left(H^0\left(L_0\right)\right) t^n.\end{aligned}$$

Applying this element to $\rho_* \tilde{f}_0$, by (34) we obtain that

$$\begin{aligned}&\rho_*\left[\overline{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]\left(\rho_* \tilde{f}_0\right) \\ &\qquad = 0 \in \sum\nolimits_{n>0} Hom\left(H^0\left(L_0\right), H^1\left(L_0\right)\right) t^n.\end{aligned} \tag{35}$$

Theorem 3.3. *Suppose that all assumptions of 3.3-3.5, in particular, the hypotheses of Lemma 3.2, hold. Suppose further that, by varying of β in 3.3 in such a way that all these assumptions continue to hold, the coefficients to t^{n+1} in all expressions*

$$\left[\overline{\partial}, e^{-L_{\beta}}\right]$$

generate $H^1\left(S^{n+1}\left(T_{X_0}\right)\right)$ for each $n \geq 0$. (For example we allow the divisor A_0 to move.) Then the maps

$$\mu^{n+1}: H^1\left(S^{n+1}T_{X_0}\right) \rightarrow \frac{Hom\left(H^0\left(L_0\right), H^1\left(L_0\right)\right)}{image\ \tilde{\mu}^n}$$

are zero for all $n \geq 0$.

Proof. Let

$$\rho_*\left[\overline{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]_n$$

denote the coefficient of t^n. Notice that the operators in $e^{\tilde{A}}$ and $e^{-L_{\tilde{\gamma}_T}}$ are 0-th order operators so that, referring to (29), there is an element

$$\sigma_n^0 \in S^n\left(T_{X_0}\right) \otimes End^0\left(H^0\left(L_0\right)\right)$$

such that

$$symbol\left(\left(\rho_*\left[\overline{\partial}, e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]\right)_{n+1}\right)$$

is given by a formula

$$(36) \qquad \left(\overline{\partial}\beta_1^n \otimes 1\right) \oplus \sigma \in S^{n+1}\left(T_{X_0}\right) \oplus \left(S^n\left(T_{X_0}\right) \otimes End^0\left(H^0\left(L_0\right)\right)\right),$$

where

$$\beta = \sum_{j>0} \beta_j t^j.$$

Let

$$A_1 \in End^0\left(H^0\left(L_0\right)\right)$$

be such that

$$e^{\tilde{A}} = 1 + A_1 t + \ldots.$$

Then
(37)

$$symbol\left(\rho_*\left[\overline{\partial}, e^{\tilde{A}} \circ e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]_{n+1} - \rho_*\left[\overline{\partial}, e^{-L_{\tilde{\beta}_T}} \circ e^{-L_{\tilde{\gamma}_T}}\right]_{n+1}\right)$$
$$= 0 + \left(\overline{\partial}\beta_1^n \otimes A_1\right).$$

Using (36) and (37) and the hypothesis that the elements

$$\overline{\partial}\beta_1^{n+1}$$

generate

$$H^1\left(S^{n+1}T_{X_0}\right),$$

by varying β and A_1 we have that the elements

$$symbol\left(\rho_*\left[\overline{\partial}, e^{\tilde{A}}\circ e^{-L_{\tilde{\beta}_T}}\circ e^{-L_{\tilde{\gamma}_T}}\right]_{n+1}\right)$$

generate

$$S^{n+1}\left(T_{X_0}\right)\oplus\left(S^n\left(T_{X_0}\right)\otimes End^0\left(H^0\left(L_0\right)\right)\right)$$

for each $n\geq 0$.

Thus, by (29) and (34), the map $\tilde{\nu}^{n+1}$ given by

$$\begin{array}{rcl} H^1\left(\mathfrak{D}_{n+1}\left(\tilde{L}_0\left(1\right)\right)\right) & \rightarrow & \dfrac{H^1\left(\tilde{L}_0\left(1\right)\right)}{image\left(\tilde{\nu}^n\right)} \\ D & \mapsto & D\left(\tilde{f}_0\right) \end{array}$$

is zero for all $n\geq 0$. Moreover, in particular the image of

$$H^1\left(\mathfrak{D}_{n+1}\left(\tilde{L}_0\left(1\right)\right)\right)\rightarrow\sum\nolimits_{n>0}H^1\left(\tilde{L}_0\left(1\right)\right)t^n$$

lies inside the image of the map

$$\begin{array}{rcl} \sum\nolimits_{n>0}H^1\left(\mathcal{O}_{\mathbb{P}_0\times X_0}\right)t^n & \rightarrow & \sum\nolimits_{n>0}H^1\left(\tilde{L}_0\left(1\right)\right)t^n \\ \alpha & \mapsto & \alpha\left(\tilde{f}_0\right). \end{array} \tag{38}$$

But, under the identifications in 3.4,

$$\rho_*\left(\tilde{f}_0\right)$$

is the tautological map

$$H^0\left(L_0\right)\otimes\mathcal{O}_{X_0}\rightarrow L_0.$$

So applying ρ_* (38) becomes

$$\sum\nolimits_{n>0}H^1\left(\mathcal{O}_{X_0}\right)t^n\rightarrow\sum\nolimits_{n>0}Hom\left(H^0\left(L_0\right),H^1\left(L_0\right)\right)t^n.$$

Since
$$\rho_* \left[\overline{\partial}, e^{-L_{\tilde{\beta}_T}}\right] = \left[\overline{\partial}, e^{-L_{\tilde{\beta}}}\right] \otimes 1,$$
we conclude that the map
$$\left[\overline{\partial}, e^{-L_{\tilde{\beta}}}\right] \otimes H^0(L_0) \to \sum\nolimits_{n>0} Hom\left(H^0(L_0), H^1(L_0)\right) t^n$$
takes values in the image of (38). So finally the hypothesis that the elements
$$\overline{\partial}\beta_1^{n+1}$$
generate $H^1\left(S^{n+1}\left(T_{X_0}\right)\right)$ gives the theorem. q.e.d.

4. Brill-Noether theory

In this last section we give a simple application of Theorem 3.3 to Brill-Noether theory. From now on we assume that X_0 is a *generic* compact Riemann surface C_0. We choose
$$A_0 = \{x_0\}$$
in §2-3 where x_0 is a general point of C_0, and let
$$C_\beta/\Delta$$
denote the family of Schiffer-type deformation associated to some vector field
$$\beta = \sum\nolimits_{j>0} \beta_j t^j,$$
where each β_j is meromorphic with poles in some neighborhood U_0 of x_0. Since C_0 is generic, there exists a line-bundle deformation
$$L/C_\beta$$
such that
$$H^0(L) \to H^0(L_0)$$
is surjective. We wish to apply Theorem 3.3 to conclude that the maps μ^{n+1} are all zero for $n \geq 0$.

Lemma 4.1. *Let β_1 range over all vector fields such that the Kodaira-Spencer class*
$$\overline{\partial}\beta_1$$

generates the kernel of the map

$$H^1\left(T_{C_0}\right) \to H^1\left(T_{C_0}\left(x_0\right)\right).$$

Then the elements

$$\overline{\partial}\left(\beta_1^{k+1}\right)$$

generate the kernel of the map

$$H^1\left(S^{k+1}T_{C_0}\right) \to H^1\left(S^{k+1}T_{C_0}\left(\left(k+1\right)x_0\right)\right).$$

Proof. Let z be a local analytic coordinate for C_0 centered on x_0. We trivialize our Schiffer-type variation of C_0 so that

$$\beta_1 = \frac{\rho}{z}\frac{\partial}{\partial z},$$

where ρ is a C^∞-function on C_0 such that

i) ρ is supported on an arbitrarily small neighborhood of x_0,

ii) in a smaller neighborhood U_0 of x_0,

$$\rho = \frac{a_{-1}}{z} + a_0 + \ldots + a_k z^k.$$

So

$$\bar{\partial}\left(\frac{\rho}{z}\right)^{k+1}\left(\frac{\partial}{\partial z}\right)^{k+1}$$

represents the symbol of $\left[\overline{\partial}, L_{\beta_1}^{k+1}\right]$. By varying the choice of the a_i in the definition of ρ we can therefore obtain symbols which generate the image of

$$\frac{S^{k+1}T_{C_0}\left(\left(k+1\right)x_0\right)}{S^{k+1}T_{C_0}}$$

in $H^1\left(S^{k+1}T_{C_0}\right)$. q.e.d.

Now if x_0 varies over a dense subset of C_0, the elements of kernel of

$$H^1\left(S^{k+1}T_{C_0}\right) \to H^1\left(S^{k+1}T_{C_0}\left(x_0\right)\right)$$

generate $H^1\left(S^{k+1}T_{C_0}\right)$. So we conclude by Theorem 3.3:

Theorem 4.2. *If C_0 is a curve of general moduli and $H^0(L_0)$ is basepoint-free, the mapping*

$$\mu^{k+1} : H^1\left(S^{k+1}T_{C_0}\right) \to \frac{Hom\left(H^0(L_0), H^1(L_0)\right)}{\sum_{k' \leq k} image\ \tilde{\mu}^{k'}}$$

given in 3.1 must be the zero map for $k \geq 0$.

To see that Petri's conjecture follows from Theorem 4.2, we reason as in §9 of [2]. Namely we consider the dual mappings

$$\mu_k : \ker \mu_{k-1} \to H^0\left(\omega_{C_0}^{k+1}\right)$$

(inductively defined beginning with the zero map

$$\mu_{-1} : H^0(L_0) \otimes H^0\left(\omega_{C_0} \otimes L_0^{\vee}\right) \to \{0\}).$$

Petri's conjecture asserts that, for our C_0 of general moduli, the mapping

$$\mu_0 : H^0(L_0) \otimes H^0\left(\omega_{C_0} \otimes L_0^{\vee}\right) \to H^0(\omega_{C_0}),$$

which is of course simply the multiplication map, is injective. To see that this follows from Theorem 4.2, let $\{s_i\}$ denote a basis for $H^0(L_0)$. Suppose now that

$$\mu_0\left(\sum s_i \otimes t_i\right) = 0.$$

Then the element

$$\sum (ds_i)t_i \in H^0\left(\omega_{C_0}^2\right)$$

is well-defined, giving the mapping μ_1, etc. Since, by Theorem 4.2, successive maps μ_k are the zero map we have, for any local trivialization of L_0 and local coordinate z near a general point x_0 on C_0, the local system of (pointwise) equations

$$\sum_i t_i(x_0) \frac{d^k s_i}{dz^k}(x_0) = 0$$

for all k, which is clearly impossible unless all the $t_i(x_0)$ are zero.

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