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A LOCAL PROPERTY OF ABSOLUTELY CONVERGENT JACOBI POLYNOMIAL SERIES

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Introduction. Fix real numbers $\alpha \geqq \beta \geqq -1/2$ and let $P_n^{(\alpha,\beta)}(x)$ denote the corresponding Jacobi polynomial of degree n in x, defined by the relation

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n \cdot n!}\left(\frac{d}{dx}\right)^n((1-x)^{n+\alpha}(1+x)^{n+\beta}) .$$

We then form the normalized polynomials $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$, so that $\sup_{-1\leq x\leq 1}|R_n^{(\alpha,\beta)}(x)| = 1$, $\forall n \geqq 0$. We let $AJ(\alpha, \beta, 0)$ denote the space of series $f(x) = \sum_{n=0}^\infty a_n R_n^{(\alpha,\beta)}(x)$ subject to the condition $\sum_{n=0}^\infty |a_n| < \infty$.

The main result of Chapter 2 of this paper states that if $f \in AJ(\alpha, \beta, 0)$ and if $0 < \varepsilon < \pi/2$ then on $[\varepsilon, \pi - \varepsilon]$ we can write

$$f(\cos\theta) = \sum_{n=0}^\infty b_n \cos(n\theta) \tag{1}$$

with

$$\sum_{n=0}^\infty |b_n|(n+1)^{\alpha+1/2} < \infty . \tag{2}$$

Conversely, if a cosine series (1) satisfies condition (2) then it represents an element of $AJ(\alpha, \beta, 0)$. The earlier paper [8] treats the case $\alpha = \beta = m + 1/2$ for an integer $m \geqq 0$.

That such a result should be possible is suggested by the work of Gatesoupe [14] on the local properties of radial Fourier transforms in $\boldsymbol{R}^n$ and that of Ricci [25] on absolutely convergent series of characters on compact semisimple Lie groups.

The space $AJ(\alpha, \beta, 0)$ can be given the structure of a Banach algebra of continuous functions on $[-1, 1]$, with the usual multiplication of functions, and this has been studied by Askey and Wainger [4], Bavinck [6], Gasper [12], and Igari and Uno [19]. It can also be viewed as the Fourier algebra of the hypergroup formed by $[-1, 1]$ when convolution of functions on $[-1, 1]$ is defined as in [5]. In Chapter 3 we show that if $\alpha \geqq 1/2$ and $-1 < x < 1$ then the singleton $\{x\}$ is not a set of synthesis for $AJ(\alpha, \beta, 0)$. The case $AJ(+1/2, +1/2, 0)$ is an example in the work of Chilana and Ross [9], namely the algebra of absolutely convergent series of characters on $SU(2)$.

We also show that when $\alpha > -1/2$ and $\alpha \geqq \beta \geqq -1/2$ nonanalytic functions operate on $AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]}$. This corresponds to [25, Thm. 2].

In the final chapter we use the preceeding results to study spectral synthesis in the Fourier algebra $K(G)$ of the compact Lie groups $G = SO(n)$ $(n \geqq 4)$; $SU(n)$ $(n \geqq 3)$; $Sp(n)$ $(n \geqq 2)$; and $F_{4(-52)}$. For example, we show that if $n \geqq 4$ and $0 < \theta < \pi$ then the double coset

$$\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & SO(n-1) \\ 0 & \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & \\ & & 0 \\ -\sin\theta & \cos\theta & \\ & 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & SO(n-1) \\ 0 & \end{pmatrix}$$

is not a set of synthesis for $K(SO(n))$. This could be considered as a "compact group version" of L. Schwartz's theorem [26] which states that if $m \geqq 3$, S^{m-1} is not a set of synthesis for the algebra of Fourier transforms on $\boldsymbol{R}^m$.

NOTATION. We let $\boldsymbol{R}$, $\boldsymbol{C}$, and $\boldsymbol{H}$ denote the real numbers, complex numbers, and quaternions, respectively. We set $\boldsymbol{T} = \boldsymbol{R}/(2\pi\boldsymbol{Z})$ and view functions on $\boldsymbol{T}$ as 2π-periodic functions on $\boldsymbol{R}$.

If $\{a_n\}_n$ and $\{b_n\}_n$ are two sequences we write $a_n \sim b_n$ $\forall n \geqq 0$ to mean that there are positive constants c_1 and c_2 so that $c_1|a_n| \leqq |b_n| \leqq c_2|a_n|$, $\forall n \geqq 0$.

1. Review of Jacobi polynomials. Our references for the properties of Jacobi polynomials are the book of Szegö [27] and the works of Askey, Gasper, and Wainger [1], [3], [4], [12] and [13]. We begin by setting up some notation. For $\alpha, \beta > -1$ and $-1 < x < 1$ let

$$W_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta \tag{1.1}$$

and

$$d\mu_{\alpha,\beta}(x) = W_{\alpha,\beta}(x)dx\ . \tag{1.2}$$

DEFINITION 1.3. For $\alpha, \beta > -1$ and an integer $n \geqq 0$, $R_n^{(\alpha,\beta)}(x)$ is the unique polynomial of degree n in x such that:

(i) for every polynomial $p(x)$ of degree less than n,

$$\int_{-1}^{1} p(x)R_n^{(\alpha,\beta)}(x)d\mu_{\alpha,\beta}(x) = 0\ ;$$

and

(ii) $R_n^{(\alpha,\beta)}(1) = 1$.

In terms of the notation of Szegö [27], $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. If $\alpha \geqq \beta \geqq -1/2$ and $n \geqq 0$ then

$$\sup_{-1\leqq x\leqq 1} |R_n^{(\alpha,\beta)}(x)| = R_n^{(\alpha,\beta)}(1) = 1 . \tag{1.4}$$

If $a \in \boldsymbol{R}$ and $n \in \boldsymbol{N}$ we use the notation

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)\cdots(a+n-1) . \tag{1.5}$$

In the case when a is not a negative integer then we can write

$$(a)_n = \Gamma(a+n)/\Gamma(a) , \qquad \forall n \in \boldsymbol{N} . \tag{1.6}$$

Recall the following properties of the Gamma function.

LEMMA 1.7. *If $a \in \boldsymbol{R}\setminus(-\boldsymbol{N})$ then*

$$\Gamma(n+a)/\Gamma(n) \sim (n+1)^a , \qquad \forall n \geqq 0 .$$

If $0 \leqq x < \infty$ then

$$2^{2x-1}\Gamma(x)\Gamma(x+1/2) = \pi^{1/2}\Gamma(2x) .$$

This latter equation is called the duplication formula. From Szegö [27, (4.3.3) and (4.1.1)] we know that for $\alpha \geqq \beta \geqq -1/2$ the sequence

$$N(\alpha, \beta, n) := \int_{-1}^{1} |R_n^{(\alpha,\beta)}|^2 d\mu_{\alpha,\beta}$$

satisfies

$$N(\alpha, \beta, n) \sim c_{\alpha,\beta}(n+1)^{-1-2\alpha} , \qquad \forall n \in \boldsymbol{N} . \tag{1.8}$$

Note the following important special cases. When $(\alpha, \beta) = (0, 0)$ we have $R_n^{(0,0)}(x) = P_n(x)$, the Legendre polynomial of degree n. If we set $x = \cos\theta$ then for $n \geqq 0$, $R_n^{(-1/2,-1/2)}(\cos\theta) = \cos(n\theta)$ and $R_n^{(1/2,1/2)}(\cos\theta) = \sin((n+1)\theta)/\{(n+1)\sin\theta\}$.

In the work below we will need some formulae connecting systems of Jacobi polynomials for different indices (α, β). For a summary of these results see the survey article of Gasper [13].

PROPOSITION 1.9. *For $\alpha, \beta, a > -1$ and $n \geqq 0$, $R_n^{(a,a)}(x)$ is equal to*

$$\sum_{k=0}^{[n/2]} \frac{n!(\alpha+1)_{n-2k}(n+2a-1)_{n-2k}(1/2)_k(a-\alpha)_k R_{n-2k}^{(\alpha,\alpha)}(x)}{(n-2k)!(2k)!(a+1)_{n-2k}(n-2k+2\alpha+1)_{n-2k}(n-2k+a+1)_k(n-2k+\alpha+3/2)_k},$$

and $R_n^{(a,\beta)}(x)$ equal to

$$\sum_{k=0}^{n} \frac{n!(\alpha+1)_k(n+a+\beta+1)_k(a-\alpha)_{n-k}(k+\beta+1)_{n-k}R_k^{(\alpha,\beta)}(x)}{k!(n-k)!(a+1)_k(k+\alpha+\beta+1)_k(k+a+1)_{n-k}(2k+\alpha+\beta+2)_{n-k}} .$$

The first of these identities is [27, (4.10.27)], due to Gegenbauer, and the second is [3, (2.8)]. We abbreviate these identities by setting

$$(1.10)\qquad R_n^{(a,b)}(x) = \sum_{k=0}^{n} g(n, k; a, b, \alpha, \beta) R_k^{(\alpha,\beta)}(x) .$$

The coefficients $g(n, k; \cdots)$ always exist and we have just written explicit descriptions of $g(n, k; a, a, \alpha, \alpha)$ and $g(n, k; a, \beta, \alpha, \beta)$.

For arbitrary $\alpha, \beta > -1$ and $n, m \geqq 0$ it is clear that there exist coefficients $H(n, m, k; \alpha, \beta)$ such that

$$(1.11)\qquad R_n^{(\alpha,\beta)}(x) \cdot R_m^{(\alpha,\beta)}(x) = \sum_{k=0}^{n+m} H(n, m, k; \alpha, \beta) R_k^{(\alpha,\beta)}(x) .$$

An elementary argument shows that $H(n, m, k; \alpha, \beta) = 0$ for $k < |n - m|$. Furthermore, Gasper [12] has shown the following to be true.

PROPOSITION 1.12. *For $\alpha \geqq \beta > -1$ and $\alpha + \beta \geqq -1$, and all $n, m \geqq 0$ the coefficients $H(n, m, k; \alpha, \beta)$ are nonnegative for $|n - m| \leqq k \leqq n + m$. In particular,*

$$\sum_{k=0}^{n+m} |H(n, m, k; \alpha, \beta)| = \sum_{k=|n-m|}^{n+m} H(n, m, k; \alpha, \beta) = 1 .$$

For further results in this direction see [1], [4], and [12].

This result enables us to equip spaces of absolutely convergent series $\sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x)$ with Banach algebra structure, as in [4] and [19].

The spaces which we consider are modelled on certain spaces of absolutely convergent Fourier series, the so called weighted algebras [20, p. 153]. We review their properties here, prior to setting up the more general algebras of absolutely convergent Jacobi polynomial series.

DEFINITION 1.13. For $\nu \geqq 0$, $A_\nu(\boldsymbol{T})$ denotes the space of absolutely convergent Fourier series

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$$

such that $\|f\|_\nu = \sum_{-\infty}^{\infty} |a_n|(|n| + 1)^\nu < \infty$.

Note that $A(\boldsymbol{T})$ is a Banach algebra of continuous functions on $\boldsymbol{T}$ and if $0 \leqq \nu_1 \leqq \nu_2$ then $A_{\nu_2}(\boldsymbol{T}) \subset A_{\nu_1}(\boldsymbol{T})$. In particular, $C^\infty(\boldsymbol{T}) \subset A_\nu(\boldsymbol{T})$, $\forall \nu \geqq 0$. We use the notation $A_\nu^e(\boldsymbol{T})$ to denote the subspace of even elements of $A_\nu(\boldsymbol{T})$, that is, cosine series.

If $\nu \geqq 1$ then elements of $A_\nu(\boldsymbol{T})$ are continuously differentiable functions on $\boldsymbol{T}$. In fact, if $n = [\nu] \geqq 1$ and $f \in A_\nu(\boldsymbol{T})$ then $f^{(n)} \in A_{\nu-n}(\boldsymbol{T}) \subseteqq A_0(\boldsymbol{T})$. One consequence of this property is that singletons $\{x\}$ are not sets of synthesis for $A_\nu(\boldsymbol{T})$, when $\nu \geqq 1$. This means that the closure of the ideal $J(x) = \{f \in A_\nu(T): f = 0$ on a neighbourhood of $x\}$ is not all of the closed ideal $I(x) = \{f \in A_\nu(T): f(x) = 0\}$. To see this, observe that

$$\overline{J(x)} \subseteqq \{f \in A_\nu(T): f'(x) = f(x) = 0\} \neq I(x) .$$

For further discussion of this behaviour see [24, Chpt. 2], [9], and [14].

Another property of $A_\nu(T)$ $(\nu > 0)$ which distinguishes these spaces from $A(T) \equiv A_0(T)$ is the fact that nonanalytic functions operate on $A_\nu(T)$. More precisely, it is known [20, p. 82] that if F is a function on $[-1, 1]$ with the property that $F \circ f \in A(T)$ for every $f \in A(T)$ with values in $[-1, 1]$ then F is analytic on $[-1, 1]$. However, if $\nu \geqq 1$ and $\mu \geqq \nu + 1/2$ then for every $F \in A_\nu(T)$ and every real-valued $f \in A_\nu(T)$,

$$F \circ f \in A_\nu(T) . \tag{1.14}$$

See [20, p. 153]. Leblanc has shown [22] that if $0 < \nu \leqq 1$ and $\mu > 1 + (1/2\nu)$ then $A_\mu(T)$ operates on $A_\nu(T)$.

2. Absolutely convergent Jacobi polynomial series.

In this section we investigate local properties of some algebras of absolutely convergent Jacobi series. A special case involving certain ultraspherical polynomials appears in [8]. Our approach is suggested by the work of Gatesoupe [14] and Ricci [25].

DEFINITION 2.1. For $\alpha \geqq \beta \geqq -1/2$ and $\lambda \geqq 0$ let $AJ(\alpha, \beta, \lambda)$ denote the space of those continuous functions f on $[-1, 1]$ whose Jacobi polynomial series

$$f(x) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x) \tag{2.2}$$

satisfies

$$\|f\|_{(\alpha,\beta,\lambda)} := \sum_{n=0}^{\infty} |a_n|(n + 1)^\lambda < \infty . \tag{2.3}$$

REMARKS 2.4. From (1.4) we know that if (2.3) is true then the series (2.2) is uniformly absolutely convergent on $[-1, 1]$. The coefficients in (2.2) are determined by

$$a_n N(\alpha, \beta, n) = \int_{-1}^{1} f R_n^{(\alpha,\beta)} d\mu_{\alpha,\beta} , \qquad \forall n \in N . \tag{2.5}$$

Clearly, if $\lambda_1 > \lambda_2$ then $AJ(\alpha, \beta, \lambda_1) \subset AJ(\alpha, \beta, \lambda_2)$. The spaces $AJ(\alpha, \beta, 0)$ have been studied by Bavinck [6] who has shown that for $\alpha \geqq \beta \geqq -1/2$ and $a \geqq b \geqq -1/2$, $AJ(\alpha, \beta, 0) \subset AJ(a, b, 0)$ provided either:

$$a = \alpha \quad \text{and} \quad b - \beta > 0 \quad \text{or} \quad \alpha - a = \beta - b > 0 . \tag{2.6}$$

Note that the spaces $AJ(-1/2, -1/2, \lambda)$ are isomorphic with $A_\lambda^e(T)$. That is, $f \in AJ(-1/2, -1/2, \lambda)$ if and only if $\theta \to f(\cos\theta)$ is an even element of $A_\lambda(T)$. Leblanc has studied weighted l^1-spaces of absolutely convergent trigonometric series, in [21] and [22].

In [4] and [12] it is shown that $AJ(\alpha, \beta, 0)$ is a Banach algebra. This is a consequence of Proposition 1.12. Similarly, one can show the following holds.

PROPOSITION 2.7. *For* $\alpha \geqq \beta \geqq -1/2$ *and* $\lambda \geqq 0$, $AJ(\alpha, \beta, \lambda)$ *is a Banach algebra of continuous functions on* $[-1, 1]$, *equipped with usual multiplication of functions.*

As mentioned in the introduction, $AJ(\alpha, \beta, 0)$ is the Fourier algebra of the hypergroup formed by equipping $[-1, 1]$ with the convolution described in [5]. This convolution generalizes that due to Bochner and Gel'fand for series of ultraspherical polynomials. The Fourier algebra of a compact abelian hypergroup is studied in [9].

We next verify the fact that smooth functions on $[-1, 1]$ provide a space of test functions contained in $AJ(\alpha, \beta, \lambda)$ for all relevant (α, β, λ).

Suppose f is an even element of $C^\infty(T)$. Then

$$f(\theta) = \sum_{n=0}^{\infty} a_n R_n^{(-1/2,-1/2)}(\cos\theta)\,, \qquad 0 \leqq \theta \leqq \pi\,,$$

and the sequence $\{a_n\}$ is rapidly decreasing. For $\alpha, \beta \geqq -1/2$,

$$R_n^{(-1/2,-1/2)} = \sum_{k=0}^{n} g(n, k; -1/2, -1/2, \alpha, \beta) R_k^{(\alpha,\beta)}$$

and

$$\sum_{k=0}^{n} |g(n, k; -1/2, -1/2, \alpha, \beta)|^2 N(\alpha, \beta, k) \leqq CN(-1/2, -1/2, n)\,,$$

since

$$\int_{-1}^{1} |R_n^{(-1/2,-1/2)}|^2 d\mu_{\alpha,\beta} = \int_{-1}^{1} W_{\alpha+1/2,\beta+1/2} |R_n^{(-1/2,-1/2)}|^2 d\mu_{-1/2,-1/2}\,.$$

From this we conclude that for $\alpha \geqq \beta \geqq -1/2$ and $\lambda \geqq 0$,

$$\begin{aligned} \|R_n^{(-1/2,-1/2)}\|_{(\alpha,\beta,\lambda)} &= \sum_{k=0}^{n} |g(n, k; -1/2, -1/2, \alpha, \beta)|(k+1)^\lambda \\ &\leqq \left(\sum_{k=0}^{n} |g(n, k; -1/2, -1/2, \alpha, \beta)|^2 (k+1)^{-1-2\alpha}\right)^{1/2} (n+1)^{\lambda+\alpha+1} \end{aligned}$$

and so

$$\|f\|_{(\alpha,\beta,\lambda)} \leqq C \sum_{n=0}^{\infty} |a_n| (n+1)^{\lambda+\alpha+1} < \infty\,.$$

Let S denote the collection of functions on $[-1, 1]$ defined by $F(\cos\theta) = f(\theta)$ for some even $f \in C^\infty(\boldsymbol{T})$.

LEMMA 2.8. *For all* $\alpha \geqq \beta \geqq -1/2$ *and* $\lambda \geqq 0$, $S \subset AJ(\alpha, \beta, \lambda)$.

The principal result of this section is the following description of the restriction of $AJ(\alpha, \beta, 0)$ to subintervals of $[-1, 1]$.

THEOREM 2.9. *If* $\alpha \geqq \beta \geqq -1/2$ *and* $0 < \varepsilon < 1$ *then*

$$AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]} = AJ(-1/2, -1/2, \alpha + 1/2)|_{[\varepsilon-1,1-\varepsilon]} .$$

When $\alpha = \beta = 1/2$ then $AJ(1/2, 1/2, 0)$ can be identified with the algebra of absolutely convergent central Fourier series on $SU(2)$ and Theorem 2.9 corresponds to [25, Thm. 1], [23], and [9, p. 327].

We prove this in several stages. Firstly, for $\alpha \geqq \beta \geqq -1/2$ we show that

$$W_{\alpha-\beta,0} \cdot AJ(\alpha, \beta, 0) \subset AJ(\beta, \beta, \alpha - \beta) \tag{2.10}$$

and

$$AJ(\beta, \beta, \alpha - \beta) \subset AJ(\alpha, \beta, 0) . \tag{2.11}$$

This reduces the problem to the case of ultraspherical polynomials. Next we fix an integer $N \geqq \beta + 1/2$ and show that for $\lambda \geqq 0$

$$W_{N,N} \cdot AJ(\beta, \beta, \lambda) \subset AJ(-1/2, -1/2, \lambda + \beta + 1/2) \tag{2.12}$$

and

$$AJ(-1/2, -1/2, \lambda + \beta + 1/2) \subset AJ(\beta, \beta, \lambda) . \tag{2.13}$$

Then

$$W_{\alpha+N-\beta,N} \cdot AJ(\alpha, \beta, 0) \subset AJ(-1/2, -1/2, \alpha + 1/2) \subset AJ(\alpha, \beta, 0) . \tag{2.14}$$

Finally fix $0 < \varepsilon < 1$ and let ϕ_ε be an element of S such that $\phi_\varepsilon(x)(1 - x)^{\alpha+N-\beta}(1 + x)^N = 1$, $\varepsilon - 1 \leqq x \leqq 1 - \varepsilon$. For each $f \in AJ(\alpha, \beta, 0)$, (2.14) implies that $\phi_\varepsilon \cdot W_{\alpha+N-\beta,N} \cdot f \in AJ(-1/2, -1/2, \alpha + 1/2)$ and $\phi_\varepsilon \cdot W_{\alpha+N-\beta,N} \cdot f|_{[\varepsilon-1,1-\varepsilon]} = f|_{[\varepsilon-1,1-\varepsilon]}$. Hence

$$AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]} \subset AJ(-1/2, -1/2, \alpha + 1/2)|_{[\varepsilon-1,1-\varepsilon]} .$$

The reverse inclusion follows from the second part of (2.14).

It remains to prove (2.10)-(2.13).

PROOF OF (2.10). We need to prove that for $k \geqq 0$,

$$\| W_{\alpha-\beta,0} \cdot R_k^{(\alpha,\beta)} \|_{(\beta,\beta,\alpha-\beta)} = 0(1) . \tag{2.16}$$

Fix k for the moment and consider the (β, β)-series $W_{\alpha-\beta,0}R_k^{(\alpha,\beta)} = \sum_{n=0}^\infty c_n R_n^{(\beta,\beta)}$, where

$$(2.17)\qquad c_n N(\beta, \beta, n) = \int_{-1}^{1} W_{\alpha-\beta,0} R_k^{(\alpha,\beta)} R_n^{(\beta,\beta)} d\mu_{\beta,\beta} = \int_{-1}^{1} R_n^{(\beta,\beta)} R_k^{(\alpha,\beta)} d\mu_{\alpha,\beta}$$
$$= g(n, k; \beta, \beta, \alpha, \beta) N(\alpha, \beta, k) .$$

In particular, $c_n = 0$ for $n < k$. Furthermore, if $\alpha - \beta \in N$ then $W_{\alpha-\beta,0}(x) R_k^{(\alpha,\beta)}(x)$ is a polynomial of degree $k + \alpha - \beta$, in which case $c_n = 0$ for $n > k + \alpha - \beta$.

Case $\alpha - \beta \in N$. Here we can write $W_{\alpha-\beta,0} R_k^{(\alpha,\beta)} = \sum_{n=k}^{k+\alpha-\beta} c_n R_n^{(\beta,\beta)}$ and observe that

$$\sum_{n=k}^{k+\alpha-\beta} |c_n|^2 N(\beta, \beta, n) = \int_{-1}^{1} (W_{\alpha-\beta,0})^2 (R_k^{(\alpha,\beta)})^2 d\mu_{\beta,\beta}$$
$$= \int_{-1}^{1} W_{\alpha-\beta,0} \cdot (R_k^{(\alpha,\beta)})^2 d\mu_{\alpha,\beta} \leqq C_{\alpha,\beta} \cdot N(\alpha, \beta, k) .$$

For any $\lambda \geqq 0$,

$$\sum_{n=k}^{k+\alpha-\beta} |c_n| (n + 1)^{\lambda+\alpha-\beta} \leqq \Big(\sum_n |c_n|^2 N(\beta, \beta, n)\Big)^{1/2} \left(\sum_{n=k}^{k+\alpha-\beta} (n + 1)^{2\lambda+2\alpha-2\beta} N(\beta, \beta, n)^{-1} \right)^{1/2}$$
$$\leqq C_{\alpha,\beta} N(\alpha, \beta, k)^{1/2} \left(\sum_{n=k}^{k+\alpha-\beta} (n + 1)^{2\lambda+2\alpha-2\beta+1+2\beta} \right)^{1/2}$$
$$\leqq C_{\alpha,\beta} (k + 1)^{-1/2-\alpha+\lambda+\alpha+1/2} ,$$

since n is limited to range over $k \leqq n \leqq k + \alpha - \beta$. This shows that for $\lambda \geqq 0$

$$(2.18)\qquad \| W_{\alpha-\beta,0} \cdot R_k^{(\alpha,\beta)} \|_{(\beta,\beta,\lambda+\alpha-\beta)} = 0((k + 1)^{\lambda}) .$$

In particular, when $\alpha - \beta \in N$,

$$(2.19)\qquad W_{\alpha-\beta,0} \cdot AJ(\alpha, \beta, \lambda) \subset AJ(\alpha, \beta, \lambda + \alpha - \beta) , \qquad \forall\lambda \geqq 0 .$$

Case $\alpha - \beta \notin N$. Now we must use the explicit description of $g(n, k; \beta, \beta, \alpha, \beta)$ given in Proposition 1.9 combined with the asymptotic properties of the Gamma function in estimating c_n. We know that

$$g(n, k; \beta, \beta, \alpha, \beta)$$
$$= \frac{\Gamma(n + 1)\Gamma(k + 1 + \alpha)\Gamma(n + k + 2\beta + 1)\Gamma(n - k + \beta - \alpha)}{\Gamma(\alpha + 1)\Gamma(n + 2\beta + 1)\Gamma(k + 1)\Gamma(n - k + 1)\Gamma(\beta - \alpha)} \times \cdots$$
$$\times \frac{\Gamma(k + \alpha + \beta + 1)\Gamma(2k + \alpha + \beta + 2)\Gamma(\beta + 1)}{\Gamma(2k + \alpha + \beta + 1)\Gamma(n + k + \alpha + \beta + 2)\Gamma(k + \beta + 1)} .$$

From Lemma 1.7 we conclude that for $n \geqq k \geqq 0$,

$$(2.20)\qquad g(n, k; \beta, \beta, \alpha, \beta)$$
$$\sim C_{\alpha,\beta} (n + 1)^{-2\beta} (k + 1)^{2\alpha+1} (n - k + 1)^{\beta-\alpha-1} (n + k + 1)^{\beta-\alpha-1} .$$

Combining this with (2.17) and (1.8) we see that

$$c_n \sim C_{\alpha,\beta}(n+1)(n-k+1)^{\beta-\alpha-1}(n+k+1)^{\beta-\alpha-1} .$$

Hence,

$$\|W_{\alpha-\beta,0}R_k^{(\alpha,\beta)}\|_{(\beta,\beta,\alpha-\beta)} \leqq C\sum_{n=k}^{\infty}(n+1)^{1+\alpha-\beta}(n+k+1)^{\beta-\alpha-1}(n-k+1)^{\beta-\alpha-1} \tag{2.21}$$
$$\leqq C\sum_{l=1}^{\infty}\left(\frac{k+l}{2k+l}\right)^{1+\alpha-\beta} l^{\beta-\alpha-1} = 0(1) .$$

In particular, $W_{\alpha-\beta,0}AJ(\alpha, \beta, 0) \subset AJ(\beta, \beta, \alpha-\beta)$, which completes the proof of (2.10).

PROOF OF (2.11). We have defined the coefficients $g(n, k; \cdots)$ by setting

$$R_n^{(\beta,\beta)} = \sum_{k=0}^{n} g(n, k; \beta, \beta, \alpha, \beta)R_k^{(\alpha,\beta)} .$$

Alternatively, the orthogonality of the $R_k^{(\alpha,\beta)}$'s implies that

$$g(n, k; \beta, \beta, \alpha, \beta)N(\alpha, \beta, k) = \int_{-1}^{1} R_n^{(\beta,\beta)}R_k^{(\alpha,\beta)}d\mu_{\alpha,\beta}$$

and if $\alpha-\beta$ is an integer we saw that this is zero when $k < n-\alpha+\beta$.

Case $\alpha-\beta \in N$. When

$$R_n^{(\beta,\beta)} = \sum_{\substack{k\geqq 0\\ k\geqq n-\alpha+\beta}}^{n} g(n, k; \cdots)R_k^{(\alpha,\beta)}$$

we see that

$$\|R_n^{(\beta,\beta)}\|_{(\alpha,\beta,\lambda)} = \sum_k |g(n, k; \cdots)|(k+1)^{\lambda}$$
$$= \sum_k |g(n, k; \cdots)|N(\alpha, \beta, k)^{1/2-1/2}(k+1)^{\lambda}$$
$$\leqq C_{\alpha,\beta}N(\beta, \beta, n)^{1/2}\Bigg(\sum_{\substack{k\geqq 0\\ k\geqq n-\alpha+\beta}}^{n} N(\alpha, \beta, k)^{-1}(k+1)^{2\lambda}\Bigg)^{1/2}$$

and so

$$\|R_n^{(\beta,\beta)}\|_{(\alpha,\beta,\lambda)} = 0((n+1)^{\lambda+\alpha-\beta}) . \tag{2.22}$$

This says that for $\alpha-\beta \in N$ and $\lambda \geqq 0$,

$$AJ(\beta, \beta, \lambda+\alpha-\beta) \subset AJ(\alpha, \beta, \lambda) . \tag{2.23}$$

Case $\alpha-\beta \notin N$. Recalling the asymptotic relation (2.20) we see that for $n \geqq 0$,

$$\|R_n^{(\beta,\beta)}\|_{(\alpha,\beta,\lambda)} \leqq C_{\alpha,\beta}\sum_{k=0}^{n}(n+1)^{-2\beta}(k+1)^{2\alpha+1+\lambda}(n-k+1)^{\beta-\alpha-1}(n+k+1)^{\beta-\alpha-1} \tag{2.24}$$
$$\leqq C_{\alpha,\beta}(n+1)^{-2\beta+2\alpha+1+\lambda+\beta-\alpha-1} \times \cdots$$

$$\times \sum_{k=0}^{n} \left(\frac{k+1}{n+1}\right)^{2\alpha+1+\lambda} \left(\frac{n+1}{n+k+1}\right)^{1+\alpha-\beta} (n-k+1)^{\beta-\alpha-1}$$
$$= 0((n+1)^{\lambda+\alpha-\beta}) .$$

Combining (2.23) and (2.24) we prove (2.11).

LEMMA 2.25. *If* $\alpha \geqq \beta \geqq -1/2$ *and* $\lambda \geqq 0$, $AJ(\beta, \beta, \lambda+\alpha-\beta) \subset AJ(\alpha, \beta, \lambda)$.

PROOF OF (2.12). We now examine the norm $\| W_{N,N} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,\lambda)}$, where $k \geqq 0$, $\beta \geqq -1/2$, and N is the smallest integer such that $N \geqq \beta + 1/2$. Observe that $W_{N,N}(x) R_k^{(\beta,\beta)}(x)$ is a polynomial of degree $(k+2N)$ in x, which means that

$$\| W_{N,N} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,\lambda)} \leqq C_\beta \cdot (k+1)^\lambda \| W_{N,N} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,0)} , \tag{2.26}$$

for all $k \geqq 0$.

In [6] it is shown that

$$W_{\mu,0} \in AJ(-1/2, -1/2, 0) , \quad \mu \geqq 0 \quad \text{and} \quad W_{0,\mu} \in AJ(-1/2, -1/2, 0) \quad \mu \geqq 0 .$$

In particular,

$$\| W_{N,N} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,0)} \leqq C_\beta \| W_{\beta+1/2,\beta+1/2} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,0)} \tag{2.27}$$

since $W_{N,N} = W_{\beta+1/2,\beta+1/2} W_{N-\beta-1/2,0} W_{0,N-\beta-1/2}$. We now have a situation similar to the proof of (2.10).

Case $\beta + 1/2 \in N$. If $W_{\beta+1/2,\beta+1/2}$ is a polynomial of degree $2\beta+1$ then for each $k \geqq 0$ there are coefficients $\{c_n\}_n$ such that

$$W_{\beta+1/2,\beta+1/2} \cdot R_k^{(\beta,\beta)} = \sum_{n=k}^{k+2\beta+1} c_n R_n^{(-1/2,-1/2)}$$

with

$$\sum_{n=k}^{k+2\beta+1} |c_n|^2 N(-1/2, -1/2, n) = \int_{-1}^{1} (W_{\beta+1/2,\beta+1/2} \cdot R_k^{(\beta,\beta)})^2 d\mu_{-1/2,-1/2} \leqq C_\beta \cdot N(\beta, \beta, k) .$$

From this we conclude that

$$\sum_{n=k}^{k+2\beta+1} |c_n| \leqq C_\beta N(\beta, \beta, k)^{1/2} \sim C_\beta (k+1)^{-\beta-1/2} .$$

Hence, for all $k \geqq 0$ and $\lambda \geqq 0$

$$\| W_{N,N} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,\lambda)} = 0((k+1)^{\lambda-\beta-1/2}) . \tag{2.28}$$

LEMMA 2.29. *If* $\beta \geqq -1/2$ *and* $\beta + 1/2 \in N$ *then*

$$W_{\beta+1/2,\beta+1/2} \cdot AJ(\beta, \beta, \lambda) \subset AJ(-1/2, -1/2, \lambda+\beta+1/2) ,$$

for every $\lambda \geqq 0$.

This corresponds to the result in [8], when $\lambda = 0$.

Case $\beta + 1/2 \notin N$. Recalling proposition 1.9 and (2.17) we see that

$$W_{\beta+1/2,\beta+1/2} \cdot R_k^{(\beta,\beta)} = \sum_{n=k}^{\infty} g(n, k; -1/2, -1/2, \beta, \beta) N(\beta, \beta, k) N(-1/2, -1/2, n)^{-1} R_n^{(-1/2,-1/2)} ,$$

for $k \geqq 0$. If $n-k$ is odd, $g(n, k; \cdots) = 0$. If $n-k$ is even, $g(n, k; -1/2, -1/2, \beta, \beta)$ is equal to

$$\frac{c(n+1)\Gamma(k+\beta+1)\Gamma(n+k)\Gamma((n-k)/2+1/2)\Gamma((n-k)/2-1/2-\beta)}{\Gamma(\beta+1)\Gamma(-1/2-\beta)\Gamma(k+1)\Gamma(n-k+1)\Gamma(2k+2\beta+1)\Gamma((n+k)/2+1/2)} \times \frac{\Gamma(k+2\beta+1)\Gamma(k+\beta+3/2)}{\Gamma((n+k)/2+\beta+3/2)} \tag{2.30}$$

$$= c_\beta \frac{(n+1)\Gamma(k+2\beta+1)\Gamma((n+k)/2)\Gamma((n-k)/2-1/2-\beta)\Gamma(k+\beta+3/2)}{\Gamma(k+1)\Gamma((n-k)/2+1)\Gamma(k+\beta+1/2)\Gamma((n+k)/2+\beta+3/2)}$$

$$\sim c_\beta(n+1)(k+1)^{2\beta+1}((n+k)/2+1)^{-\beta-3/2}((n-k)/2+1)^{-\beta-3/2} .$$

Then, for $k \geqq 0$ we see that

$$\begin{aligned} &\| W_{\beta+1/2,\beta+1/2} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,0)} \\ &\leqq c_\beta \sum_{\substack{n=k \\ (n-k)\text{ even}}}^{\infty} (n+1)((n+k)/2+1)^{-\beta-3/2}((n-k)/2+1)^{-\beta-3/2} \\ &\leqq c_\beta(k+1)^{-\beta-1/2} \sum_{n=k}^{\infty} ((n+1)/(n+k+2))(n-k+1)^{-\beta-3/2} \\ &= 0((k+1)^{-\beta-1/2}) . \end{aligned} \tag{2.31}$$

In (2.26) we can write $\| W_{N,N} \cdot R_k^{(\beta,\beta)} \|_{(-1/2,-1/2,\lambda)} = 0((k+1)^{\lambda-\beta-1/2})$.

LEMMA 2.32. *If $\beta \geqq -1/2$ and N is the least integer such that $N \geqq \beta + 1/2$, then*

$$W_{N,N} \cdot AJ(\beta, \beta, \lambda) \subset AJ(-1/2, -1/2, \lambda + \beta + 1/2) , \qquad \forall\lambda \geqq 0 .$$

3. Consequences. Fix $\alpha \geqq \beta \geqq -1/2$ and $0 < \varepsilon < 1$. We have shown that $AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]} = AJ(-1/2, -1/2, \alpha + 1/2)|_{[\varepsilon-1,1-\varepsilon]}$. If $\alpha \geqq 1/2$ we know that $AJ(-1/2, -1/2, \alpha + 1/2)|_{[\varepsilon-1,1-\varepsilon]} \subseteqq AJ(-1/2, -1/2, 1)|_{[\varepsilon-1,1-\varepsilon]}$ and so the elements of $AJ(\alpha, \beta, 0)$ are differentiable on $]-1, 1[$. If $f \in AJ(\alpha, \beta, 0)$ and $\varepsilon - 1 \leqq x \leqq 1 - \varepsilon$, then

$$|f'(x)| \leqq C_{\alpha,\beta,\varepsilon} \| f \|_{(\alpha,\beta,0)} . \tag{3.1}$$

THEOREM 3.2. *If $\alpha \geqq \beta \geqq -1/2$, $\alpha \geqq 1/2$, and $-1 < x_0 < 1$ then $\{x_0\}$ is not a set of spectral synthesis for $AJ(\alpha, \beta, 0)$.*

PROOF. As in the work of Chilana and Ross [9] observe that

$J(x_0) = \{f \in AJ(\alpha, \beta, 0): f = 0$ on a neighbourhood of $x_0\}$ is contained in $\{f \in AJ(\alpha, \beta, 0): f(x_0) = f'(x_0) = 0\}$ and this is a proper *closed* subspace of $I(x_0) = \{f \in AJ(\alpha, \beta, 0): f(x_0) = 0\}$.

Hence $I(x_0)$ is larger than the closure of $J(x_0)$. q.e.d.

We can also provide examples of nonanalytic functions which operate on $AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]}$, analogous to [25].

THEOREM 3.3. *If* $\alpha \geqq \beta \geqq -1/2$, $\alpha \geqq 1/2$, $0 < \varepsilon < 1$, $F \in A_{\alpha+1}(T)$ *and if* f *is a real valued element of* $AJ(\alpha, \beta, 0)$ *then*

$$F \circ f|_{[\varepsilon-1,1-\varepsilon]} \in AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]} .$$

PROOF. From Theorem 2.9 we know that there is a real-valued $g \in AJ(-1/2, -1/2, \alpha + 1/2)$ such that $f|_{[\varepsilon-1,1-\varepsilon]} = g|_{[\varepsilon-1,1-\varepsilon]}$. Then $\theta \rightarrow g(\cos\theta)$ is an element of $A^e_{\alpha+1/2}(T)$ and from [20, p. 153] we know that $\theta \rightarrow F(g(\cos\theta))$ is an element of $A^e_{\alpha+1/2}(T)$. Finally note that $F \circ g \in AJ(-1/2, -1/2, \alpha + 1/2) \subset AJ(\alpha, \beta, 0)$ and $F \circ g|_{[\varepsilon-1,1-\varepsilon]} = F \circ f|_{[\varepsilon-1,1-\varepsilon]}$.
q.e.d.

Similarly, we can treat the case $-1/2 < \alpha < 1/2$.

THEOREM 3.4. *If* $1/2 > \alpha \geqq \beta \geqq -1/2$ *and* $\alpha > -1/2$, $0 < \varepsilon < 1$, $F \in A_{(2\alpha+2)/(2\alpha+1)}(T)$, *and if* f *is a real-valued element of* $AJ(\alpha, \beta, 0)$ *then* $F \circ f|_{[\varepsilon-1,1-\varepsilon]} \in AJ(\alpha, \beta, 0)|_{[\varepsilon-1,1-\varepsilon]}$.

Apply [21] in place of [20] in the proof of Theorem 3.3.

In [4] Askey and Wainger prove a Wiener-Lévy theorem for $AJ(\alpha, \beta, 0)$.

Theorems 3.3 and 3.4 state that if $\alpha \geqq \beta \geqq -1/2$ and $\alpha > -1/2$ then closed subintervals of $]-1, 1[$ are *not* sets of analyticity for $AJ(\alpha, \beta, 0)$, in contrast with the case of $A(T)$. See [20, pp. 80 and 84].

4. **Compact rank one symmetric spaces.** We wish to apply the results of Chapter 2 to demonstrate the failure of spectral synthesis for the Fourier algebras of the classical compact groups $SO(n)$ ($n \geqq 4$), $SU(n)$ ($n \geqq 3$), and $Sp(n)$. First we recall some facts from harmonic analysis on compact groups [18] and the theory of zonal spherical functions [10].

For the moment let G denote a compact Hausdorff group with dual object $\hat{G}$ and equip G with normalized Haar measure m_G. To each $\sigma \in \hat{G}$ fix a representation $(\pi^\sigma, \mathscr{H}^\sigma) \in \sigma$ and set $d_\sigma = \dim \mathscr{H}^\sigma$ and $\chi_\sigma = \operatorname{tr}(\pi^\sigma)$. Let H be a closed subgroup of G, with normalized Haar measure m_H. We assume that the pair (G, H) has the following property: *for each* $\sigma \in \hat{G}$

$$^{H}\mathscr{H}^{\sigma} = \{\xi \in \mathscr{H}^{\sigma} : \pi^{\sigma}(x)\xi = \xi, \forall x \in H\}$$

is either zero or one-dimensional. Let $\hat{G}_H$ be the collection of σ in $\hat{G}$ such that $^{H}\mathscr{H}^{\sigma} \neq \{0\}$. Associated to such a pair (G, H) are a family of special functions, indexed by $\hat{G}_H$. These are the zonal spherical functions, defined by setting

$$\phi_\sigma(x) = \chi_\sigma * m_H(x) , \qquad x \in G , \quad \sigma \in \hat{G}_H .$$

The properties of $\{\phi_\sigma\}$ are examined in [10]. In particular, if $\sigma \in \hat{G}_H$,

$$\phi_\sigma(h_1 x h_2) = \phi_\sigma(x) , \qquad \forall x \in G , \quad h_1, h_2 \in H .$$

Functions with this property are called *bi-H-invariant*. The fact that $\dim(^{H}\mathscr{H}^{\sigma}) = 1$ implies that $\phi_\sigma(1) = 1 = \|\phi_\sigma\|_\infty$. The Fourier algebra of G is defined to be $K(G) = L^2(G) * L^2(G)$, [18, (34.15)]. It is sometimes denoted $A(G)$ and its properties are described in [18, §34]. $K(G)$ is an algebra of continuous functions on G and is equipped with the norm

$$\|f\|_K = \inf\{\|\psi_1\|_2 \|\psi_2\|_2 : f = \psi_1 * \psi_2\} . \tag{4.1}$$

There is an alternative description of the norm on $K(G)$ in terms of absolutely convergent Fourier series on G, [18, (34.4)].

We are interested in the subspace of bi-H-invariant elements of $K(G)$, which we denote by $^{H}K(G)^{H}$. It is a fact that $^{H}K(G)^{H}$ consists of series $f(x) = \sum_{\sigma \in \hat{G}_H} a_\sigma \phi_\sigma(x)$, with $\|f\|_K = \sum_\sigma |a_\sigma| < \infty$.

There is a projection $P: K(G) \to {}^{H}K(G)^{H}$ defined in the following manner. If f is a continuous function on G set $Pf(x) = m_H * f * m_H(x)$.

LEMMA 4.2. *If $f \in K(G)$ then $Pf \in {}^{H}K(G)^{H}$ and $\|Pf\|_K \leqq \|f\|_K$. If $f \in {}^{H}K(G)^{H}$ then $Pf = f$.*

PROOF. If $f \in K(G)$ and $\varepsilon > 0$ there exists $\psi_1, \psi_2 \in L^2(G)$ with $f = \psi_1 * \psi_2$ and $\|f\|_K \geqq \|\psi_1\|_2 \|\psi_2\|_2 - \varepsilon$. From the definition of P, $Pf = (m_H * \psi_1) * (\psi_2 * m_H)$ which shows that $Pf \in L^2(G) * L^2(G)$. Furthermore,

$$\|Pf\|_K \leqq \|m_H * \psi_1\|_2 \|\psi_2 * m_H\|_2 \leqq \|\psi_1\|_2 \|\psi_2\|_2 \leqq \|f\|_K + \varepsilon .$$

The ε was arbitrary, hence $\|Pf\|_K \leqq \|f\|_K$. The last part of the lemma is obvious. q.e.d.

DEFINITION 4.3. If E is a closed subset of G we let

$$I(E) = \{f \in K(G) : f(x) = 0 \ \forall x \in E\}$$

and $J(E) = \{f \in K(G): f = 0$ on a neighbourhood of $E\}$. We say that E is a set of synthesis for $K(G)$ if $I(E)$ is the closure of $J(E)$ in $K(G)$.

We now restrict our attention to some special groups, namely those corresponding to the compact rank-one Riemannian symmetric spaces. The possibilities are tabulated as in Table 1, see [2].

TABLE 1

G	H	G/H
$SO(n)$	$\{1\} \times SO(n-1)$	S^{n-1}
$SO(n)$	$S(\{\pm 1\} \times 0(n))$	$P^{n-1}(\boldsymbol{R})$
$SU(n)$	$S(\boldsymbol{T} \times U(n-1))$	$P^{n-1}(\boldsymbol{C})$
$Sp(n)$	$Sp(1) \times Sp(n-1)$	$P^{n-1}(\boldsymbol{H})$
$F_{4(-52)}$	$SO(9)$	P^2(Cayley).

If $k = \boldsymbol{R}, \boldsymbol{C}$, or $\boldsymbol{H}$, $P^m(k)$ denotes the space of k-lines in $k^{m+1} \cdot P^2$(Cayley) is the Cayley projective plane. The geometry of these spaces is described in [7].

In each case listed here there is a closed subgroup of G isomorphic to $\boldsymbol{T}$, which we will denote by A, such that

$$G = HAH\ . \tag{4.4}$$

Let $a: \boldsymbol{T} \to A$ be this isomorphism. Then if $\theta \in \boldsymbol{T}$ there exist $h_1, h_2 \in H$ with

$$h_1 a(\theta) h_2 = a(-\theta)\ . \tag{4.5}$$

On account of (4.4) and (4.5) it follows that every bi-H-invariant function is completely determined by its restriction to $A_+ = \{a(\theta): 0 \leqq \theta \leqq \pi\}$. Furthermore, the set $H(\operatorname{int} A_+)H$ is an open set of full measure in G.

For example, if $G = SO(n)$ and $H = \{1\} \times SO(n-1)$, with $n \geqq 3$, we can take

$$A = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & 0 \\ & 0 & I \end{pmatrix} : \ 0 \leqq \theta \leqq 2\pi \right\}\ .$$

For G and H as above, $\hat{G}_H$ and the zonal spherical functions have been completely determined, [16] and [11]. We can identify $\hat{G}_H$ with N and to each $n \in N$ the corresponding zonal spherical function is

$$\phi_n(a(\theta)) = R_n^{(\alpha,\beta)}(\cos\theta)\ , \qquad 0 \leqq \theta \leqq \pi\ , \tag{4.6}$$

where the indices (α, β) depend only on G/H.

The possible values of (α, β) are as in Table 2. See [2] for details. Note that if $d = \dim(G/H)$ then $\alpha = (d-2)/2$ and $\alpha \geqq \beta \geqq -1/2$. From the discussion above and (4.6) we see that for (G, H, α, β) as in Table 2 the correspondence $T: {}^H\boldsymbol{K}(G)^H \to AJ(\alpha, \beta, 0)$

$$Tf(x) = f(a(\arccos(x))) , \qquad -1 \leqq x \leqq 1 ,$$

is an isometric isomorphism.

TABLE 2

G/H	$\dim(G/H)$	α	β
$S^m(m \geqq 2)$	m	$(m-2)/2$	$(m-2)/2$
$P^m(\boldsymbol{R})$	m	$(m-2)/2$	$-1/2$
$P^m(\boldsymbol{C})$	$2m$	$(m-1)$	0
$P^m(\boldsymbol{H})$	$4m$	$2m-1$	1
P^2(Cayley)	16	7	3

In particular, suppose that G/H is a d-dimensional compact rank-one Riemannian symmetric space and $0 < \varepsilon < \pi/2$. Then every $f \in {}^H\boldsymbol{K}(G)^H$, when restricted to $\{a(\theta): \varepsilon \leqq \theta \leqq \pi - \varepsilon\}$, can be written as

$$f(a(\theta)) = \sum_{n=0}^{\infty} b_n \cos(n\theta) , \qquad \varepsilon \leqq \theta \leqq \pi - \varepsilon ,$$

with

$$\sum_{n=0}^{\infty} |b_n|(n+1)^{(d-1)/2} \leqq C\|f\|_K . \tag{4.7}$$

This is a consequence of Theorem 2.9.

Hence, if $d \geqq 3$, $\theta \to f(a(\theta))$ is differentiable on $]0, \pi[$. As in Chapter 3, we wish to use this to demonstrate the existence of sets of nonsynthesis.

THEOREM 4.8. *If G and H are as in Table 1, if the dimension of G/H is greater than two, and if $0 < \theta_0 < \pi$ then the double coset $Ha(\theta_0)H$ is not a set of synthesis for $\boldsymbol{K}(G)$.*

To prove this we will need the following lemma.

LEMMA 4.9. *If G and H are as in Table 1, $0 < \theta_0 < \pi$, and if U is a neighbourhood of $Ha(\theta_0)H$ in G then there exists $\delta > 0$ such that U contains*

$$H.\{a(\theta): |\theta - \theta_0| < \delta\}.H .$$

This follows from [15, Lemma VII 7.1].

Now fix θ_0 as in the statement of the theorem. Suppose that $E = H.a(\theta_0).H$ and $f \in J(E)$. Then Lemma 4.9 implies that there is a $\delta > 0$ such that $Pf(a(\theta)) = 0$ for all $|\theta - \theta_0| < \delta$. Hence $(d/d\theta)(Pf(a(\theta)))|_{\theta=\theta_0} = 0$. Since $d \geqq 3$, (4.7) tells us that we can define a bounded linear functional Λ on $\boldsymbol{K}(G)$ by setting

$$\Lambda(f) = (d/d\theta)(Pf(a(\theta)))|_{\theta=\theta_0} . \tag{4.10}$$

We have just seen that $J(E) \subseteq \ker(\Lambda)$, and so $\overline{J(E)} \subseteq \ker(\Lambda)$.

However, $I(E)$ is not contained in $\ker(\Lambda)$. For example, the function Ψ defined by

$$\Psi(h_1 a(\theta) h_2) = \cos(\theta) - \cos(\theta_0), \qquad h_1 h_2 \in H, \tag{4.11}$$

is in $I(E) \cap ({}^H\boldsymbol{K}(G)^H)$ but $\Lambda(\Psi) = -\sin(\theta_0) \neq 0$, on account of the choice of θ_0. This completes the proof of the theorem.

Observe that we could define a collection of bounded functionals Λ_j $(0 \leqq j \leqq [(d-1)/2])$ by setting

$$\Lambda_j(f) = (d/d\theta)^j (Pf(a(\theta)))|_{\theta=\theta_0}, \qquad 1 \leqq j \leqq [(d-1)/2],$$

and $\Lambda_0(f) = Pf(a(\theta_0))$. Then the spaces

$$i_j(\theta_0) = \{f \in \boldsymbol{K}(G) : \Lambda_l(f) = 0, \quad 0 \leqq l \leqq j\}$$

are all closed subspaces of $\boldsymbol{K}(G)$ containing $J(E)$ and

$$\overline{J(E)} \subset i_{[(d-1)/2]}(\theta_0) \subsetneqq \cdots \subsetneqq i_1(\theta_0) \subsetneqq I(E).$$

This property is similar to [28, Thm. 3].

The theorem of Herz [17] that the circle is a set of synthesis for the algebra of Fourier transforms on $\boldsymbol{R}^2$ suggests that the case of $SO(3)/SO(2)$ could be different from the higher dimensional cases described in Theorem 4.8.

In [25, Thm. 2] Ricci shows that nonanalytic functions operate locally on $\boldsymbol{K}^z(G)$, the subalgebra of central elements of $\boldsymbol{K}(G)$, when G is a compact connected semisimple Lie group.

THEOREM 4.12. *Let G/H be a compact rank-one Riemannian symmetric space of dimension $d > 1$. Let $x_0 \in H.\mathrm{int}(A_+).H$. Then there is a neighbourhood U of x_0 in G such that $A_{d/2}(\boldsymbol{T})$ operates on the real-valued elements of $({}^H\boldsymbol{K}(G)^H)|_U$.*

PROOF. Our hypothesis is that $x_0 = h_1 a(\theta_0) h_2$, for some $0 < \theta_0 < \pi$ and $h_1, h_2 \in H$. Let $2\delta = \min\{\theta_0, |\theta_0 - \pi/2|\}$ and put $U = H.\{a(\theta) : |\theta - \theta_0| < \delta\}.H$, an open set in G. Then $({}^H\boldsymbol{K}(G)^H)|_U$ is isomorphic with $AJ(\alpha, \beta, 0)|_I$, where I is the interval $\cos(\{\theta : |\theta - \theta_0| < \delta\})$. Now apply Theorem 3.3. q.e.d.

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