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A LOCAL REGULARIZATION OPERATOR FOR TRIANGULAR AND QUADRILATERAL FINITE ELEMENTS*

C. BERNARDI† AND V. GIRAULT†

Abstract. This paper develops a local regularization operator on triangular or quadrilateral finite elements built on structured or unstructured meshes. This operator is a variant of the regularization operator of Clément; however, ours is constructed via a local projection in a reference domain. We prove in this paper that it has the same optimal approximation properties as the standard interpolation operator, and we present some applications.

Key words. regularization operator, triangular finite elements, quadrilateral finite elements

AMS subject classifications. Primary, 65D05; Secondary, 65N30

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Introduction. Let Ω be a two-dimensional bounded open set with a polygonal boundary Γ. Let $\mathcal{T}_h$ be a triangulation or quadrangulation of $\overline{\Omega}$, and let Θ_h be a standard associated finite element space. The purpose of this paper is to construct an operator R_h that associates, with any function u in $L^1(\Omega)$, an element $R_h(u)$ in Θ_h and satisfies the same local approximation properties as the usual interpolation operator when u is sufficiently smooth. Since this operator must also act on functions that are not necessarily continuous, it replaces the nodal values of the function that is interpolated by adequate averages.

For triangular meshes, such operators were introduced by Clément in [7] and generalized by Bernardi in [2]. However, in contrast to [7], the averages in the present paper are computed in some reference domain; this idea was used in [2] to treat curved (isoparametric) triangles or simplices and also allows for an extension to quadrilateral meshes. In contrast to [2], they are computed on spaces of piecewise polynomial functions. Indeed, we will show by a simple counterexample that this is necessary to recover the usual interpolation error when the function that must be approximated is smooth.

Several modified versions of these operators exist; see [4, Chap. 4]. For instance, Scott and Zhang [14] use averages on the boundary of the elements, in particular when the associated degrees of freedom are on the boundary. The advantages are that, on one hand, the corresponding operator preserves the nullity of traces and that, on the other hand, it leaves invariant the functions of the discrete space. However, the drawback is that it is only defined on more regular functions, i.e., sufficiently smooth to have a trace on the boundary of elements. For this reason, we prefer first to construct a general operator and second to modify it in order that the new operator preserves the nullity of traces.

This paper is organized as follows. In section 1, we make precise the notation and we recall some basic results. We have chosen to treat separately, in sections 2 and 3, the discussion of the averaging process on triangular finite elements and on

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†Laboratoire d'Analyse Numérique, C.N.R.S. et Université Pierre et Marie Curie, B.C. 187, 4 Place Jussieu, 75252 Paris cedex 05, France (bernardi@ann.jussieu.fr, girault@ann.jussieu.fr). Part of the work of the second author was done while visiting the Indian Institute of Science in Bangalore (India) and was supported by the CEFIPRA Contract IFC/501-1/93/1553.

quadrilateral finite elements, because the techniques involved are somewhat different, especially in the case of non-Cartesian quadrilateral meshes. In section 4, the error estimates for these averages are used to derive the error estimates for the corresponding regularization operator. Section 5 is devoted to some applications.

1. Preliminaries and notation. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\mathcal{O}$. We denote by $|\mathcal{O}|$ the measure of $\mathcal{O}$. For any nonnegative integer m and any number p with $1 \leq p \leq \infty$, we use the standard Sobolev spaces

$$W^{m,p}(\mathcal{O}) = \Big\{ v \in L^p(\mathcal{O})\,;\, \frac{\partial^k v}{\partial x_1^i \partial x_2^{k-i}} \in L^p(\mathcal{O})\ ,\ 0 \leq i \leq k\ ,\ 1 \leq k \leq m \Big\},$$

equipped with the two seminorms

$$|v|_{W^{m,p}(\mathcal{O})} = \left(\sum_{i=0}^{m} \Big\| \frac{\partial^m v}{\partial x_1^i \partial x_2^{m-i}} \Big\|_{L^p(\mathcal{O})}^p \right)^{1/p},$$

$$[v]_{W^{m,p}(\mathcal{O})} = \Big(\Big\| \frac{\partial^m v}{\partial x_1^m} \Big\|_{L^p(\mathcal{O})}^p + \Big\| \frac{\partial^m v}{\partial x_2^m} \Big\|_{L^p(\mathcal{O})}^p \Big)^{1/p}$$

and norm

$$\|v\|_{W^{m,p}(\mathcal{O})} = \left(\sum_{k=0}^{m} |v|_{W^{k,p}(\mathcal{O})}^p \right)^{1/p},$$

with the usual modification for $p = \infty$. By interpolation, this definition can be extended to nonintegral values of m. In particular, for $1 \leq p < \infty$, fractional order spaces include the trace space of functions of $W^{1,p}(\mathcal{O})$, that is, $W^{1-1/p,p}(\partial\mathcal{O})$, equipped with the norm

$$\|\mu\|_{W^{1-1/p,p}(\partial\mathcal{O})} = \inf_{v \in W^{1,p}(\mathcal{O}), v_{|\partial\mathcal{O}} = \mu} \|v\|_{W^{1,p}(\mathcal{O})}\,.$$

The reader is referred to Lions and Magenes [12, Chap. 1] for fractional-order Sobolev spaces.

Finally, let us recall two fundamental results of polynomial interpolation. For any nonnegative integer k, let $\mathbb{P}_k$ be the space of polynomials in two variables of *total* degree less than or equal to k, and let $\mathbb{Q}_k$ be the space of polynomials in two variables of degree less than or equal to k in *each* variable. Note that $\mathbb{P}_k$ and $\mathbb{Q}_k$ coincide for $k = 0$, but otherwise $\mathbb{Q}_k$ is a subspace of $\mathbb{P}_{2k}$. For any nonnegative integers k and ℓ, the polynomial spaces $\mathbb{P}_k$ and $\mathbb{Q}_k$ are contained in $W^{\ell,p}(\mathcal{O})$, and we can define the quotient spaces $W^{\ell,p}(\mathcal{O})/\mathbb{P}_k$ and $W^{\ell,p}(\mathcal{O})/\mathbb{Q}_k$, which are also Banach spaces equipped with the quotient norms

$$\forall \dot{v} \in W^{\ell,p}(\mathcal{O})/\mathbb{P}_k,\ \|\dot{v}\|_{W^{\ell,p}(\mathcal{O})/\mathbb{P}_k} = \inf_{r \in \mathbb{P}_k} \|v + r\|_{W^{\ell,p}(\mathcal{O})},$$

$$\forall \dot{v} \in W^{\ell,p}(\mathcal{O})/\mathbb{Q}_k,\ \|\dot{v}\|_{W^{\ell,p}(\mathcal{O})/\mathbb{Q}_k} = \inf_{r \in \mathbb{Q}_k} \|v + r\|_{W^{\ell,p}(\mathcal{O})}\,.$$

The next two theorems state important properties of these quotient spaces. The first one is proven in Deny and Lions [8] (cf. also Nečas [13, Chap. 1]) and the second one in Ciarlet and Raviart [6]. A more general result, in the finite union of star-shaped domains with respect to balls, is proven by Dupont and Scott [9] and also by Durán [10] in a constructive way. This construction, inspired by Sobolev's explicit representation of a function as a polynomial plus a remainder term, is based on the representation of a function as an averaged Taylor's series. We refer to [4] for more details.

THEOREM 1.1. *Assume that $\mathcal{O}$ is a bounded and connected open set in $\mathbb{R}^2$ with a Lipschitz-continuous boundary. For each integer $k \geq 0$ and number p with $1 \leq p \leq \infty$, there exists a constant C such that*

$$\forall v \in W^{k+1,p}(\mathcal{O}),\ \|\dot{v}\|_{W^{k+1,p}(\mathcal{O})/\mathbb{P}_k} \leq C\, |v|_{W^{k+1,p}(\mathcal{O})}\,. \tag{1.1}$$

THEOREM 1.2. *Assume that $\mathcal{O}$ is a bounded and connected open set in $\mathbb{R}^2$ with a Lipschitz-continuous boundary. For each integer $k \geq 0$ and number p, with $1 \leq p \leq \infty$, there exists a constant C such that*

$$\forall v \in W^{k+1,p}(\mathcal{O}),\ \|\dot{v}\|_{W^{k+1,p}(\mathcal{O})/\mathbb{Q}_k} \leq C\, [v]_{W^{k+1,p}(\mathcal{O})}\,. \tag{1.2}$$

2. A projection operator on triangular meshes. Let h be a positive discretization parameter. Recall (cf. Ciarlet [5, Chap. II]) that a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ is a partition of $\overline{\Omega}$ into nondegenerate triangles T with diameter bounded by h, such that each pair of triangles T_1 and T_2 of $\mathcal{T}_h$ are either disjoint or share a vertex or a complete side. We denote by h_T the diameter of T, by ρ_T the diameter of the circle inscribed in T, and we set

$$\sigma_T = \frac{h_T}{\rho_T}\,.$$

We assume that the family of triangulations $(\mathcal{T}_h)_h$ is regular, i.e., there exists a constant σ, independent of h, such that

$$\forall T \in \mathcal{T}_h,\ \sigma_T \leq \sigma.$$

Let us fix a positive integer k, and let Θ_h be the standard finite element space

$$\Theta_h = \Big\{\theta_h \in \mathcal{C}^0(\overline{\Omega});\ \forall T \in \mathcal{T}_h,\ \theta_{h|T} \in \mathbb{P}_k\Big\}. \tag{2.1}$$

This definition must be completed by specifying the degrees of freedom of the functions of Θ_h: for the sake of simplicity, we assume that, in each triangle T, the degrees of freedom of a function θ_h in Θ_h are the values of θ_h on the principal lattice of order k, as in the example of Figure 1. In other words, the degrees of freedom of θ_h are its values at a set of particular nodes of the triangulation $\mathcal{T}_h$. Let N be the number of these nodes and let $\{\boldsymbol{a}_i,\, 1 \leq i \leq N\}$ denote this set of nodes. For any node $\boldsymbol{a}_i$, let the macroelement Δ_i be the union of the triangles of $\mathcal{T}_h$ that share this node $\boldsymbol{a}_i$, as in Figure 2.

Remark 1. The results below still hold for more general degrees of freedom defined by linear functionals, if these functionals are continuous on functions in $\mathcal{C}^0(\overline{\Omega})$. But our proofs are not valid for Hermite-type finite elements, for instance.

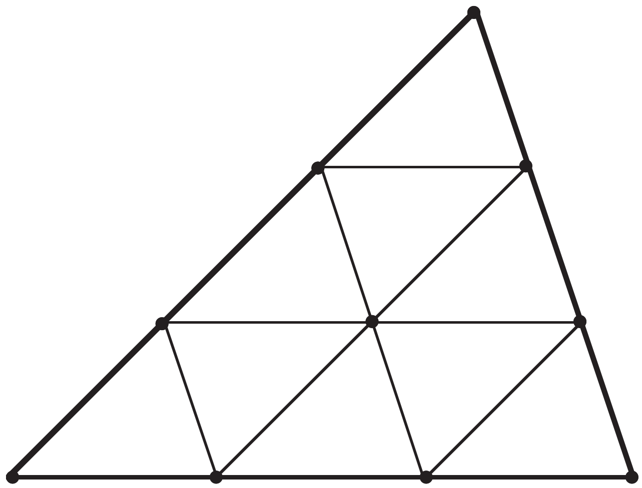

Fig. 1

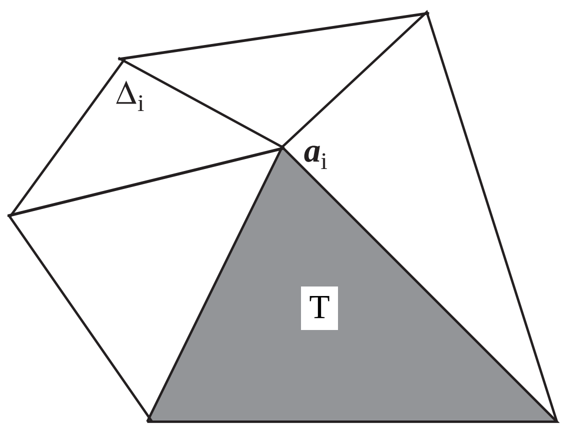

Fig. 2

We set

$$h_{\Delta_i} = \sup_{T \subset \Delta_i} h_T, \qquad \rho_{\Delta_i} = \inf_{T \subset \Delta_i} \rho_T, \qquad \sigma_{\Delta_i} = \frac{h_{\Delta_i}}{\rho_{\Delta_i}}.$$

Since the family of triangulations $(\mathcal{T}_h)_h$ is regular, it can be proved that (cf. Bernardi [2], Clément [7])

(i) there exists a constant L, independent of h, such that, for $1 \le i \le N$, Δ_i consists of at most L triangles T (more precisely, if $\boldsymbol{a}_i$ lies in the interior of T, then Δ_i coincides with T; if $\boldsymbol{a}_i$ lies on one side of T, then Δ_i consists of either two triangles, or only one if that side is a part of Γ, and if $\boldsymbol{a}_i$ is a vertex of T, then Δ_i has at most L triangles);

(ii) there exists a constant $\hat{c}_1$, independent of h, such that, for $1 \le i \le N$,

$$\forall T \subset \Delta_i,\ h_{\Delta_i} \le \hat{c}_1\, h_T; \tag{2.2}$$

(iii) there exists a constant $\hat{c}_2$, independent of h, such that, for $1 \le i \le N$,

$$\sigma_{\Delta_i} \le \hat{c}_2\, \sigma\,. \tag{2.3}$$

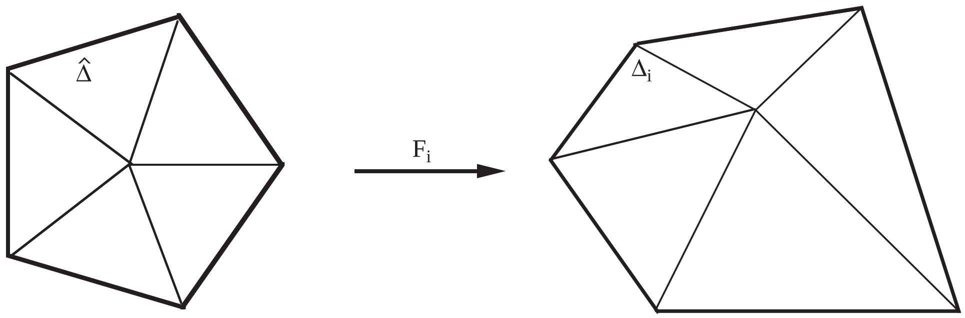

FIG. 3

Note also that

(iv) there exists a constant K, independent of h (in fact, $K = (k+1)(k+2)/2$), such that any T in $\mathcal{T}_h$ belongs to at most K macroelements Δ_i.

Consider a macroelement Δ_i made of, say, J triangles; we associate with Δ_i a reference macroelement $\hat{\Delta}_i$, made of J equal isosceles reference unit triangles $\hat{T}_j$, as in Figure 3. Owing to property (i), there exists only a fixed number $\hat{L}$ of different reference macroelements $\hat{\Delta}_i$, for $1 \le i \le N$, where $\hat{L}$ is independent of h. Therefore, since all the geometric characteristics of these reference macroelements can be bounded by constants independent of h, to alleviate notation we shall not distinguish them and suppress their index i, thus denoting them indifferently by $\hat{\Delta}$. It can be easily proved that, for each macroelement Δ_i, there exists a continuous and invertible mapping F_i that is affine on each reference triangle $\hat{T}$ of $\hat{\Delta}$:

$$\forall \hat{\boldsymbol{x}} \in \hat{T},\ F_i(\hat{\boldsymbol{x}}) = B_T \hat{\boldsymbol{x}} + \boldsymbol{b}_T,$$

such that

$$\Delta_i = F_i(\hat{\Delta})\,.$$

It follows from the above construction that each matrix B_T is nonsingular and

$$\|B_T\| \le \hat{c}_3\, h_T, \quad \|B_T^{-1}\| \le \frac{\hat{c}_4}{\rho_T}, \quad \hat{c}_5 \rho_T^2 \le |\det(B_T)| \le \hat{c}_6\, h_T^2. \tag{2.4}$$

We associate with Θ_h the local finite element spaces

$$\Theta(\hat{\Delta}) = \Big\{ \hat{\theta} \in \mathcal{C}^0(\hat{\Delta})\,;\, \forall \hat{T} \subset \hat{\Delta},\ \hat{\theta}_{|\hat{T}} \in \mathbb{P}_k \Big\}\,, \tag{2.5}$$

$$\Theta(\Delta_i) = \Big\{ \theta \in \mathcal{C}^0(\Delta_i)\,;\, \forall T \subset \Delta_i,\ \theta_{|T} \in \mathbb{P}_k \Big\}\,. \tag{2.6}$$

Then, for any function $\hat{u}$ in $L^1(\hat{\Delta})$, we define $\hat{r}(\hat{u})$ in $\Theta(\hat{\Delta})$ by

$$\forall \hat{\theta} \in \Theta(\hat{\Delta}),\ \int_{\hat{\Delta}} (\hat{r}(\hat{u}) - \hat{u})\hat{\theta}\, d\hat{\boldsymbol{x}} = 0\,, \tag{2.7}$$

and, for any function u in $L^1(\Delta_i)$, we define $r_i(u)$ in $\Theta(\Delta_i)$ by

$$r_i(u) \circ F_i = \hat{r}(u \circ F_i)\,, \tag{2.8}$$

a relation that is often denoted symbolically in the literature by

$$\widehat{r_i(u)} = \hat{r}(\hat{u})\,.$$

Clearly, $\hat{r}$ is a projection operator from $L^1(\hat{\Delta})$ onto $\Theta(\hat{\Delta})$ (orthogonal in $L^2(\hat{\Delta})$). But in general, the operator r_i, which is continuous from $L^1(\Delta_i)$ onto $\Theta(\Delta_i)$, is not an orthogonal projection operator for the scalar product of $L^2(\Delta_i)$. Our first theorem establishes an L^p-error estimate for r_i.

THEOREM 2.1. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations. For any integers k and ℓ with $k \geq 1$ and $0 \leq \ell \leq k+1$ and any number p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that, for any macroelement Δ_i, any triangle T contained in Δ_i, and any function u in $W^{\ell,p}(\Delta_i)$, the following inequality holds:*

$$\|u - r_i(u)\|_{L^p(T)} \leq C\, h_T^{\ell} |u|_{W^{\ell,p}(\Delta_i)}\,. \tag{2.9}$$

Proof. The discussion depends upon the value of ℓ. First suppose that ℓ is equal to zero, i.e., that u belongs to $L^p(\Delta_i)$. Let us fix a triangle T_0 in Δ_i; we have

$$\|u - r_i(u)\|_{L^p(T_0)} = |\det(B_{T_0})|^{1/p} \|\hat{u} - \hat{r}(\hat{u})\|_{L^p(\hat{T})} \leq |\det(B_{T_0})|^{1/p} \|\hat{u} - \hat{r}(\hat{u})\|_{L^p(\hat{\Delta})}\,.$$

But by the definition (2.7),

$$\|\hat{r}(\hat{u})\|^2_{L^2(\hat{\Delta})} \leq \|\hat{u}\|_{L^p(\hat{\Delta})}\, \|\hat{r}(\hat{u})\|_{L^{p'}(\hat{\Delta})}\,,$$

where p' denotes the dual exponent of p:

$$\frac{1}{p} + \frac{1}{p'} = 1\,.$$

Note that all norms are equivalent on the finite element space $\Theta(\hat{\Delta})$, since it has a finite dimension, and that the equivalence constants are bounded by a fixed constant (as $\hat{\Delta}$ can only take a fixed number of configurations). Therefore, since $\hat{r}(\hat{u})$ belongs to $\Theta(\hat{\Delta})$, for each number p with $1 \leq p \leq \infty$, there exist positive constants $\hat{c}_p$ and $\hat{C}_p$, which depend only on p and the dimension of $\Theta(\hat{\Delta})$, such that

$$\hat{c}_p \|\hat{r}(\hat{u})\|_{L^p(\hat{\Delta})} \leq \|\hat{r}(\hat{u})\|_{L^2(\hat{\Delta})} \leq \hat{C}_p \|\hat{r}(\hat{u})\|_{L^p(\hat{\Delta})}\,. \tag{2.10}$$

Hence,

$$\|\hat{r}(\hat{u})\|_{L^p(\hat{\Delta})} \leq \frac{1}{\hat{c}_{p'}\hat{c}_p} \|\hat{u}\|_{L^p(\hat{\Delta})}\,, \tag{2.11}$$

which proves that $\hat{r}$ is stable in $L^p(\hat{\Delta})$ for all p with $1 \leq p \leq \infty$. As a consequence,

$$\begin{aligned}\|u - r_i(u)\|_{L^p(T_0)} &\leq |\det(B_{T_0})|^{1/p} \Big(\|\hat{u}\|_{L^p(\hat{T})} + \frac{1}{\hat{c}_{p'}\hat{c}_p} \|\hat{u}\|_{L^p(\hat{\Delta})} \Big) \\ &\leq \Big(1 + \frac{1}{\hat{c}_{p'}\hat{c}_p}\Big) |\det(B_{T_0})|^{1/p} \|\hat{u}\|_{L^p(\hat{\Delta})}\,.\end{aligned}$$

But

$$\|\hat{u}\|_{L^p(\hat{\Delta})} = \Bigg(\sum_{T \subset \Delta} \frac{1}{|\det(B_T)|} \|u\|^p_{L^p(T)} \Bigg)^{1/p}\,.$$

Therefore, using (2.4) and the definition of σ_{Δ_i}, we obtain

$$\|u - r_i(u)\|_{L^p(T_0)} \le \hat{C}_1\Big(1 + \frac{1}{\hat{c}_{p'}\hat{c}_p}\Big)\sigma_{\Delta_i}^{2/p}\|u\|_{L^p(\Delta_i)}\,. \tag{2.12}$$

Now, consider the case where ℓ is equal to one, and take u in $W^{1,p}(\Delta_i)$. As $\hat{r}$ is a projection, observe that

$$\forall\hat{\theta} \in \Theta(\hat{\Delta}),\ \hat{u} - \hat{r}(\hat{u}) = \hat{u} - \hat{\theta} - \hat{r}(\hat{u} - \hat{\theta})\,.$$

Therefore, (2.11) yields, for all numbers p, $1 \le p \le \infty$,

$$\forall\hat{\theta} \in \Theta(\hat{\Delta}),\ \|\hat{u} - \hat{r}(\hat{u})\|_{L^p(\hat{\Delta})} \le \Big(1 + \frac{1}{\hat{c}_{p'}\hat{c}_p}\Big)\|\hat{u} - \hat{\theta}\|_{L^p(\hat{\Delta})}\,. \tag{2.13}$$

Let $\hat{\theta}$ run through the constant functions on $\hat{\Delta}$. Then Theorem 1.1 with $k = 0$ and (2.13) give

$$\|\hat{u} - \hat{r}(\hat{u})\|_{L^p(\hat{\Delta})} \le \hat{C}_2\Big(1 + \frac{1}{\hat{c}_{p'}\hat{c}_p}\Big)|\hat{u}|_{W^{1,p}(\hat{\Delta})}\,,$$

where the constant $\hat{C}_2$ depends only on $\hat{\Delta}$. But

$$|\hat{u}|_{W^{1,p}(\hat{\Delta})} \le \left(\sum_{T\subset\Delta} \frac{\|B_T\|^p}{|\det(B_T)|}|u|^p_{W^{1,p}(T)}\right)^{1/p}. \tag{2.14}$$

Therefore,

$$\|u - r_i(u)\|_{L^p(T_0)} \le \hat{C}_3\sigma_{\Delta_i}^{2/p}\,h_{\Delta_i}|u|_{W^{1,p}(\Delta_i)}\,. \tag{2.15}$$

Finally, let ℓ be ≥ 2 and take u in $W^{\ell,p}(\Delta_i)$. Then $\hat{u}$ is continuous, and we can choose in (2.13) $\hat{\theta}$ equal to $\hat{I}(\hat{u})$, the standard interpolant of $\hat{u}$ in $\Theta(\hat{\Delta})$. Furthermore, $\hat{u}$ belongs to $W^{\ell,p}(\hat{T})$ for all $\hat{T}$ contained in $\hat{\Delta}$, and as $\ell \le k+1$, it follows from Theorem 1.1 that

$$\|\hat{u} - \hat{I}(\hat{u})\|_{L^p(\hat{\Delta})} = \left(\sum_{\hat{T}\subset\hat{\Delta}} \|\hat{u} - \hat{I}(\hat{u})\|^p_{L^p(\hat{T})}\right)^{1/p} \le \hat{C}_4\left(\sum_{\hat{T}\subset\hat{\Delta}} |\hat{u}|^p_{W^{\ell,p}(\hat{T})}\right)^{1/p}.$$

Hence, we easily derive from (2.13) that

$$\|u - r_i(u)\|_{L^p(T_0)} \le \hat{C}_5\sigma_{\Delta_i}^{2/p}\,h^{\ell}_{\Delta_i}\left(\sum_{T\subset\Delta_i} |u|^p_{W^{\ell,p}(T)}\right)^{1/p} \le \hat{C}_5\sigma_{\Delta_i}^{2/p}\,h^{\ell}_{\Delta_i}|u|_{W^{\ell,p}(\Delta_i)},$$

since u belongs to $W^{\ell,p}(\Delta_i)$. ∎

The next theorem uses the argument of Theorem 2.1 to derive a $W^{1,p}$-error estimate for r_i.

THEOREM 2.2. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations. For any integers k and ℓ with $k \ge 1$ and $1 \le \ell \le k+1$ and any number p with $1 \le p \le \infty$, there exists a constant C, independent of h, such that, for any macroelement Δ_i, any triangle T contained in Δ_i, and any function u in $W^{\ell,p}(\Delta_i)$, we have*

$$|u - r_i(u)|_{W^{1,p}(T)} \le C\,h_T^{\ell-1}|u|_{W^{\ell,p}(\Delta_i)}. \tag{2.16}$$

Proof. Here again, the discussion depends upon the value of ℓ. Take first ℓ equal to one, and u in $W^{1,p}(\Delta_i)$. We have

$$|u - r_i(u)|_{W^{1,p}(T_0)} \leq |\det(B_{T_0})|^{1/p} \|B_{T_0}^{-1}\| \, |\hat{u} - \hat{r}(\hat{u})|_{W^{1,p}(\hat{\Delta})}\,.$$

For any $\hat{\theta}$ in $\Theta(\hat{\Delta})$, we can write

$$|\hat{u} - \hat{r}(\hat{u})|_{W^{1,p}(\hat{\Delta})} \leq |\hat{u} - \hat{\theta}|_{W^{1,p}(\hat{\Delta})} + |\hat{r}(\hat{u} - \hat{\theta})|_{W^{1,p}(\hat{\Delta})} \leq |\hat{u} - \hat{\theta}|_{W^{1,p}(\hat{\Delta})} + \hat{C}_1 \|\hat{r}(\hat{u} - \hat{\theta})\|_{L^p(\hat{\Delta})}$$

because all norms are equivalent on $\Theta(\hat{\Delta})$ and the equivalence constant $\hat{C}_1$ depends only on $\hat{\Delta}$. Then (2.11) implies that

$$\forall \hat{\theta} \in \Theta(\hat{\Delta}), \ |\hat{u} - \hat{r}(\hat{u})|_{W^{1,p}(\hat{\Delta})} \leq \Big(1 + \frac{\hat{C}_1}{\hat{c}_{p'}\hat{c}_p}\Big) \|\hat{u} - \hat{\theta}\|_{W^{1,p}(\hat{\Delta})}\,. \tag{2.17}$$

As previously, letting $\hat{\theta}$ run through the constant functions yields

$$|\hat{u} - \hat{r}(\hat{u})|_{W^{1,p}(\hat{\Delta})} \leq \hat{C}_2 |\hat{u}|_{W^{1,p}(\hat{\Delta})}. \tag{2.18}$$

Therefore,

$$|u - r_i(u)|_{W^{1,p}(T_0)} \leq \hat{C}_2 |\det(B_{T_0})|^{1/p} \|B_{T_0}^{-1}\| |\hat{u}|_{W^{1,p}(\hat{\Delta})} \leq \hat{C}_3 \sigma_{\Delta_i}^{1+2/p} |u|_{W^{1,p}(\Delta_i)}.$$

When ℓ is ≥ 2, we choose $\hat{\theta} = \hat{I}(\hat{u})$ in (2.17). This choice gives

$$|\hat{u} - \hat{r}(\hat{u})|_{W^{1,p}(\hat{\Delta})} \leq \hat{C}_4 \bigg(\sum_{\hat{T} \subset \hat{\Delta}} |\hat{u}|^p_{W^{\ell,p}(\hat{T})} \bigg)^{1/p}.$$

Therefore,

$$|u - r_i(u)|_{W^{1,p}(T_0)} \leq \hat{C}_5 \sigma_{\Delta_i}^{1+2/p} \, h_{\Delta_i}^{\ell-1} |u|_{W^{\ell,p}(\Delta_i)}\,.$$

This proves the theorem. ∎

Observe that when ℓ is ≥ 2, the $W^{\ell,p}$-norm of $\hat{u}$ is never taken on $\hat{\Delta}$ but only separately on each $\hat{T}$. The reason for this is that, although u belongs to $W^{\ell,p}(\Delta_i)$, $\hat{u} = u \circ F_i$ does not belong, in general, to $W^{\ell,p}(\hat{\Delta})$. This lack of regularity explains why $\hat{\theta}$ is chosen in the local finite element space $\Theta(\hat{\Delta})$ and not in $\mathbb{P}_k$. In fact, the following counterexample shows that this last choice does not yield the estimates of Theorems 2.1 and 2.2.

A counterexample. Let h be any positive real number; define the two consecutive intervals $I_1 = [0, 2h]$ and $I_2 = [2h, 3h]$, and set $\Delta = I_1 \cup I_2$. We associate with Δ the reference macroelement $\hat{\Delta} = \hat{I}_1 \cup \hat{I}_2$, where $\hat{I}_1 = [-1, 0]$ and $\hat{I}_2 = [0, 1]$. The continuous piecewise affine mapping F that maps $\hat{\Delta}$ onto Δ is

$$F(t) = \begin{cases} 2h(1+t) & \text{on } \ \hat{I}_1, \\ 2h + ht & \text{on } \ \hat{I}_2. \end{cases}$$

Now, consider the function $v(x) = x$, and let $p = \hat{r}(\hat{v})$ be the projection of $\hat{v} = v \circ F$ onto $\mathbb{P}_1$ for the $L^2(\hat{\Delta})$ scalar product; i.e.,

$$\int_{-1}^{1} p(t)\, dt = \int_{-1}^{1} (v \circ F)(t)\, dt \quad \text{and} \quad \int_{-1}^{1} t\, p(t)\, dt = \int_{-1}^{1} t\, (v \circ F)(t)\, dt.$$

An easy calculation gives

$$p(t) = \frac{7}{4}\,h + \frac{3}{2}\,h\,t,$$

and

$$(v \circ F)(t) - p(t) = \begin{cases} h(\frac{1}{4} + \frac{1}{2}t) & \text{on } \hat{I}_1, \\ h(\frac{1}{4} - \frac{1}{2}t) & \text{on } \hat{I}_2. \end{cases}$$

Then, on one hand,

$$\|v - p \circ F^{-1}\|_{L^2(\Delta)} = \frac{1}{4}\,h^{3/2},$$

and on the other hand,

$$|v|_{H^2(\Delta)} = 0, \quad |v|_{H^1(\Delta)} = \sqrt{3\,h}, \quad \|v\|_{L^2(\Delta)} = 3\,h^{3/2}\,.$$

As a consequence,

$$\frac{\|v - p \circ F^{-1}\|_{L^2(\Delta)}}{\|v\|_{H^2(\Delta)}} = \frac{h}{4\sqrt{3}(1+3h^2)^{1/2}},$$

which is exactly of the order of h and not of the order of h^2.

Remark 2. The results of this section can readily be extended to tetrahedral triangulations of three-dimensional domains with polyhedral boundaries.

Remark 3. The statement of Theorem 2.1 (resp., Theorem 2.2) extends to the case where u belongs to $W^{\ell,q}(\Delta_i)$ for any q such that $W^{\ell,q}(\Delta_i)$ is continuously embedded in $L^p(\Delta_i)$ (resp., $W^{1,p}(\Delta_i)$). More precisely, under the assumptions of Theorem 2.1, if u belongs to $W^{\ell,q}(\Delta_i)$, the following bounds hold:

$$\begin{aligned} &\text{if } q \ge p,\ \|u - r_i(u)\|_{L^p(\Delta_i)} \le C\,h_{\Delta_i}^{\ell}\,h_{\Delta_i}^{2/p-2/q}|u|_{W^{\ell,q}(\Delta_i)}; \\ &\text{if } q < p,\ \|u - r_i(u)\|_{L^p(\Delta_i)} \le C\,h_{\Delta_i}^{\ell}\,\frac{1}{\rho_{\Delta_i}^{2/q-2/p}}|u|_{W^{\ell,q}(\Delta_i)}. \end{aligned} \tag{2.19}$$

Theorem 2.2 has a similar extension. Note that these local estimates are optimal. However, for $q \ge p$, summing up the first estimate on all macroelements and using the Hölder's inequality does not lead to a global optimal estimate, while for $q < p$ summing up the second bound leads to an optimal estimate, of order $h^{\ell+2/p-2/q}$, thanks to the Jensen's inequality.

Remark 4. The argument used in Theorem 2.2 for proving (2.18) can be readily extended to show that

$$|\hat{u} - \hat{r}(\hat{u})|_{W^{s,p}(\hat{\Delta})} \le \hat{C}_2'|\hat{u}|_{W^{t,p}(\hat{\Delta})},$$

for any real numbers s and t with $0 \le s \le 1$ and $s \le t \le 1$. It also holds for $1 < t \le 2$ by letting $\hat{\theta}$ run through piecewise affine functions. So, the following estimate holds for any real numbers s and t with $0 \le s \le 1$ and $s \le t \le k+1$ and any number p with $1 \le p \le \infty$, provided that the function u belongs to $W^{t,p}(\Delta_i)$:

$$|u - r_i(u)|_{W^{s,p}(T)} \le C\,h_T^{t-s}\,|u|_{W^{t,p}(\Delta_i)}. \tag{2.20}$$

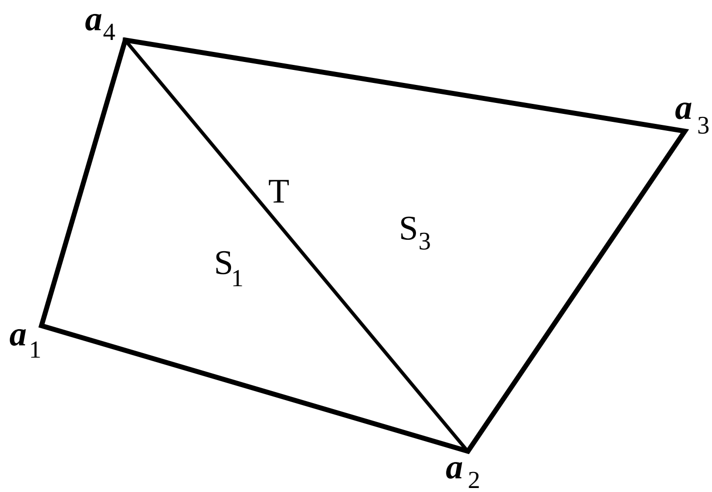

Fig. 4

This estimate can also be derived from the principal theorem of interpolation between Banach spaces (see [12, Chap. 1]) with the seminorm $|\cdot|_{W^{t,p}(\Delta_i)}$ replaced by the norm $\|\cdot\|_{W^{t,p}(\Delta_i)}$ in the right-hand side. The proof of the following result, concerning the approximation on sides f of elements T, relies on similar arguments: for any real numbers s, t, and p with $0 \leq s \leq 1$, $s + \frac{1}{p} < t \leq k+1$, and $1 \leq p < \infty$, if the function u belongs to $W^{t,p}(\Delta_i)$, we have the estimate

$$|u - r_i(u)|_{W^{s,p}(f)} \leq C\, h_T^{t-s-\frac{1}{p}}\, |u|_{W^{t,p}(\Delta_i)}. \tag{2.21}$$

3. A projection operator on quadrilateral meshes. Let $\mathcal{T}_h$ be a quadrangulation of $\overline{\Omega}$ made of convex and nondegenerate quadrilaterals T (i.e., not reduced to triangles) with diameter bounded by h. Let T be one of these quadrilaterals, let $\boldsymbol{a}_i$ be its vertices, $1 \leq i \leq 4$, numbered counterclockwise, and let S_i denote its subtriangle with vertices $\boldsymbol{a}_{i-1}$, $\boldsymbol{a}_i$, $\boldsymbol{a}_{i+1}$, the indices being numbered modulo four, as in Figure 4. Let h_i be the diameter of S_i, and ρ_i the diameter of its inscribed circle. We set

$$h_T = \sup_{1\leq i\leq 4} h_i, \quad \rho_T = 2 \inf_{1\leq i\leq 4} \rho_i, \quad \text{and} \quad \sigma_T = \frac{h_T}{\rho_T}.$$

Clearly, h_T is the diameter of T, and σ_T is a measure of the nondegeneracy of T. Here also, we assume that the family of quadrangulations $(\mathcal{T}_h)_h$ is regular, i.e., there exists a constant σ, independent of h, such that

$$\forall\, T \in \mathcal{T}_h,\ \sigma_T \leq \sigma.$$

In contrast to triangular finite element spaces, in the case of quadrilaterals, the finite elements are defined first on the reference square $\hat{T} = [0,1] \times [0,1]$ and, after they are transformed into functions (generally, not polynomials), defined on the quadrilateral T by a transformation from T onto $\hat{T}$. More precisely (cf. [6]), as T is convex and nondegenerate, there exists an invertible, bilinear mapping $\mathcal{F}_T$ (i.e., with components in $\mathbb{Q}_1$) that maps $\hat{T}$ onto T with $\boldsymbol{a}_i = \mathcal{F}_T(\hat{\boldsymbol{a}}_i)$, $1 \leq i \leq 4$, where $\hat{\boldsymbol{a}}_1 = (0,0)$, $\hat{\boldsymbol{a}}_2 = (1,0)$, $\hat{\boldsymbol{a}}_3 = (1,1)$, and $\hat{\boldsymbol{a}}_4 = (0,1)$ are the vertices of $\hat{T}$. Let $D\,\mathcal{F}_T$

and J_T (resp., $D\,\mathcal{F}_T^{-1}$ and J_T^{-1}) denote the Jacobian matrix and the Jacobian of $\mathcal{F}_T$ (resp., $\mathcal{F}_T^{-1}$). In the case of quadrilaterals none of these quantities are constant, but they satisfy the following bounds:

$$\|J_T\|_{L^\infty(\hat{T})} = 2 \sup_{1\le i\le 4} |S_i| \le \frac{\sqrt{3}}{2} h_T^2, \quad \|J_T^{-1}\|_{L^\infty(T)} = \frac{1}{2\,\inf_{1\le i\le 4}|S_i|} < \frac{8}{\pi \rho_T^2}\,, \tag{3.1}$$

$$\|D\,\mathcal{F}_T\|_{L^\infty(\hat{T})} \le C_1\, h_T, \quad \|D\,\mathcal{F}_T^{-1}\|_{L^\infty(T)} \le C_2 \frac{\sigma_T}{\rho_T}\,. \tag{3.2}$$

Then we define the function space $\mathcal{Q}_k(T)$ by

$$\mathcal{Q}_k(T) = \left\{ q = \hat{q} \circ \mathcal{F}_T^{-1}\,;\ \hat{q} \in \mathbb{Q}_k \right\}.$$

The corresponding standard finite element space, for a positive integer k, is

$$\Theta_h = \left\{ \theta_h \in \mathcal{C}^0(\overline{\Omega});\ \forall T \in \mathcal{T}_h,\ \theta_{h|T} \in \mathcal{Q}_k(T) \right\}. \tag{3.3}$$

Here also, for the sake of simplicity, we assume that, in each T, the degrees of freedom of any function of Θ_h are its values at the principal lattice of order k. Let N be the number of nodes where these degrees of freedom are defined, and let $\{\boldsymbol{a}_i, 1 \le i \le N\}$, denote this set of nodes. Here again, for any node $\boldsymbol{a}_i$, let the macroelement Δ_i be the union of the quadrilaterals of $\mathcal{T}_h$ that share this node $\boldsymbol{a}_i$, and define

$$h_{\Delta_i} = \sup_{T\subset\Delta_i} h_T, \quad \rho_{\Delta_i} = \inf_{T\subset\Delta_i} \rho_T, \quad \sigma_{\Delta_i} = \frac{h_{\Delta_i}}{\rho_{\Delta_i}}\,.$$

If the mesh is Cartesian, the situation is simpler than that of the previous section, because for all nodes $\boldsymbol{a}_i$, Δ_i consists of one, two, or four quadrilaterals (or possibly three if $\boldsymbol{a}_i$ is a boundary node) and the reference macroelement associated with any Δ_i is made of at most four unit squares. But we do not necessarily choose a Cartesian mesh, and at a node where the mesh is not Cartesian, the reference macroelement cannot consist of unit squares. Indeed, let $\boldsymbol{a}_i$ denote a node where the mesh is not Cartesian, and suppose that the corresponding macroelement Δ_i has J elements. Consider one element T in Δ_i, and to simplify the discussion, let $\boldsymbol{a}_i = \boldsymbol{a}_1$, as in the example of Figure 5, and let $S = S_1$ and $S' = S_3$ be the two corresponding subtriangles of T. Let D_i be the auxiliary macroelement consisting of all these subtriangles S with common vertex $\boldsymbol{a}_i$, and let $\tilde{D}_i$ be the corresponding auxiliary reference macroelement consisting of J equal isosceles unit triangles as in Figure 3.

Let $\tilde{S}$ be one of these triangles with vertices denoted by $\tilde{\boldsymbol{a}}_1 = (0,0)$, $\tilde{\boldsymbol{a}}_2$, and $\tilde{\boldsymbol{a}}_4$, as in Figure 5. Since T is convex and not reduced to a triangle, there exists a unique affine invertible mapping F_S such that $S = F_S(\tilde{S})$ and $\boldsymbol{a}_i = F_S(\tilde{\boldsymbol{a}}_i)$, $i = 1, 2, 4$:

$$\boldsymbol{x} = F_S(\tilde{\boldsymbol{x}}) = B_S\,\tilde{\boldsymbol{x}} + \boldsymbol{a}_1\,.$$

We construct an auxiliary reference element $\tilde{T}$ by means of the mapping F_S in the following way. Let $\tilde{\boldsymbol{a}}_3 = F_S^{-1}(\boldsymbol{a}_3)$, and let $\tilde{S}'$ denote the triangle with vertices $\tilde{\boldsymbol{a}}_2$, $\tilde{\boldsymbol{a}}_3$, and $\tilde{\boldsymbol{a}}_4$. We associate with T the auxiliary reference element $\tilde{T} = \tilde{S} \cup \tilde{S}'$. Clearly, $\tilde{T}$ is also convex and not reduced to a triangle, and therefore, there exists a unique bilinear mapping $\mathcal{F}_{\tilde{T}}$ such that $\tilde{T} = \mathcal{F}_{\tilde{T}}(\hat{T})$ and $\tilde{\boldsymbol{a}}_i = \mathcal{F}_{\tilde{T}}(\hat{\boldsymbol{a}}_i)$, $1 \le i \le 4$. In fact,

$$\mathcal{F}_T = F_S \circ \mathcal{F}_{\tilde{T}}.$$

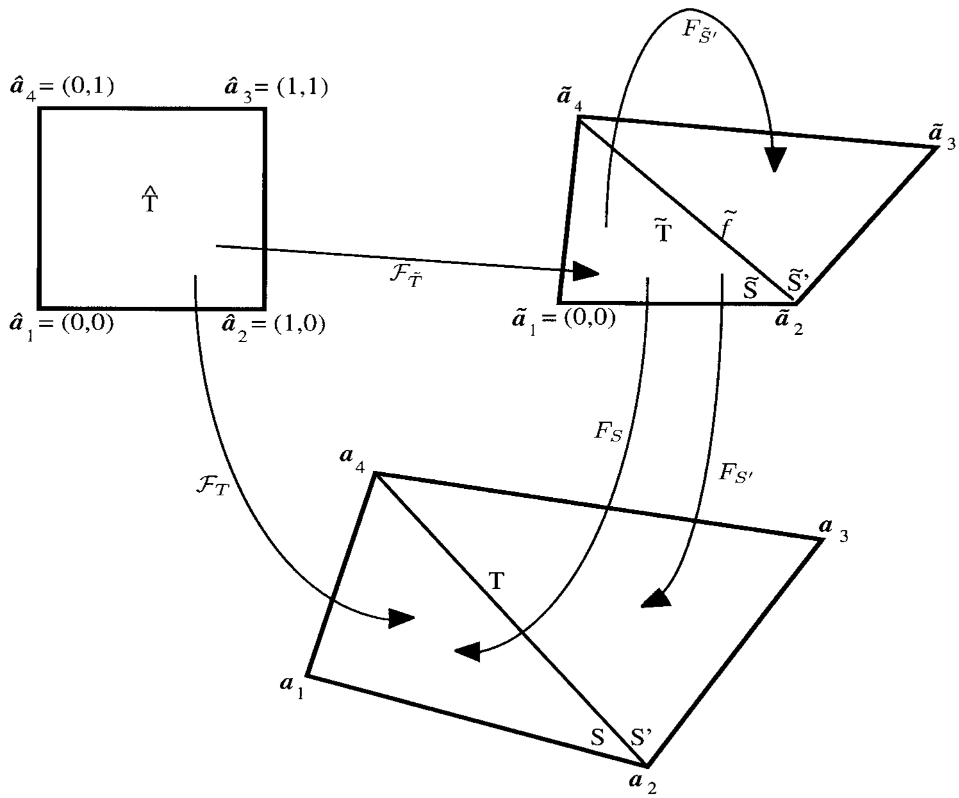

FIG. 5

Let $\tilde{D}'_i$ be the union of the triangles $\tilde{S}'$ associated with all the triangles $\tilde{S}$ in $\tilde{D}_i$; we take for reference macroelement

$$\tilde{\Delta}_i = \tilde{D}_i \cup \tilde{D}'_i. \tag{3.4}$$

Observe that $\tilde{\Delta}_i$ is a variable macroelement because the triangles $\tilde{S}'$ constituting $\tilde{D}'_i$ do not have a regular shape; as a consequence, we cannot apply directly on $\tilde{\Delta}_i$ any result that depends upon the shape of the domain. In order to take into account the geometry of $\tilde{D}'_i$, we introduce first the affine invertible mapping $F_{\tilde{S}'}$ that maps $\tilde{S}$ onto $\tilde{S}'$ and leaves invariant $\tilde{f}$, the diagonal separating $\tilde{S}$ and $\tilde{S}'$; i.e., $\tilde{S}' = F_{\tilde{S}'}(\tilde{S})$, $\tilde{f} = F_{\tilde{S}'}(\tilde{f})$, and $\tilde{\boldsymbol{a}}_3 = F_{\tilde{S}'}(\tilde{\boldsymbol{a}}_1)$:

$$\tilde{\boldsymbol{y}} = F_{\tilde{S}'}(\tilde{\boldsymbol{x}}) = B_{\tilde{S}'}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{a}}_3\,.$$

And finally, let $F_{S'}$ denote the affine invertible mapping such that $S' = F_{S'}(\tilde{S})$ and $\boldsymbol{a}_3 = F_{S'}(\tilde{\boldsymbol{a}}_1)$:

$$\boldsymbol{x} = F_{S'}(\tilde{\boldsymbol{x}}) = B_{S'}\tilde{\boldsymbol{x}} + \boldsymbol{a}_3\,.$$

Note that $S' = F_S(\tilde{S}') = F_S \circ F_{\tilde{S}'}(\tilde{S})$:

$$\boldsymbol{x} = B_S B_{\tilde{S}'}\tilde{\boldsymbol{x}} + B_S\tilde{\boldsymbol{a}}_3 + \boldsymbol{a}_1 = B_S B_{\tilde{S}'}\tilde{\boldsymbol{x}} + \boldsymbol{a}_3\,;$$

therefore,

$$B_{S'} = B_S B_{\tilde{S}'}\,,$$

and this equality allows one to estimate the geometrical parameters related to $B_{\tilde{S}'}$. Indeed, denoting by h_U the diameter of any triangle U, by ρ_U the diameter of the circle inscribed in U, and setting naturally $\sigma_U = h_U/\rho_U$, we have

$$\|B_S\| \le \frac{h_S}{\rho_{\tilde{S}}}, \quad \|B_S^{-1}\| \le \frac{h_{\tilde{S}}}{\rho_S}, \quad |\det(B_S)| = \frac{|S|}{|\tilde{S}|},$$
$$\|B_{S'}\| \le \frac{h_{S'}}{\rho_{\tilde{S}}}, \quad \|B_{S'}^{-1}\| \le \frac{h_{\tilde{S}}}{\rho_{S'}}, \quad |\det(B_{S'})| = \frac{|S'|}{|\tilde{S}|}.$$

Thus, as $B_{\tilde{S}'} = B_S^{-1} B_{S'}$, we obtain

$$\|B_{\tilde{S}'}\| \le \sigma_{\tilde{S}} \frac{h_{S'}}{\rho_S} \le 2\sigma_{\tilde{S}}\sigma_T,\ \|B_{\tilde{S}'}^{-1}\| \le 2\sigma_{\tilde{S}}\sigma_T,\ |\det(B_{\tilde{S}'})| = \frac{|S'|}{|S|} \le 4\frac{\sqrt{3}}{\pi}\sigma_T^2, \tag{3.5}$$

and since the family of quadrangulations is regular, these three quantities can be bounded independently of h.

Similarly, the fact that $\mathcal{F}_{\tilde{T}} = F_S^{-1} \circ \mathcal{F}_T$, and hence $D\mathcal{F}_{\tilde{T}} = B_S^{-1} \cdot D\mathcal{F}_T$, allows one to estimate the geometrical parameters related to $\mathcal{F}_{\tilde{T}}$:

$$\begin{aligned} &\|D\mathcal{F}_{\tilde{T}}\|_{L^\infty(\hat{T})} \le C_3\sigma_T, \quad \|D\mathcal{F}_{\tilde{T}}^{-1}\|_{L^\infty(\tilde{T})} \le C_4\sigma_T^2,\\ &\|J_{\tilde{T}}\|_{L^\infty(\hat{T})} \le C_5\sigma_T^2, \quad \|J_{\tilde{T}}^{-1}\|_{L^\infty(\tilde{T})} \le C_6\sigma_T^2. \end{aligned} \tag{3.6}$$

Owing to the above construction, there exists a continuous and invertible mapping F_i, that is affine on each "reference" quadrilateral $\tilde{T}$ of $\tilde{\Delta}_i$ and coincides with F_S on $\tilde{T}$:

$$\forall \tilde{\boldsymbol{x}} \in \tilde{T},\ F_i(\tilde{\boldsymbol{x}}) = F_S(\tilde{\boldsymbol{x}}).$$

Moreover, F_i is such that

$$\Delta_i = F_i(\tilde{\Delta}_i).$$

Since the family of quadrangulations is regular, properties (i)–(iii) of section 2 obviously hold here and, as in the preceding section, all geometric constants of $\tilde{\Delta}_i$ can be bounded by constants independent of i; therefore, we drop the index i. Similarly, property (iv) holds with $K = (k+1)^2$.

Then we associate with Θ_h the local finite element spaces

$$\Theta(\tilde{\Delta}) = \left\{\tilde{\theta} \in \mathcal{C}^0(\tilde{\Delta});\, \forall \tilde{T} \subset \tilde{\Delta},\ \tilde{\theta}_{|\tilde{T}} \in \mathcal{Q}_k(\tilde{T})\right\}, \tag{3.7}$$

$$\Theta(\Delta_i) = \left\{\theta \in \mathcal{C}^0(\Delta_i);\, \forall T \subset \Delta_i,\ \theta_{|T} \in \mathcal{Q}_k(T)\right\}, \tag{3.8}$$

and we define the projection operator $\tilde{r}$ in analogy to the preceding section. More precisely, for any function $\tilde{u}$ in $L^1(\tilde{\Delta})$, we define $\tilde{r}(\tilde{u})$ in $\Theta(\tilde{\Delta})$ by

$$\forall \tilde{\theta} \in \Theta(\tilde{\Delta}),\ \int_{\tilde{\Delta}} (\tilde{r}(\tilde{u}) - \tilde{u})\tilde{\theta}\, d\tilde{\boldsymbol{x}} = 0\,, \tag{3.9}$$

and for any function u in $L^1(\Delta_i)$, we define $r_i(u)$ in $\Theta(\Delta_i)$ by

$$r_i(u) \circ F_i = \tilde{r}(u \circ F_i)\,, \tag{3.10}$$

which we denote symbolically by $\widetilde{r_i(u)} = \tilde{r}(\tilde{u})$.

Looking back at the proofs of the previous section, we see that we need two equivalences of norms satisfied by functions of $\Theta(\tilde{\Delta})$, and we think of applying Theorem 1.1 on $\tilde{\Delta}$ (observe that Theorem 1.1 is relevant here because the mapping F_i is piecewise affine). But since $\tilde{\Delta}$ is composed of variable quadrilaterals, these equivalences are no longer simple consequences of the finite dimension of $\Theta(\tilde{\Delta})$, and neither does Theorem 1.1 apply directly on $\tilde{\Delta}$. These results are established in the next three lemmas.

LEMMA 3.1. *Assume that $(\mathcal{T}_h)_h$ is a regular family of quadrangulations. For each number p with $1 \le p \le \infty$, there exist positive constants $\hat{c}_p$ and $\hat{C}_p$, independent of h, such that, for all $\tilde{\Delta}$, we have the following equivalence:*

$$\forall \tilde{\theta} \in \Theta(\tilde{\Delta}),\ \hat{c}_p \|\tilde{\theta}\|_{L^p(\tilde{\Delta})} \le \|\tilde{\theta}\|_{L^2(\tilde{\Delta})} \le \hat{C}_p \|\tilde{\theta}\|_{L^p(\tilde{\Delta})}\,. \tag{3.11}$$

Proof. We have

$$\begin{aligned}\|\tilde{\theta}\|_{L^p(\tilde{\Delta})} &= \Big(\sum_{\tilde{T}\subset\tilde{\Delta}} \|\tilde{\theta}\|^p_{L^p(\tilde{T})}\Big)^{1/p} \le \Big(\sum_{\tilde{T}\subset\tilde{\Delta}} \|J_{\tilde{T}}\|_{L^\infty(\hat{T})} \|\tilde{\theta}\circ\mathcal{F}_{\tilde{T}}\|^p_{L^p(\hat{T})}\Big)^{1/p} \\ &\le \hat{C}_1 \sigma_{\Delta_i}^{2/p} \Big(\sum_{\tilde{T}\subset\tilde{\Delta}} \|\tilde{\theta}\circ\mathcal{F}_{\tilde{T}}\|^p_{L^p(\hat{T})}\Big)^{1/p}\end{aligned}$$

by applying (3.6). But since $\tilde{\theta}\circ\mathcal{F}_{\tilde{T}}$ belongs to the finite-dimensional space $\mathbb{Q}_k$ on the reference square $\hat{T}$, there exists a constant $\hat{C}_2$ such that

$$\|\tilde{\theta}\circ\mathcal{F}_{\tilde{T}}\|_{L^p(\hat{T})} \le \hat{C}_2 \|\tilde{\theta}\circ\mathcal{F}_{\tilde{T}}\|_{L^2(\hat{T})}\,.$$

Thus, reverting to each $\tilde{T}$ and applying again (3.6), we obtain

$$\|\tilde{\theta}\|_{L^p(\tilde{\Delta})} \le \hat{C}_3 \sigma_{\Delta_i}^{2/p+1} \|\tilde{\theta}\|_{L^2(\tilde{\Delta})}\,.$$

This establishes the first part of (3.11). The proof of the second part is similar. ◻

LEMMA 3.2. *Assume that $(\mathcal{T}_h)_h$ is a regular family of quadrangulations. For each number p with $1 \le p \le \infty$, there exists a positive constant $\hat{C}_p$, independent of h, such that, for all $\tilde{\Delta}$, we have the following inverse inequality:*

$$\forall \tilde{\theta} \in \Theta(\tilde{\Delta}),\ |\tilde{\theta}|_{W^{1,p}(\tilde{\Delta})} \le \hat{C}_p \|\tilde{\theta}\|_{L^p(\tilde{\Delta})}\,. \tag{3.12}$$

Proof. We have

$$|\tilde{\theta}|_{W^{1,p}(\tilde{\Delta})} \le \Big(\sum_{\tilde{T}\subset\tilde{\Delta}} \|J_{\tilde{T}}\|_{L^\infty(\hat{T})} \|D\mathcal{F}_{\tilde{T}}\|^p_{L^\infty(\hat{T})} |\tilde{\theta}\circ\mathcal{F}_{\tilde{T}}|^p_{W^{1,p}(\hat{T})}\Big)^{1/p}\,.$$

Again, since $\tilde{\theta} \circ \mathcal{F}_{\tilde{T}}$ belongs to the finite-dimensional space $\mathbb{Q}_k$ on the reference square $\hat{T}$, there exists a constant $\hat{C}_1$ such that

$$|\tilde{\theta} \circ \mathcal{F}_{\tilde{T}}|_{W^{1,p}(\hat{T})} \leq \hat{C}_1 \|\tilde{\theta} \circ \mathcal{F}_{\tilde{T}}\|_{L^p(\hat{T})}\,.$$

Therefore, in view of (3.6), we obtain

$$|\tilde{\theta}|_{W^{1,p}(\tilde{\Delta})} \leq \hat{C}_2 \sigma_{\Delta_i}^{4/p+1} \|\tilde{\theta}\|_{L^p(\tilde{\Delta})}\,,$$

thus proving (3.12). ∎

The next result is a special case of some inequalities of [9] and [10]; we give the proof for the sake of completeness.

LEMMA 3.3. *Assume that $(\mathcal{T}_h)_h$ is a regular family of quadrangulations. For any function $\tilde{\theta}$ in $L^1(\tilde{\Delta})$, define the average*

$$c(\tilde{\theta}) = \frac{1}{|\tilde{D}|} \int_{\tilde{D}} \tilde{\theta}\, d\tilde{\boldsymbol{x}}\,.$$

For each number p with $1 \leq p \leq \infty$, there exists a positive constant $\hat{C}_p$, independent of h, such that for all $\tilde{\Delta}$, we have

$$\forall \tilde{\theta} \in W^{1,p}(\tilde{\Delta}),\ \|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{\Delta})} \leq \hat{C}_p |\tilde{\theta}|_{W^{1,p}(\tilde{\Delta})}\,. \tag{3.13}$$

Proof. We write

$$\|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{\Delta})} \leq (\|\tilde{\theta} - c(\tilde{\theta})\|^p_{L^p(\tilde{D})} + \|\tilde{\theta} - c(\tilde{\theta})\|^p_{L^p(\tilde{D}')})^{1/p}\,. \tag{3.14}$$

Since $\tilde{D}$ has a regular shape that can assume only a fixed number of configurations, and $c(\tilde{\theta}) = \tilde{\theta}$ when $\tilde{\theta}$ is a constant function, we can apply Theorem 1.1 with $k = 0$ on $\tilde{D}$: there exists a constant $\hat{C}_1$, independent of $\tilde{D}$, such that

$$\|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{D})} \leq \hat{C}_1 |\tilde{\theta}|_{W^{1,p}(\tilde{D})}\,. \tag{3.15}$$

It remains to estimate the second term of (3.14). Consider a triangle $\tilde{S}'$ in $\tilde{D}'$, and let us switch to $\tilde{S}$:

$$\|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{S}')} = |\det(B_{\tilde{S}'})|^{1/p} \|\tilde{\theta} \circ F_{\tilde{S}'} - c(\tilde{\theta})\|_{L^p(\tilde{S})}\,.$$

But since $\tilde{S}$ has a regular shape, there exists a constant $\hat{C}_2$ such that

$$\forall v \in W^{1,p}(\tilde{S}),\ \|v\|_{W^{1,p}(\tilde{S})} \leq \hat{C}_2 (|v|^p_{W^{1,p}(\tilde{S})} + \|v\|^p_{L^p(\tilde{f})})^{1/p}.$$

Therefore,

$$\|\tilde{\theta} \circ F_{\tilde{S}'} - c(\tilde{\theta})\|_{L^p(\tilde{S})} \leq \hat{C}_2 (|\tilde{\theta} \circ F_{\tilde{S}'}|^p_{W^{1,p}(\tilde{S})} + \|\tilde{\theta} \circ F_{\tilde{S}'} - c(\tilde{\theta})\|^p_{L^p(\tilde{f})})^{1/p}\,.$$

On one hand, for any function v, $v \circ F_{\tilde{S}'}$ coincides with v on $\tilde{f}$ because $F_{\tilde{S}'}$ reduces to the identity mapping on $\tilde{f}$, and hence,

$$\|\tilde{\theta} \circ F_{\tilde{S}'} - c(\tilde{\theta})\|_{L^p(\tilde{f})} = \|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{f})} \leq \hat{C}_3 \|\tilde{\theta} - c(\tilde{\theta})\|_{W^{1,p}(\tilde{S})}$$

by applying a trace theorem on $\tilde{S}$. On the other hand, reverting to $\tilde{S}'$,

$$|\tilde{\theta} \circ F_{\tilde{S}'}|_{W^{1,p}(\tilde{S})} \leq |\det(B_{\tilde{S}'})|^{-1/p} \, \|B_{\tilde{S}'}\| |\tilde{\theta}|_{W^{1,p}(\tilde{S}')} \leq 2\sigma_{\tilde{S}}\sigma_T |\det(B_{\tilde{S}'})|^{-1/p} |\tilde{\theta}|_{W^{1,p}(\tilde{S}')} .$$

Hence,

$$\|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{S}')} \leq \hat{C}_4(\sigma_T^p |\tilde{\theta}|^p_{W^{1,p}(\tilde{S}')} + |\det(B_{\tilde{S}'})| \|\tilde{\theta} - c(\tilde{\theta})\|^p_{W^{1,p}(\tilde{S})})^{1/p},$$

and summing over all triangles of $\tilde{D}'$,

$$\|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{D}')} \leq \hat{C}_5(\sigma^p_{\Delta_i} |\tilde{\theta}|^p_{W^{1,p}(\tilde{D}')} + \sigma^2_{\Delta_i} \|\tilde{\theta} - c(\tilde{\theta})\|^p_{W^{1,p}(\tilde{D})})^{1/p}.$$

Then (3.15) gives

$$\|\tilde{\theta} - c(\tilde{\theta})\|_{L^p(\tilde{D}')} \leq \hat{C}_6 \sigma_{\Delta_i}^{\max(2/p,1)} |\tilde{\theta}|_{W^{1,p}(\tilde{\Delta})} ,$$

thus proving (3.13). □

The following theorems are analogues of Theorems 2.1 and 2.2. We skip their proofs because, owing to Lemmas 3.1, 3.2, and 3.3, they are very similar to those of Theorems 2.1 and 2.2.

THEOREM 3.4. *Assume that $(\mathcal{T}_h)_h$ is a regular family of quadrangulations. For any integers k and ℓ with $k \geq 1$ and $0 \leq \ell \leq k+1$ and any number p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that, for any macroelement Δ_i, any quadrilateral T contained in Δ_i, and any function u in $W^{\ell,p}(\Delta_i)$, the following inequality holds:*

$$\|u - r_i(u)\|_{L^p(T)} \leq C\, h_T^\ell |u|_{W^{\ell,p}(\Delta_i)}. \tag{3.16}$$

THEOREM 3.5. *Assume that $(\mathcal{T}_h)_h$ is a regular family of quadrangulations. For any integers k and ℓ with $k \geq 1$ and $1 \leq \ell \leq k+1$ and any number p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that, for any macroelement Δ_i, any quadrilateral T contained in Δ_i, and any function u in $W^{\ell,p}(\Delta_i)$, we have*

$$|u - r_i(u)|_{W^{1,p}(T)} \leq C\, h_T^{\ell-1} |u|_{W^{\ell,p}(\Delta_i)}. \tag{3.17}$$

Remark 5. Similar arguments to those of section 2 yield that, under the same assumptions, estimates (2.19) to (2.21) still hold for quadrilateral meshes.

4. A regularizing operator. We shall first study the regularization of functions with no imposed value on the boundary. Let $\mathcal{T}_h$ be a triangulation or quadrangulation of $\overline{\Omega}$ as defined in sections 2 or 3, let Θ_h be the finite element space defined by (2.1) or (3.3) for some positive integer k, and let $\{\boldsymbol{a}_i, 1 \leq i \leq N\}$ denote the set of nodes of $\mathcal{T}_h$ where the degrees of freedom of the functions of Θ_h are defined. For $1 \leq i \leq N$, let φ_i denote the basis function of Θ_h that takes the value one at the node $\boldsymbol{a}_i$ and zero at all other nodes.

For any number p with $1 \leq p \leq \infty$ and any nonnegative integer ℓ, let u be a given function in $W^{\ell,p}(\Omega)$, and for any integer i with $1 \leq i \leq N$, let $r_i(u)$ be defined by (2.8) or (3.10). Then we define the regularizing operator R_h from $W^{\ell,p}(\Omega)$ into Θ_h by

$$\forall u \in W^{\ell,p}(\Omega),\ R_h(u)(\boldsymbol{x}) = \sum_{i=1}^N [r_i(u)](\boldsymbol{a}_i)\varphi_i(\boldsymbol{x}). \tag{4.1}$$

Clearly R_h is continuous from $W^{\ell,p}(\Omega)$ into Θ_h. The next two theorems establish error estimates satisfied by R_h.

THEOREM 4.1. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations or quadrangulations of $\overline{\Omega}$. For any integers k and ℓ with $k \geq 1$ and $0 \leq \ell \leq k+1$ and any number p with $1 \leq p \leq \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{\ell,p}(\Omega),\ \forall T \in \mathcal{T}_h,\ \|u - R_h(u)\|_{L^p(T)} \leq C\, h_T^{\ell} |u|_{W^{\ell,p}(\Delta_T)}\,, \tag{4.2}$$

where Δ_T denotes the union of all elements in $\mathcal{T}_h$ which share at least a corner with T.

Proof. Suppose that Δ_T contains n macroelements Δ_i, which, for the sake of simplicity, we number from 1 to n. Then, owing to the support of the basis functions φ_i, we have

$$R_h(u)|_T = \sum_{i=1}^{n} [r_i(u)](\boldsymbol{a}_i)\varphi_i|_T\,.$$

Therefore, we can write

$$[u - R_h(u)]|_T = u|_T - \sum_{i=1}^{n}[r_1(u)](\boldsymbol{a}_i)\varphi_i|_T - \sum_{i=2}^{n}[r_i(u) - r_1(u)](\boldsymbol{a}_i)\varphi_i|_T\,.$$

But

$$\sum_{i=1}^{n}[r_1(u)](\boldsymbol{a}_i)\varphi_i|_T = r_1(u)|_T\,,$$

hence,

$$[u - R_h(u)]|_T = [u - r_1(u)]|_T - \sum_{i=2}^{n}[r_i(u) - r_1(u)](\boldsymbol{a}_i)\varphi_i|_T\,. \tag{4.3}$$

Therefore,

$$\|u - R_h(u)\|_{L^p(T)} \leq \|u - r_1(u)\|_{L^p(T)} + \sum_{i=2}^{n} |[r_i(u) - r_1(u)](\boldsymbol{a}_i)|\, \|\varphi_i\|_{L^p(T)}\,.$$

The first term on the right-hand side is estimated by Theorems 2.1 or 3.4, and it remains to evaluate the sum. On the one hand, if T is a triangle,

$$\|\varphi_i\|_{L^p(T)} = |\det(B_T)|^{1/p} \|\hat{\varphi}_i\|_{L^p(\hat{T})}\,,$$

or if T is a quadrilateral,

$$\|\varphi_i\|_{L^p(T)} \leq \|J_T\|_{L^\infty(\hat{T})}^{1/p} \|\hat{\varphi}_i\|_{L^p(\hat{T})}\,.$$

In both cases, $\|\hat{\varphi}_i\|_{L^p(\hat{T})}$ is a constant independent of T and h. Thus we have

$$\|\varphi_i\|_{L^p(T)} \leq \hat{C}_1\, |\det(B_T)|^{1/p} \quad \text{or} \quad \|\varphi_i\|_{L^p(T)} \leq \hat{C}_2 \|J_T\|_{L^\infty(\hat{T})}^{1/p}\,, \tag{4.4}$$

according to whether T is a triangle or a quadrilateral. On the other hand,

$$\begin{aligned} |[r_i(u) - r_1(u)](\boldsymbol{a}_i)| &\leq \|r_i(u) - r_1(u)\|_{L^\infty(T)} = \|\widehat{r_i(u)} - \widehat{r_1(u)}\|_{L^\infty(\hat{T})} \\ &\leq \hat{C}_3 \|\widehat{r_i(u)} - \widehat{r_1(u)}\|_{L^p(\hat{T})} \end{aligned}$$

since $\widehat{r_i(u)} - \widehat{r_1(u)}$ belongs to a space of finite and fixed dimension on the reference set $\hat{T}$. Therefore,

$$|[r_i(u) - r_1(u)](\boldsymbol{a}_i)| \le \hat{C}_3(\|\widehat{r_i(u)} - \hat{u}\|_{L^p(\hat{T})} + \|\widehat{r_1(u)} - \hat{u}\|_{L^p(\hat{T})}).$$

Then, if T is a triangle, by virtue of Theorem 2.1 we have

$$\begin{aligned} \|\widehat{r_i(u)} - \hat{u}\|_{L^p(\hat{T})} &= |\det(B_T)|^{-1/p}\|r_i(u) - u\|_{L^p(T)} \\ &\le \hat{C}_4\, h_T^\ell |\det(B_T)|^{-1/p}|u|_{W^{\ell,p}(\Delta_i)}, \end{aligned} \tag{4.5}$$

and if T is a quadrilateral, by virtue of Theorem 3.4 we have

$$\begin{aligned} \|\widehat{r_i(u)} - \hat{u}\|_{L^p(\hat{T})} &\le \|J_T^{-1}\|_{L^\infty(T)}^{1/p}\|r_i(u) - u\|_{L^p(T)} \\ &\le \hat{C}_5\, h_T^\ell \|J_T^{-1}\|_{L^\infty(T)}^{1/p}|u|_{W^{\ell,p}(\Delta_i)}. \end{aligned} \tag{4.6}$$

Therefore, combining (4.5) or (4.6) with (4.4), we obtain, if T is a triangle,

$$|[r_i(u) - r_1(u)](\boldsymbol{a}_i)|\|\varphi_i\|_{L^p(T)} \le \hat{C}_6\, h_T^\ell(|u|_{W^{\ell,p}(\Delta_i)} + |u|_{W^{\ell,p}(\Delta_1)}),$$

or, if T is a quadrilateral,

$$\begin{aligned} |[r_i(u) - r_1(u)](\boldsymbol{a}_i)|\|\varphi_i\|_{L^p(T)} &\le \hat{C}_7\, h_T^\ell \|J_T\|_{L^\infty(\hat{T})}^{1/p}\|J_T^{-1}\|_{L^\infty(T)}^{1/p}(|u|_{W^{\ell,p}(\Delta_i)} + |u|_{W^{\ell,p}(\Delta_1)}) \\ &\le \hat{C}_8\, \sigma_T^{2/p}\, h_T^\ell(|u|_{W^{\ell,p}(\Delta_i)} + |u|_{W^{\ell,p}(\Delta_1)}). \end{aligned}$$

This proves the theorem. ∎

In view of Theorems 2.2 and 3.5, the argument of Theorem 4.1 can easily be extended to prove the following estimate.

THEOREM 4.2. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations or quadrangulations of $\overline{\Omega}$. For any integers k and ℓ with $k \ge 1$ and $1 \le \ell \le k+1$ and any number p with $1 \le p \le \infty$, there exists a constant C, independent of h, such that*

$$\forall\, u \in W^{\ell,p}(\Omega),\ \forall\, T \in \mathcal{T}_h,\ |u - R_h(u)|_{W^{1,p}(T)} \le C\, h_T^{\ell-1}|u|_{W^{\ell,p}(\Delta_T)}. \tag{4.7}$$

Remark 6. Clearly, the statement of Theorem 4.1 (resp., Theorem 4.2) extends immediately to the case where u belongs to $W^{\ell,q}(\Omega)$, for any q such that $W^{\ell,q}(\Omega)$ is continuously embedded into $L^p(\Omega)$ (resp., $W^{1,p}(\Omega)$). For instance, in view of (2.19), under the assumptions of Theorem 4.1, if u belongs to $W^{\ell,q}(\Omega)$, we have, for all T in $\mathcal{T}_h$,

$$\begin{aligned} &\text{if } q \ge p,\ \|u - R_h(u)\|_{L^p(T)} \le C\, h_T^\ell\, h_T^{2/p-2/q}|u|_{W^{\ell,q}(\Delta_T)}; \\ &\text{if } q < p,\ \|u - R_h(u)\|_{L^p(T)} \le C\, h_T^\ell\, \frac{1}{\rho_T^{2/q-2/p}}|u|_{W^{\ell,q}(\Delta_T)}. \end{aligned} \tag{4.8}$$

Remark 7. Combining the final remarks of sections 2 and 3 with the inequality, valid for any nonnegative real number s

$$|\varphi_i|_{W^{s,p}(T)} \le c\, h_T^{\frac{2}{p}-s},$$

we derive that, for any real numbers s and t with $0 \le s \le 1$ and $s \le t \le k+1$ and any number p with $1 \le p \le \infty$, the following estimate holds for any function u in $W^{t,p}(\Omega)$:

$$|u - R_h(u)|_{W^{s,p}(T)} \le C\, h_T^{t-s}\, \|u\|_{W^{t,p}(\Delta_T)}\,. \tag{4.9}$$

Similarly, if f is a side of an element T, for any real numbers s, t, and p with $0 \le s \le 1$, $s + \frac{1}{p} < t \le k+1$, and $1 \le p < \infty$, the following estimate holds for any function u in $W^{t,p}(\Omega)$:

$$|u - R_h(u)|_{W^{s,p}(f)} \le C\, h_T^{t-s-\frac{1}{p}}\, \|u\|_{W^{t,p}(\Delta_T)}\,. \tag{4.10}$$

Now we turn to the regularization of functions with imposed values on some part of the boundary. More precisely, let Γ_0 denote a connected subset of Γ with positive measure. For $1 \le p < \infty$ and $\ell \ge 1$, we want to approximate functions of the space

$$W^{\ell,p}_{\Gamma_0}(\Omega) = \Big\{ v \in W^{\ell,p}(\Omega)\,;\, v = 0 \;\text{ on } \Gamma_0 \Big\}.$$

To this end, we assume that $\overline{\Omega}$ is triangulated or quadrangulated so that the end points of Γ_0 coincide with nodes of the triangulation. Then we number first the nodes of $\mathcal{T}_h$ that lie on Γ_0, say from 1 to N_0, and next we number the remaining nodes from N_0+1 to N. To ensure that the finite element functions vanish on Γ_0, we consider the finite element space spanned by the set of basis functions $\{\varphi_i;\, N_0+1 \le i \le N\}$, that is,

$$\Theta_{h,\Gamma_0} = \Big\{ \theta_h \in \mathcal{C}^0(\overline{\Omega});\, \forall T \in \mathcal{T}_h,\, \theta_h|_T \in \mathbb{P}_k \;\text{ or }\; \mathcal{Q}_k(T) \;\text{ and }\; \theta_h = 0 \;\text{ on } \Gamma_0 \Big\}.$$

The regularization operator R_{h,Γ_0} is defined by

$$\forall\, u \in W^{\ell,p}_{\Gamma_0}(\Omega),\; R_{h,\Gamma_0}(u)(\boldsymbol{x}) = \sum_{i=N_0+1}^{N} [r_i(u)](\boldsymbol{a}_i)\varphi_i(\boldsymbol{x}). \tag{4.11}$$

Clearly, R_{h,Γ_0} is continuous from $W^{\ell,p}_{\Gamma_0}(\Omega)$ into Θ_{h,Γ_0}. Let us show that it satisfies the analogues of Theorems 4.1 and 4.2.

THEOREM 4.3. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations or quadrangulations of $\overline{\Omega}$. For any integers k and ℓ with $k \ge 1$ and $1 \le \ell \le k+1$ and any real number p with $1 \le p < \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{\ell,p}_{\Gamma_0}(\Omega),\; \forall T \in \mathcal{T}_h,\; \|u - R_{h,\Gamma_0}(u)\|_{L^p(T)} \le C\, h_T^{\ell} |u|_{W^{\ell,p}(\Delta_T)}. \tag{4.12}$$

Proof. It suffices to consider the elements T that have some nodes on Γ_0. Suppose again that Δ_T contains n macroelements Δ_i. We agree to number first, say from 1 to n_0, the nodes $\boldsymbol{a}_i$ that lie on Γ_0 and from n_0+1 to n the remaining nodes. Then, owing to the support of the basis functions φ_i, we have

$$R_{h,\Gamma_0}(u)|_T = \sum_{i=n_0+1}^{n} [r_i(u)](\boldsymbol{a}_i)\varphi_i|_T = R_h(u)|_T - \sum_{i=1}^{n_0} [r_i(u)](\boldsymbol{a}_i)\varphi_i|_T\,.$$

Hence,

$$\|u - R_{h,\Gamma_0}(u)\|_{L^p(T)} \le \|u - R_h(u)\|_{L^p(T)} + \sum_{i=1}^{n_0} |[r_i(u)](\boldsymbol{a}_i)|\, \|\varphi_i\|_{L^p(T)}\,. \tag{4.13}$$

Next, we observe that each boundary node $\boldsymbol{a}_i$, $1 \le i \le n_0$, belongs to a side f of a triangle or quadrilateral T' contained in Δ_i (T' does not necessarily coincide with T). Thus, we have

$$\begin{aligned}|[r_i(u)](\boldsymbol{a}_i)| &\le \|r_i(u)\|_{L^\infty(f)} = \|\widehat{r_i(u)}\|_{L^\infty(\hat f)}\\ &\le \hat C_1 \|\widehat{r_i(u)}\|_{L^p(\hat f)} = \hat C_1 \|\widehat{r_i(u)} - \hat u\|_{L^p(\hat f)} \le \hat C_2 \|\widehat{r_i(u)} - \hat u\|_{W^{1,p}(\hat T)}\,.\end{aligned}$$

Here we have used first the fact that $\widehat{r_i(u)}$ belongs to a finite-dimensional space on $\hat T$, next the fact that $\hat u$ vanishes on $\hat f$, and finally the trace theorem on $\hat T$. Therefore, if T' is a triangle,

$$|[r_i(u)](\boldsymbol{a}_i)| \le \hat C_3 |\det(B_{T'})|^{-1/p} (\|r_i(u) - u\|_{L^p(T')} + \|B_{T'}\|\, |r_i(u) - u|_{W^{1,p}(T')})\,, \tag{4.14}$$

or if T' is a quadrilateral,

$$\begin{aligned}|[r_i(u)](\boldsymbol{a}_i)| \le &\hat C_4 |J_{T'}^{-1}|_{L^\infty(T')}^{1/p} (\|r_i(u) - u\|_{L^p(T')}\\ &+ \|D\mathcal{F}_{T'}\|_{L^\infty(\hat T)}\, |r_i(u) - u|_{W^{1,p}(T')})\,.\end{aligned} \tag{4.15}$$

Then (4.12) follows readily from (4.14) or (4.15) combined with (4.4), the fact that T and T' belong to the same macroelement Δ_i, and Theorems 2.1 and 2.2 or 3.4 and 3.5. □

Since the gist of the above proof consists in deriving an upper bound for $|[r_i(u)](\boldsymbol{a}_i)|$, it is clear that this proof can be easily adapted to establish the next result.

THEOREM 4.4. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations or quadrangulations of $\overline{\Omega}$. For any integers k and ℓ with $k \ge 1$ and $1 \le \ell \le k+1$ and any real number p with $1 \le p < \infty$, there exists a constant C, independent of h, such that*

$$\forall u \in W^{\ell,p}_{\Gamma_0}(\Omega),\ \forall T \in \mathcal{T}_h,\ \|u - R_{h,\Gamma_0}(u)\|_{W^{1,p}(T)} \le C\, h_T^{\ell-1} |u|_{W^{\ell,p}(\Delta_T)}. \tag{4.16}$$

Remark 8. Estimates (4.8) to (4.10) still hold with the operator R_h replaced by R_{h,Γ_0}.

Thus, we have exhibited two regularization operators, the second one being designed for handling functions that vanish on part of the boundary. They have optimal approximation properties in a large number of Sobolev norms, and the optimality concerns both the order of the approximation and its local behavior (the ratio of the diameter of Δ_T to the diameter of T is bounded independently of h).

5. Applications to a lifting operator and residual error indicators. A regularizing operator is a very useful theoretical tool. Among its best-known applications is the proof of the "inf-sup" condition that must be satisfied by spaces that discretize the Stokes or Navier–Stokes problem; cf. [11, Chapter II], for instance. But it is far from being its only application, and to illustrate this point, we have chosen to describe, on one hand, the construction of a discrete lifting operator that was suggested by O. Widlund [17] and, on the other hand, the derivation of optimal estimates for a family of residual indicators.

Construction of a lifting operator. Again let Γ_0 denote a subset of Γ with positive measure, and assume that $\overline{\Omega}$ is triangulated, by a regular family of triangulations (or quadrangulations) $(\mathcal{T}_h)_h$, in such a way that the end points of Γ_0 coincide with nodes of the triangulation. Here also, we number first the nodes of $\mathcal{T}_h$ that lie on Γ_0, say from 1 to N_0, and next we number the remaining nodes from N_0+1 to N. Then we associate with $\mathcal{T}_h$ the finite element space Θ_h, defined by (2.1) or (3.3) for some integer $k \geq 1$, and we denote by W_h the space of traces on Γ_0 of all functions of Θ_h. For $1 \leq p < \infty$, we wish to construct an operator L_h from W_h into Θ_h that lifts the trace (for all w_h in W_h, the trace of $L_h(w_h)$ on Γ_0 coincides with w_h) and that is continuous with a norm independent of h. To this end, we introduce first a standard lifting operator L that is continuous from $W^{1-1/p,p}(\Gamma_0)$ into $W^{1,p}(\Omega)$. Next, we regularize $L(w_h)$ by the operator R_h defined by (4.1). Finally, since the values of $R_h(L(w_h))$ do not necessarily coincide with those of w_h on Γ_0, we correct them by the technique of the preceding section; thus we set

$$L_h(w_h)(\boldsymbol{x}) = \sum_{i=1}^{N_0} w_h(\boldsymbol{a}_i)\varphi_i(\boldsymbol{x}) + \sum_{i=N_0+1}^{N} [R_h(L(w_h))](\boldsymbol{a}_i)\varphi_i(\boldsymbol{x})\,. \tag{5.1}$$

Obviously, the trace of $L_h(w_h)$ on Γ_0 coincides with w_h. The next theorem establishes the uniform stability of L_h.

THEOREM 5.1. *Assume that $(\mathcal{T}_h)_h$ is a regular family of triangulations or quadrangulations of $\overline{\Omega}$, and let L_h be defined by* (5.1). *For any integer $k \geq 1$ and any real number p with $1 \leq p < \infty$, there exists a constant C, independent of h, such that*

$$\forall w_h \in W_h,\ \|L_h(w_h)\|_{W^{1,p}(\Omega)} \leq C\,\|w_h\|_{W^{1-1/p,p}(\Gamma_0)}. \tag{5.2}$$

Proof. For any w_h in W_h, we write

$$\|L_h(w_h)\|_{W^{1,p}(\Omega)} \leq \|R_h(L(w_h))\|_{W^{1,p}(\Omega)} + \|R_h(L(w_h)) - L_h(w_h)\|_{W^{1,p}(\Omega)}\,. \tag{5.3}$$

The first term is estimated by Theorems 4.1 and 4.2 with $\ell = 1$ and by the standard property of the lifting operator L,

$$\|R_h(L(w_h))\|_{W^{1,p}(\Omega)} \leq C_1\|L(w_h)\|_{W^{1,p}(\Omega)} \leq C_2\|w_h\|_{W^{1-1/p,p}(\Gamma_0)}. \tag{5.4}$$

By construction, the second term has the expression

$$R_h(L(w_h)) - L_h(w_h) = \sum_{i=1}^{N_0}[R_h(L(w_h)) - w_h](\boldsymbol{a}_i)\varphi_i.$$

Then we proceed as in the preceding section. Let T be an element of $\mathcal{T}_h$ such that some nodes of T lie on Γ_0. Suppose again that Δ_T contains n macroelements Δ_i. We agree to number first, say from 1 to n_0, the nodes $\boldsymbol{a}_i$ that lie on Γ_0 and from n_0+1 to n the remaining nodes. Then, owing to the support of the basis functions φ_i, we have

$$\|R_h(L(w_h)) - L_h(w_h)\|_{W^{1,p}(T)} \leq \sum_{i=1}^{n_0} |[R_h(L(w_h)) - w_h](\boldsymbol{a}_i)|\, \|\varphi_i\|_{W^{1,p}(T)}\,.$$

On one hand, if T is a triangle,

$$\|\varphi_i\|_{W^{1,p}(T)} \leq |\det(B_T)|^{1/p}(\|\hat{\varphi}_i\|^p_{L^p(\hat{T})} + \|B_T^{-1}\|^p|\hat{\varphi}_i|^p_{W^{1,p}(\hat{T})})^{1/p}\,,$$

and since the leading term is the one with the factor $\|B_T^{-1}\|$, we can write

$$\|\varphi_i\|_{W^{1,p}(T)} \leq \hat{C}_1|\det(B_T)|^{1/p}\|B_T^{-1}\|\,. \tag{5.5}$$

Similarly, if T is a quadrilateral, we have

$$\|\varphi_i\|_{W^{1,p}(T)} \le \hat{C}_2 \|J_T\|^{1/p}_{L^\infty(\hat{T})} \|D\mathcal{F}_T^{-1}\|_{L^\infty(T)} . \tag{5.6}$$

On the other hand, to simplify the discussion, we assume that $\boldsymbol{a}_i$ belongs to a side f of T that lies on Γ_0. Then

$$\begin{aligned} |[R_h(L(w_h)) - w_h](\boldsymbol{a}_i)| &\le \|R_h(L(w_h)) - w_h\|_{L^\infty(f)} = \|R_h\widehat{(L(w_h)}) - \hat{w}_h\|_{L^\infty(\hat{f})} \\ &\le \hat{C}_3 \|R_h\widehat{(L(w_h)}) - \hat{w}_h\|_{L^p(\hat{f})} \le \hat{C}_4 \|R_h\widehat{(L(w_h)}) - \widehat{L(w_h)}\|_{W^{1,p}(\hat{T})} , \end{aligned}$$

where in the last inequality, we have used the fact that

$$\widehat{L(w_h)} = \hat{w}_h \quad \text{on } \hat{f}$$

in order to apply the trace theorem on $\hat{T}$. Therefore, if T is a triangle,

$$\begin{aligned} |[R_h(L(w_h)) - w_h](\boldsymbol{a}_i)| \le \hat{C}_5 |\det(B_T)|^{-1/p} (&\|R_h(L(w_h)) - L(w_h)\|^p_{L^p(T)} \\ &+ \|B_T\|^p |R_h(L(w_h)) - L(w_h)|^p_{W^{1,p}(T)})^{1/p} , \end{aligned} \tag{5.7}$$

and if T is a quadrilateral,

$$\begin{aligned} |[R_h(L(w_h)) - w_h](\boldsymbol{a}_i)| \le \hat{C}_6 \|J_T^{-1}\|^{1/p}_{L^\infty(T)} (&\|R_h(L(w_h)) - L(w_h)\|^p_{L^p(T)} \\ &+ \|D\mathcal{F}_T\|^p_{L^\infty(\hat{T})} |R_h(L(w_h)) - L(w_h)|^p_{W^{1,p}(T)})^{1/p} . \end{aligned} \tag{5.8}$$

Then (5.2) follows from (5.3), (5.4), and (5.7) combined with (5.5) if T is a triangle, or (5.8) combined with (5.6) if T is a quadrilateral, together with Theorems 4.1 and 4.2. $\square$

Residual indicators on a quadrilateral mesh. The residual indicators for the Poisson equation are known to satisfy optimal estimates when associated with a standard conforming discretization on a triangular mesh (or tetrahedral mesh in three dimensions); see [16] or [3]. The aim of this section is to extend these results to the case of any mesh made of convex quadrilaterals.

So, assuming that the data g belong to $L^2(\Omega)$, we consider the Poisson equation

$$\begin{cases} -\Delta u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

Since the domain Ω has a polygonal boundary, we introduce a regular family $(\mathcal{T}_h)_h$ of quadrangulations of $\overline{\Omega}$, and, for the discrete space Θ_h defined in (3.3), we set

$$\Theta_h^0 = \Theta_h \cap H_0^1(\Omega). \tag{5.10}$$

Then, the discrete problem reads

(5.11) *Find u_h in Θ_h^0 such that*

$$\forall v_h \in \Theta_h^0, \quad \int_\Omega \mathbf{grad}\, u_h \,.\, \mathbf{grad}\, v_h \, d\mathbf{x} = \int_\Omega g(\mathbf{x}) v_h(\mathbf{x})\, d\mathbf{x}.$$

The standard a priori estimate is

$$|u - u_h|_{H^1(\Omega)} \le c\, h^{\ell-1}\, |u|_{H^\ell(\Omega)}$$

when the solution u is supposed to belong to $H^\ell(\Omega)$, $1 \le \ell \le k+1$.

Next, for a nonnegative integer m, we introduce the finite element space

$$\Lambda_h = \Big\{ \lambda_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ \lambda_{h|T} \in \mathcal{Q}_m(T) \Big\},$$

and we choose an approximation g_h of the data g in Λ_h. Also, with each quadrilateral T of $\mathcal{T}_h$, we associate the set $\mathcal{E}_T$ of sides of T which are not contained in the boundary of Ω and we denote by h_f the length of each f in $\mathcal{E}_T$.

We are now in a position to define the family of indicators $(\eta_T)_{T\in\mathcal{T}_h}$:

$$\eta_T = h_T \, \|g_h + \Delta u_h\|_{L^2(T)} + \frac{1}{2} \sum_{f\in\mathcal{E}_T} h_f^{\frac{1}{2}} \, \Big\| \Big[\frac{\partial u_h}{\partial n} \Big] \Big\|_{L^2(f)}, \tag{5.12}$$

where $[\frac{\partial u_h}{\partial n}]$ denotes the jump of $\frac{\partial u_h}{\partial n}$ across f. The following two theorems sum up the optimal properties of these indicators.

THEOREM 5.2. *The family of indicators defined in* (5.12) *satisfies*

$$|u - u_h|_{H^1(\Omega)} \le c \left(\sum_{T\in\mathcal{T}_h} \big(\eta_T^2 + h_T^2 \, \|g - g_h\|^2_{L^2(T)} \big) \right)^{\frac{1}{2}}. \tag{5.13}$$

Proof. It relies on the formula

$$|u - u_h|_{H^1(\Omega)} = \sup_{w\in H_0^1(\Omega)} \frac{\int_\Omega \mathbf{grad}\,(u-u_h)\,.\,\mathbf{grad}\,w\,d\mathbf{x}}{|w|_{H^1(\Omega)}}. \tag{5.14}$$

It follows from (5.11) that, for any w_h in Θ_h^0,

$$\begin{aligned}\int_\Omega \mathbf{grad}\,(u-u_h)\,.\,\mathbf{grad}\,w\,d\mathbf{x} &= \int_\Omega \mathbf{grad}\,(u-u_h)\,.\,\mathbf{grad}\,(w-w_h)\,d\mathbf{x}\\ &= \sum_{T\in\mathcal{T}_h} \int_T \mathbf{grad}\,(u-u_h)\,.\,\mathbf{grad}\,(w-w_h)\,d\mathbf{x},\end{aligned}$$

so that integrating by parts and using a Cauchy–Schwarz inequality leads to

$$\begin{aligned}&\int_\Omega \mathbf{grad}\,(u-u_h)\,.\,\mathbf{grad}\,w\,d\mathbf{x}\\ &\le \sum_{T\in\mathcal{T}_h} \left(\|g+\Delta u_h\|_{L^2(T)} \|w-w_h\|_{L^2(T)} + \frac{1}{2} \sum_{f\in\mathcal{E}_T} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{L^2(f)} \|w-w_h\|_{L^2(f)} \right).\end{aligned}$$

Now, we take $w_h = R_h^0 w$, where R_h^0 denotes the regularization operator R_{h,Γ_0} for $\Gamma_0 = \partial\Omega$, and we derive from (4.16) and the analogue of (4.10) that

$$\begin{aligned}&\int_\Omega \mathbf{grad}\,(u-u_h)\,.\,\mathbf{grad}\,w\,d\mathbf{x}\\ &\le\ c \sum_{T\in\mathcal{T}_h} \left(h_T \, \|g+\Delta u_h\|_{L^2(T)} + \frac{1}{2} \sum_{f\in\mathcal{E}_T} h_f^{\frac{1}{2}} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{L^2(f)} \right) |w|_{H^1(\Delta_T)}.\end{aligned}$$

Using once more a Cauchy–Schwarz inequality and also inserting g_h with a triangular inequality, we obtain (5.13). ∎

The arguments for deriving an upper bound for the η_T are strictly the same as in the case of a triangular mesh, so we refer to Verfürth [15], [16], [3] for the proof of the following theorem.

THEOREM 5.3. *The family of indicators defined in* (5.12) *satisfies, for all* T *in* $\mathcal{T}_h$,

$$\eta_T \leq c\left(|u-u_h|_{H^1(\Omega_T)} + \|g-g_h\|_{L^2(\Omega_T)}\right), \tag{5.15}$$

where Ω_T *denotes the union of all quadrilaterals in* $\mathcal{T}_h$ *which share at least a side with* T.

The main consequence of Theorem 5.2 is that a bound for the error can be computed explicitly, up to a multiplicative constant, without any further assumption on the regularity of the exact solution. However, the constant is not easy to evaluate; we refer to Babuška, Durán, and Rodríguez [1] for interesting tentatives in this direction. And, by combining the two theorems, each indicator η_T appears to be fairly representative of the local error, thus leading to an efficient refinement of the mesh. Complete results are given by Verfürth (see [16] and the references therein).

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