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A LOCAL SIGNATURE FOR FIBERED 4-MANIFOLDS WITH A FINITE GROUP ACTION

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Abstract. Let p be a finite regular covering on a 2-sphere with at least three branch points. In this paper, we construct a local signature for the class of fibered 4-manifolds whose general fibers are isomorphic to the covering p.

1. Introduction. Let E and B be closed oriented smooth manifolds of dimension 4 and 2, respectively. Let Σ_g denote a closed oriented surface of genus $g \geq 1$. Assume that a smooth surjective map $f : E \to B$ has finitely many critical values $\{b_l\}_{l=1}^n$, and the fiber $f^{-1}(b)$ on $b \in B - \{b_l\}_{l=1}^n$ is connected. Then, its restriction $E - \{f^{-1}(b_l)\}_{l=1}^n \to B - \{b_l\}_{l=1}^n$ is an oriented fiber bundle whose fiber is diffeomorphic to Σ_g. We call the triple (f, E, B) satisfying these conditions a fibered 4-manifold of genus g. Inverse images of a regular value and a singular value under f are called a regular fiber and a singular fiber, respectively.

For a fibered 4-manifold $f : E \to B$, denote by $\Delta_l \subset B$ a closed neighborhood of the critical value b_l of f. We assume $\Delta_1, \ldots, \Delta_n$ are mutually disjoint. Denote by E_l the inverse image of Δ_l under f, and the restriction $f|_{E_l}$ by f_l. On some classes of fibered 4-manifolds, the signature of the fibered 4-manifold $f : E \to B$ is described as the sum

$$\operatorname{Sign} E = \sum_{l=1}^{n} \sigma_{\mathrm{loc}}(f_l, E_l, \Delta_l)$$

of local invariants $\sigma_{\mathrm{loc}}(f_l, E_l, \Delta_l) \in \boldsymbol{R}$, each of which depends on the neighborhood E_l of the singular fiber $f^{-1}(b_l)$. To be precise, these local invariants are defined as a function σ_{loc} on the set of singular fiber germs which arise in the class of fibered 4-manifolds. We call this function the local signature.

One of the motivations for the study of local signatures is that it is closely related to a cion of the Meyer cocycle, an important 2-cocycle of the mapping class group of the surface Σ_g. This function is related to several invariants including the eta invariant of the signature operator (see Atiyah [4], Iida [14]) and the Casson invariant of homology 3-spheres (see Morita [22]). In algebraic geometry, the local signature is related to the slope equality problem. This is studied in order to describe the geography of algebraic surfaces of general type (see Ashikaga-Konno [3]).

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For many classes of fibered 4-manifolds, local signatures are constructed and calculated in various fields including topology, algebraic geometry, and complex analysis. For the fibered 4-manifolds of genus 1 and 2, Matsumoto [18, Theorem 6], [19, Theorem 3.3] constructed local signatures using the cobounding function of the Meyer cocycle. He also calculated the values for Lefschetz singular fiber germs. Ueno [27] also constructed and calculated the local signature for the genus 2 fibered 4-manifolds using the even theta constant. For the fibered 4-manifolds of genus more than 2 whose monodromies are in the whole mapping class group, there does not exist a local signature. However, if we consider some restricted class, there still exists local signatures. For example, Endo [9, Theorem 4.4] constructed it for hyperelliptic fibered 4-manifolds (see also Morifuji [21]). Arakawa and Ashikaga [1, Proposition 4.7] also constructed it in the setting of algebraic geometry, and Terasoma proved that these two local signatures are equal. Local signatures for many kinds of restricted classes of fibered 4-manifolds are listed in Ashikaga-Endo [2] and Ashikaga-Konno [3]. See also Iida [14], Kuno [16], [17] and Yoshikawa [28].

The purpose of this paper is to construct the local signatures for the classes of fibered 4-manifolds which have a fiber-preserving finite group action. We also assume that the quotient space of their general fiber is a sphere with at least 3 branch points. We should mention that Furuta [10] constructed a local signature for broader classes of fibered 4-manifolds than ours in the manner of differential geometry. Furthermore, Nakata [23] calculated it on Lefschetz fiber germs of the hyperelliptic fibered 4-manifold. But the local signature in this paper is easier to compute than that of Furuta. In general, a local signature of this class is not unique. The author does not know whether the local signature in this paper coincides with Furuta's.

Let G be a finite group, and Σ a closed surface. We call a finite regular covering $p : \Sigma_g \to \Sigma$ a G-covering if its deck transformation group is isomorphic to G. For a G-covering p, Birman-Hilden [7] defined a group which we call the symmetric mapping class group. The monodromy group of a fibered 4-manifold of the G-covering p is considered as a subgroup of this group. The local signature induces a cobounding function of the pullback of the Meyer cocycle in the symmetric mapping class group. Using this cobounding function, we will construct the local signature which can be applied to a broader class of fibered 4-manifolds than our class.

This paper is organized as follows. In Section 2, we define a class of fibered 4-manifolds for which we will construct the local signature (Theorem 1.1). We also define a broader class of fibered 4-manifolds, for which Furuta constructed a local signature in a different way. We also construct a local signature for the broader class (Proposition 5.6).

For a fibered 4-manifold $f : E \to B$ of the G-covering p, let $E^h \subset E$ denote the fixed point set in E for $h \in G$. Since general fibers of the fibered 4-manifold are isomorphic to the G-covering p, the subspace $E^h \cap (E - \{f^{-1}(b_l)\}_{l=1}^n)$ of E^h can be considered as a (not necessarily connected) covering space of $B - \{b_l\}_{l=1}^n$. We call a component S of E^h horizontal if $S \cap (E - \{f^{-1}(b_l)\}_{l=1}^n)$ is a covering space of $B - \{b_l\}_{l=1}^n$, and vertical if S is contained in a singular fiber. As will be shown in Lemma 3.1, any component of the fixed point set E^h is either horizontal or vertical. For $\psi \in [0, \pi]$, denote by $I(h, \psi)$ the union of the horizontal components whose normal bundles rotate $\pm\psi$ under the action of $h \in G$. Let $J(f_l, h)$ denote the union of vertical components contained in a fiber germ $f^{-1}(b_l)$. In Section 3, we prove the main theorem (Theorem 1.1) as below assuming that the sum of normal euler numbers $\chi(N(S))$ of the connected component $S \subset I(h, \psi)$ for $h \in G$ localizes as

$$\sum_{S \subset I(h,\psi)} \chi(N(S)) = \sum_{l=1}^{n} \chi_{\mathrm{loc}}^{h,\psi}([f_l, E_l, \Delta_l]) .$$

Here $\chi_{\mathrm{loc}}^{h,\psi}([f_l, E_l, \Delta_l])$ is a rational number which depends on the fiber germ $[f_l, E_l, \Delta_l]$ in the fibered 4-manifold $f : E \to B$.

Let $f : E \to \Delta$ be a fiber germ of the G-covering p on a closed 2-disk Δ. The fixed point set E^h consists of 0-dimensional components and 2-dimensional components. Let $\{P_j\}$ and $\{F_j\}$ be the discrete fixed points and 2-dimensional components in $J(f, h)$, respectively. Denote the rotating angle of the normal bundle of F_j by $\pm\psi_j$ for $\psi_j \in [0, \pi]$. Identifying the neighborhood of P_j and $\boldsymbol{C}^2$, choose $\varphi_j, \varphi_j' \in [0, 2\pi]$ so that the action of h is written as $(z, w) \mapsto (e^{\sqrt{-1}\varphi_j} z, e^{\sqrt{-1}\varphi_j'} w)$ in a suitable coordinate. With the local normal euler number $\chi_{\mathrm{loc}}^{h,\psi}$ and these connected components of the fixed point set, the local signature σ_{loc} is described as follows.

THEOREM 1.1. *Let $p : \Sigma_g \to S^2$ be a G-covering with at least three branch points. The signatures of fibered* 4*-manifolds of the G-covering p localizes. Our local signature is written as*

$$\begin{aligned}
\sigma_{\mathrm{loc}}([f, E, \Delta]) := {} & |G| \operatorname{Sign}(E/G) \\
& + \sum_{h \neq 1 \in G} \Bigg(- \sum_{\psi \in [0,\pi]} \chi_{\mathrm{loc}}^{h,\psi}([f, E, \Delta]) \operatorname{cosec}^2\left(\frac{\psi}{2}\right) \\
& \qquad + \sum_{P_j \subset J(f,h)} \cot\left(\frac{\varphi_j}{2}\right) \cot\left(\frac{\varphi_j{}'}{2}\right) \\
& \qquad - \sum_{F_j \subset J(f,h)} \chi(N(F_j)) \operatorname{cosec}^2\left(\frac{\psi_j}{2}\right) \Bigg) .
\end{aligned}$$

The key tools to prove this theorem are the localization of the normal euler number and the G-signature theorem (Atiyah-Singer [5, Theorem (6.12)]). In Section 4, we construct a local euler number using the multi-section on the normal bundle of $I(h, \psi)$ made by Furuta, and complete the proof of the main theorem (Theorem 1.1).

In the rest of paper, we will consider a cobounding function of the pullback of the Meyer cocycle in the symmetric mapping class group $\mathcal{M}_g(p)$ of the G-covering p. In Section 5, we construct a local signature for broader class of fibered 4-manifolds of the G-covering p when p satisfies some condition. To do this, we describe the local signature using the cobounding function $\phi : \mathcal{M}_g(p) \to \boldsymbol{Q}$ of the pullback of the Meyer cocycle in the symmetric mapping class group. In Section 6, we give a standard generating system of the symmetric mapping class group of a G-covering when G is abelian. Let $d \geq 2$ and $m \geq 3$ be integers such that m is divided by d, and A a finite set $\{\alpha_i\}_{i=1}^m$ in S^2. For each $i = 1, 2, \ldots, m$, choose a loop γ_{α_i} which rotates around a point α_i counterclockwise once. Define a surjective homomorphism $k : H_1(S^2 - A) \to \mathbf{Z}_d$ by mapping each homology class $[\gamma_{\alpha_i}]$ to 1 mod d. Let $p_1 : \Sigma_g \to S^2$ be the $\mathbf{Z}_d$-covering on S^2 which has the branch set A in S^2 and the monodromy homomorphism k. In Section 7, we calculate the local signature of a fiber germ $f : E \to \Delta$ of the $\mathbf{Z}_d$-covering p_1. Its monodromy is the inverse of an element $\hat{\sigma}_{ij}$ in the standard generating system of the symmetric mapping class group for the $\mathbf{Z}_d$-covering p_1. We also calculate the value of the cobounding function $\phi(\hat{\sigma}_{ij})$.

PROPOSITION 1.2. *Let $f : E \to \Delta_1$ be the representative of a fiber germ in $S_g^{p_1}$ constructed in the proof of Lemma* 7.1. *Then we have*

$$\sigma_{\mathrm{loc}}([f, E, \Delta_1]) = -\frac{(d-1)(d+1)m}{3d(m-1)}$$

and

$$\phi(\hat{\sigma}_{ij}) = \frac{(d-1)(d+1)m}{3d(m-1)},$$

where $\hat{\sigma}_{ij}$ is the generator of $\mathcal{M}_g(p_1)$ defined in Section 6.

If $d = 2$, i.e., $m = 2g + 2$, the value $\phi(\hat{\sigma}_{ij})$ for $\hat{\sigma}_{ij} \in \mathcal{M}_g(p_1)$ coincides with that of the Meyer function on the hyperelliptic mapping class group obtained by Endo [9, Lemma 3.2].

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2. Fibered 4-manifolds of the G-covering p.

Denote by Σ_g a closed oriented surface of genus $g \geq 1$. Let G be a finite group, and $p : \Sigma_g \to S^2$ a G-covering, that is, a finite regular covering whose deck transformation group $\mathrm{Deck}(p)$ is isomorphic to G. In the sequel, we fix an isomorphism between the deck transformation group $\mathrm{Deck}(p)$ and G. In this section, we define two kinds of fibered 4-manifolds of the G-covering p in Definitions 2.1 and 2.2. Later in Subsection 3.2, we will construct a local signature for the fibered 4-manifolds in Definition 2.2.

Let $C(p)$ denote the centralizer of $\mathrm{Deck}(p)$ in the orientation-preserving diffeomorphism group $\mathrm{Diff}_+ \Sigma_g$ of the surface Σ_g.

DEFINITION 2.1. Let E and B be compact oriented manifolds of dimension 4 and 2, and $f : E \to B$ a smooth surjective map. A triple (f, E, B) is called a fibered 4-manifold of the G-covering p in a broad sense if it satisfies

(i) $\partial E = f^{-1}(\partial B)$,

(ii) $f : E \to B$ has finitely many critical values $\{b_l\}_{l=1}^n$ in $\mathrm{Int}\, B$, and the restriction $E - \{f^{-1}(b_l)\}_{l=1}^n \to B - \{b_l\}_{l=1}^n$ is a smooth oriented Σ_g-bundle,

(iii) the structure group of the Σ_g-bundle $E - \{f^{-1}(b_l)\}_{l=1}^n \to B - \{b_l\}_{l=1}^n$ is contained in $C(p)$.

The fibered 4-manifold of a G-covering is a generalization of hyperelliptic fibrations in [9], that is, a fibered 4-manifold of a $\mathbf{Z}_2$-covering $\Sigma_g \to S^2$ on a sphere. The natural action of G on Σ_g gives rise to a smooth fiberwise G-action on $E - \{f^{-1}(b_l)\}_{l=1}^n$, since the structure group of the fiber bundle $E - \{f^{-1}(b_l)\}_{l=1}^n$ is contained in $C(p)$. Note that for each regular value $b \in B$, the covering $f^{-1}(b) \to f^{-1}(b)/G$ is isomorphic to the G-covering p.

On a class of fibered 4-manifolds of a covering which is not necessarily regular, Furuta has already constructed a local signature. For a more detailed setting, see Furuta [10]. He constructed a canonical multi-section on the relative tangent bundle of the fiber bundle $E - \{f^{-1}(b_l)\}_{l=1}^n \to B - \{b_l\}_{l=1}^n$, using the fact that any fiber has at least 3 branch points. He made a connection on the tangent bundle TE by using the multi-section, and showed that its Pontrjagin form vanishes outside neighborhoods of the singular fibers $\{f^{-1}(b_l)\}_{l=1}^n$. Thus, the signature localizes.

We also use this multi-section to construct our local signature. But, in this paper, we consider a narrower class of fibered 4-manifolds in order to make a local signature without using the connection which is easy to compute. A fibered 4-manifold of the G-covering p (in a narrow sense) is defined as follows.

DEFINITION 2.2. A triple (f, E, B) is called a fibered 4-manifold of the G-covering p (in a narrow sense), if it satisfies

(i) the map $f : E \to B$ is a fibered 4-manifold of the G-covering p in the broad sense,
(ii) the natural G-action on $E - \{f^{-1}(b_l)\}_{l=1}^n$ extends to a smooth action on E.

In this paper, we simply call it a fibered 4-manifold of the G-covering p. Our local signature and local euler number are defined as functions on the set of fiber germs which arise in these fibered 4-manifolds. The set of fiber germs is defined as follows. Denote by Δ a closed 2-disk. Consider fibered 4-manifolds (f,E,Δ) of the G-covering p which have unique critical value in Δ. Let (f_1,E_1,Δ_1) and (f_2,E_2,Δ_2) be such fibered 4-manifolds with critical values b_1 and b_2, respectively. These fibered 4-manifolds are defined to be equivalent if there exist closed 2-disks $\Delta_1' \subset \Delta_1$ and $\Delta_2' \subset \Delta_2$ including the critical values, an orientation-preserving diffeomorphism $\varphi : (\Delta_1', b_1) \to (\Delta_2', b_2)$, and a G-equivariant orientation-preserving diffeomorphism $\tilde{\varphi} : f_1^{-1}(\Delta_1) \to f_2^{-1}(\Delta_2)$ such that

$$\varphi f_1 = f_2 \tilde{\varphi} .$$

We call the equivalent class a fiber germ of fibered 4-manifolds of the G-covering p, and denote the set of equivalent classes by S_g^p.

3. A local signature on the class of fibered 4-manifolds of the G-covering p in the narrow sense. Let G be a finite group, and $p : \Sigma_g \to S^2$ a G-covering as in Section 2. Let (f, E, B) be a fibered 4-manifold of the G-covering p. We assume that E and B have no boundaries. For $h \in G$ and $\psi \in [0, \pi]$, denote by $I(h, \psi)$ the union of all horizontal components of the fixed point set E^h whose normal bundles are rotated $\pm\psi$ by the action of $h \in G$. Let $\{[f_l, E_l, \Delta_l]\}_{l=1}^n$ be the fiber germs in $f : E \to B$, and denote by $\chi(N(S))$ the normal euler number of a connected component S in $I(h, \psi)$. If there exists a function

$$\chi_{\mathrm{loc}}^{h,\psi} : S_g^p \to \boldsymbol{Q}$$

on the set of fiber germs such that

$$\sum_{S \subset I(h,\psi)} \chi(N(S)) = \sum_{l=1}^{n} \chi_{\mathrm{loc}}^{h,\psi}([f_l, E_l, \Delta_l]),$$

we say that the normal euler number of the horizontal components $I(h, \psi)$ localizes. In this section, we prove the main theorem (Theorem 1.1) assuming that this number localizes. We will construct a local euler number later in Definition 4.2.

3.1. The fixed point set of the G-action. In Subsection 3.2, we will apply the G-signature theorem on the total space E of the fibered 4-manifold (f, E, B) to construct the local signature. It will enable us to calculate the G-signature in terms of the fixed point sets of the G-action on E. Hence, we investigate the fixed point set in this subsection.

For $h \in G$, the fixed point set E^h is a pairwise disjoint collection of closed submanifolds (see, for example, Conner [8, p. 72]). Since the group G preserves the orientation of E, it consists of closed 2-manifolds $\{S_i\}$ and 0-manifolds $\{P_j\}$. In Introduction, we defined two kinds of components of E^h called vertical and horizontal.

LEMMA 3.1. *Any component of E^h is either vertical or horizontal.*

PROOF. Let S be a component of E^h which is not vertical. Then, there exists a regular value $b \in B$ of f such that $S \cap f^{-1}(b)$ is not empty. Let $N(b)$ denote a neighborhood of b. If we endow the natural G-action on the first factor of $\Sigma_g \times D^2$, a local trivialization $f^{-1}(N(b)) \cong \Sigma_g \times D^2$ preserving the $C(p)$ structure is G-equivariant. Since any component of $f^{-1}(N(b)) \cap E^h$ is a 2-dimensional manifold, S is a 2-dimensional submanifold of E^h. Moreover, $S \cap f^{-1}(B - \{b_l\}_{l=1}^n)$ is a (not necessarily connected) covering space of $B - \{b_l\}_{l=1}^n$. $\square$

The followings are examples of fibered 4-manifolds with boundary of $\boldsymbol{Z}_2$-coverings which have horizontal components and a vertical component, respectively.

EXAMPLE 3.2 (a fibered 4-manifold with horizontal components). Let $[x : y : z]$ be a homogeneous coordinate of $\boldsymbol{CP}^2$ and b a coordinate of a unit disk D^2 in $\boldsymbol{C}$. Consider the singular surface

$$E_1 = \{([x : y : z], b) \in \boldsymbol{CP}^2 \times D^2 \,;\; y^2 z = x(x - z)(x - 2bz)\}.$$

Let $f : E_1 \to D^2$ denote the map defined by $([x : y : z], b) \mapsto b$. If we blow up E_1 at

$$([x : y : z], b) = ([1 : 0 : 1], 1/2), ([1 : 0 : 0], 0),$$

the proper transform E_1' of E_1 is a smooth compact 4-manifold, and we can extend the map f to a smooth map $E_1' \to D^2$ naturally. Endow the action of $\boldsymbol{Z}_2$ on E_1 by

$$([x : y : z], b) \mapsto ([x : -y : z], b),$$

which preserves each fiber of f. This group action also extends naturally onto E_1', and the fixed point set in E_1' consists of the proper transforms of $\{x = y = 0\}$, $\{x = z, y = 0\}$, and $\{x = 2bz, y = 0\}$, which are horizontal components.

EXAMPLE 3.3 (a fibered 4-manifold with a vertical component). Consider S^1 as the quotient space $\boldsymbol{R}/\boldsymbol{Z}$. Let $E_2 = S^1 \times S^1 \times D^2$, and denote its coordinate by (s, t, b). Define an involution on E_2 by $(s, t, b) \mapsto (s + 1/2, t, -b)$, and denote its quotient space by E_2'. The map $f : E_2' \to D^2$ defined by $[s, t, b] \mapsto b^2$ is a fibered 4-manifold whose fiber is $S^1 \times S^1$. Endow the action of $\boldsymbol{Z}_2$ on E_2' by

$$[s, t, b] \mapsto [s + 1/2, t, b]\,,$$

which preserves each fiber of f. The fixed point set consists of one vertical component $\{[s, t, 0] \in E_2'\,;\ s, t \in S^1\}$.

3.2. Localizations of signature and the normal euler number. We deduce the following lemma from the G-Signature theorem.

LEMMA 3.4. *Let $f : E \to B$ be a fibered 4-manifold of the G-covering p. If the normal euler number of the horizontal components $I(h, \psi)$ localizes for any $h \in G$ and $\psi \in [0, \pi]$, the signature of E also localizes.*

To prove this lemma, we prepare some facts about the G-signature. For a $\boldsymbol{Q}$-vector space V on which G acts, denote by $\mathrm{tr}(h, V)$ the trace of the action of $h \in G$. Let X be a closed oriented 4-manifold on which G acts preserving the orientation. The G-signature $\mathrm{Sign}(h, X)$ for $h \in G$ is a rational number defined by

$$\mathrm{Sign}(h, X) := \mathrm{tr}(h, H_2^+(X;\, \boldsymbol{Q})) - \mathrm{tr}(h, H_2^-(X;\, \boldsymbol{Q}))\,,$$

where $H_2^+(X;\, \boldsymbol{Q})$ and $H_2^-(X;\, \boldsymbol{Q})$ denote the subspaces of $H_2(X;\, \boldsymbol{Q})$ spanned by the eigenvectors of positive and negative eigenvalues with respect to the intersection form, respectively. For the details, see Atiyah-Singer [5, Section 6] (see also Hirzebruch [12] and Gordon [11]). If we consider the action of the formal sum $\sum_{h \in G} h$ on $H_2(X;\, \boldsymbol{Q})$, we obtain

$$\mathrm{Sign}\, X = -\sum_{h \neq 1 \in G} \mathrm{Sign}(h, X) + |G|\, \mathrm{Sign}(X/G)\,. \tag{1}$$

The fixed point set of $h \in G$ is the disjoint sum of closed 2-manifolds $\{S_i\}$ and 0-manifolds $\{P_j\}$. Denote the rotation angle of the normal bundle of S_i by $\pm\psi_i$, where $\psi_i \in [0, \pi]$. Identifying neighborhoods of P_j and the origin in $\boldsymbol{C}^2$, choose $\varphi_j, \varphi_j' \in [0, 2\pi]$ so that the action of h is written as $(z, w) \mapsto (e^{\sqrt{-1}\varphi_j} z, e^{\sqrt{-1}\varphi_j'} w)$ in a suitable coordinate. Then, the G-signature $\mathrm{Sign}(h, X)$ is written in terms of these rotation angles and the normal euler number of S_i as follows.

THEOREM 3.5 (Atiyah-Singer [5, Theorem (6.12)]).

$$\mathrm{Sign}(h, X) = \sum_i \chi(N(S_i))\, \mathrm{cosec}^2 \left(\frac{\psi_i}{2}\right) - \sum_j \cot\left(\frac{\varphi_j}{2}\right) \cot\left(\frac{\varphi_j'}{2}\right). \tag{2}$$

PROOF OF LEMMA 3.4. Choose representatives $\{f_l, E_l, \Delta_l\}_{l=1}^n$ of all fiber germs in the fibered 4-manifold $f : E \to B$. Note that the complement of the fiber germs $\bigcup_{l=1}^n E_l/G$ in the quotient space E/G is a S^2-bundle. Since the signature of a S^2-bundle on a compact 2-manifold vanishes, we have

$$\begin{aligned}\operatorname{Sign}(E/G) &= \operatorname{Sign}\left(\left(E - \coprod_{l=1}^{n} \operatorname{Int} E_l\right)/G\right) + \sum_{l=1}^{n} \operatorname{Sign}(E_l/G) \\ &= \sum_{l=1}^{n} \operatorname{Sign}(E_l/G)\end{aligned} \tag{3}$$

by the Novikov additivity. Substituting (2) and (3) to (1), we see that the signature of E localizes if and only if the right-hand side of (2) localizes. By the definition, any vertical component F is contained in a singular fiber. Hence, the normal euler number $\chi(N(F))$ depends only on the singular fiber germ. Thus, if the normal euler numbers of $I(h, \psi)$ localizes, the signature of E also localizes. $\square$

By the equations (1), (2), and (3), we can write the local signature as Theorem 1.1.

REMARK 3.6. In the proof, we do not consider the localization of the normal euler number of each horizontal component but the sum of the normal euler numbers of the horizontal components $I(h, \psi)$.

4. The local normal euler number of the set of horizontal components. Recall that G is a finite group, and $p : \Sigma_g \to S^2$ is a G-covering. Denote by m the order of the branch set of the G-covering p. In this section, we assume that the order m is at least 3. Let $f_0 : E_0 \to B_0$ be an oriented Σ_g-bundle on an oriented manifold B_0 with structure group in $C(p)$. The group G acts on the total space E_0 fiberwise as stated after Definition 2.1. Let $\bar{J}_0 \subset E_0/G$ denote the branched locus of the branched covering space $E_0 \to E_0/G$. Since $f_0^{-1}(b) \to f_0^{-1}(b)/G$ is isomorphic to the covering p for any $b \in B_0$, the induced smooth map $E_0/G \to B_0$ by f_0 is an oriented S^2-bundle, and $\bar{J}_0$ is a submanifold of E_0/G.

For a fiber bundle $Y \to Z$, we denote by $T(Y/Z)$ the subbundle $\operatorname{Ker}(TY \to TZ)$ of the tangent bundle $TY \to Y$, and call it the relative tangent bundle of $Y \to Z$. In Subsection 4.1, we will review a canonical multi-section of the relative tangent bundle $T((E_0/G)/B)|_{\bar{J}_0}$ constructed by Furuta. As in Section 3, let $f : E \to B$ be a fibered 4-manifold of the G-covering p, and $\{f_l, E_l, \Delta_l\}_{l=1}^n$ representatives of the fiber germs in the fibered 4-manifold. Applying Subsection 4.1 to the Σ_g-bundle

$$E - \coprod_{l=1}^{n} \operatorname{Int} E_l \to B - \coprod_{l=1}^{n} \operatorname{Int} \Delta_l\,,$$

we obtain the canonical multi-section of the relative tangent bundle of the S^2-bundle

$$\left(E - \coprod_{l=1}^{n} \operatorname{Int} E_l\right)/G \to B - \coprod_{l=1}^{n} \operatorname{Int} \Delta_l$$

over the branched locus of the covering

$$E - \coprod_{l=1}^{n} \operatorname{Int} E_l \to \left(E - \coprod_{l=1}^{n} \operatorname{Int} E_l \right) / G \, .$$

We show that the normal euler number of $I(h, \psi)$ in the fibered 4-manifold $f : E \to B$ localizes in Subsection 4.2 by means of this multi-section.

4.1. A multi-section of the relative tangent bundle of a $\boldsymbol{CP}^1$-bundle. Denote by $\bar{E}_0$ the quotient spaces of E_0 under the G-action. We review a canonical multi-section of the relative tangent bundle $T(\bar{E}_0/B_0) \to \bar{E}_0$ over the branched locus $\bar{J}_0 \subset \bar{E}_0$ constructed by Furuta.

Fix a fiberwise complex structure on the S^2-bundle $\bar{f}_0 : \bar{E}_0 \to B_0$.

LEMMA 4.1 (Furuta [10, Lemma 2]). *When the order m of the branch set in S^2 of the G-covering p is at least* 3, *there exists a canonical section $\bar{s}$ of the complex line bundle $T(\bar{E}_0/B_0)|_{\bar{J}_0}^{\otimes(m-1)(m-2)} \to \bar{J}_0$. Moreover, the homotopy class of the section $\bar{s}$ does not depend on the choice of the complex structure.*

Actually, he also constructed a canonical multi-section of $T(\bar{E}_0/B_0)|_{\bar{E}_0 - \bar{J}_0}$, which we do not need in this paper.

PROOF. For $b \in B_0$, the intersection of $\bar{J}_0$ and the fiber $\bar{f}_0^{-1}(b)$ is a finite point set. Number them as $\{\alpha_i(b)\}_{i=1}^m = \bar{J}_0 \cap \bar{f}_0^{-1}(b)$. We will construct a tangent vector at $\alpha_i(b)$ for all distinct i, j, k in $\{1, 2, \dots, m\}$. Define an isomorphism

$$t_b^{ijk} : \boldsymbol{CP}^1 \to \bar{f}_0^{-1}(b)$$

by mapping 0, 1, and ∞, to $\alpha_i(b)$, $\alpha_j(b)$, and $\alpha_k(b)$, respectively. Since $\bar{f}_0^{-1}(b)$ has the complex structure, this isomorphism is unique. In this way, we obtain the tangent vector $t_{b*}^{ijk}(d/dz)$ at $\alpha_i(b)$, where z is the inhomogeneous coordinate in $\boldsymbol{CP}^1$. Then we have

$$\bigotimes_{j,k} t_{b*}^{ijk} \left(\frac{d}{dz} \right) \in T_{\alpha_i(b)}(\bar{E}_0/B_0)^{\otimes(m-1)(m-2)} \, ,$$

where j, k run over $\{1, 2, \dots, m\}$ with i, j, k distinct. Thus, we obtain a non-zero section $\bar{s}$ of the bundle $T(\bar{E}_0/B_0)^{\otimes(m-1)(m-2)}|_{\bar{J}_0}$. Since the space of fiberwise complex structures on the S^2-bundle $\bar{E}_0 \to B_0$ is contractible, the homotopy class of the nonzero section $\bar{s}$ is unique up to homotopy. □

4.2. The local euler number. We prove that the normal euler number of the horizontal components $I(h, \psi)$ localizes. Let S be a compact connected surface with nonempty boundary, and $V(S) \to S$ a vector bundle. Assume that $V(S)$ is oriented as a manifold, and that we are given a nonzero section $s : \partial S \to V(S)|_{\partial S}$. We introduce an integer $n(s, V(S))$ for the section s in order to show the localization of the normal euler number of $I(h, \psi)$.

In the following, all homology groups are with integral coefficients if not specified. Let $s_0 : S \to V(S)$ be the zero section. If we extend the section s to a section $\tilde{s}$ of $V(S) \to S$, we know from the exact sequence

$$0 = H_2(V(S)) \to H_2(V(S), V(S) - s_0(S)) \to H_1(V(S) - s_0(S)) \to H_1(V(S))$$

that the homology class $[\tilde{s}] \in H_2(V(S), V(S) - s_0(S))$ is independent of the choice of $\tilde{s}$. Denote by $[s_0]$ the homology class of the zero section in $H_2(V(S), V(S)|_{\partial S})$. Let $D(S)$ be a unit disk bundle in $V(S)$, and $S(S)$ its sphere bundle. Then we have natural isomorphisms $H_2(V(S), V(S)|_{\partial S}) \cong H_2(D(S), D(S)|_{\partial S})$ and $H_2(V(S), V(S) - s_0(S)) \cong H_2(D(S), S(S))$ induced by the inclusions. Define the number $n(s, V(S)) := [\tilde{s}] \cdot [s_0] \in \boldsymbol{Z}$ by using the intersection form $H_2(D(S), D(S)|_{\partial S}) \times H_2(D(S), S(S)) \to \boldsymbol{Z}$.

We go back to a fibered 4-manifold of the G-covering p. Let $f : E \to \Delta$ be a representative of a fiber germ $[f, E, \Delta] \in S_g^p$. Let S denote a horizontal component of the fixed point set E^h for $h \neq 1 \in G$, and let H denote the subgroup of G consisting of elements which fix S pointwise.

Let $q : E \to E/G$, $q_H : E \to E/H$, and $q_{G/H} : E/H \to E/G$ denote the quotient maps of the group action. Endow a metric g_r on $T(E/H)|_{q_H(S)}$ whose restriction on $T(E/H)|_{q_H(\partial S)}$ makes the relative tangent bundle $T((\partial E/H)/\partial\Delta)|_{q_H(\partial S)}$ and the tangent bundle $T(q_H(S))|_{q_H(\partial S)}$ orthogonal. Then we have an isomorphism

$$T((\partial E/H)/\partial\Delta)|_{q_H(\partial S)} \cong N(q_H(S))|_{q_H(\partial S)}\,.$$

We can construct a canonical section on the bundle $T((\partial E/H)/\partial\Delta)|_{q_H(\partial S)}$ as follows. Since the intersection of the branched locus of $q_{G/H} : E/H \to E/G$ and $q(\partial S) \subset E/G$ is empty, we have a canonical isomorphism

$$T((\partial E/H)/\partial\Delta)|_{q_H(\partial S)} \cong q_{G/H}^* T((\partial E/G)/\partial\Delta)|_{q(\partial S)}\,.$$

By pulling back the multi-section $(\bar{s})^{\otimes(m-1)(m-2)}$ constructed in Subsection 4.1 of the relative tangent bundle $T((\partial E/G)/\partial\Delta)|_{q(\partial S)}$, we have a canonical section of the bundle $T((\partial E/H)/\partial\Delta)^{\otimes(m-1)(m-2)}|_{q_H(\partial S)}$. By the isomorphism

$$T((\partial E/H)/\partial\Delta)|_{q_H(\partial S)} \cong N(q_H(S))|_{q_H(\partial S)}\,,$$

we obtain a section $s_{\partial S} : q_H(\partial S) \to N(q_H(S))^{\otimes(m-1)(m-2)}|_{q_H(\partial S)}$. Our local euler number is described in terms of this section $s_{\partial S}$ as follows.

Definition 4.2. Define a map $\chi_{\mathrm{loc}}^{h,\psi} : S_g^p \to \boldsymbol{Q}$ by

$$\begin{array}{ccc} S_g^p & \to & \boldsymbol{Q} \\ {[f, E, \Delta]} & \mapsto & \displaystyle\sum_{S \subset I(h,\psi)} \frac{1}{r_S(m-1)(m-2)} n(s_{\partial S}, N(q_H(S))^{\otimes(m-1)(m-2)})\,, \end{array}$$

where r_S is the order of the subgroup H of G consisting of elements which fix S pointwise.

Since the homotopy class of the nonzero section $s_{\partial S}$ of the normal bundle $N(q_H(S))^{\otimes(m-1)(m-2)}|_{q_H(\partial S)}$ does not depend on the choices of the complex structure on the S^2-bundle $\partial E/G \to \partial\Delta$ and the metric g_r, this map is well-defined.

THEOREM 4.3. *For any fibered* 4*-manifold* $f : E \to B$ *of the* G*-covering* p *with singular fiber germs* $\{[f_l, E_l, \Delta_l]\}_{l=1}^n$, *we have*

$$\sum_{S \subset I(h,\psi)} \chi(N(S)) = \sum_{l=1}^{n} \chi_{\mathrm{loc}}^{h,\psi}([f_l, E_l, \Delta_l]) .$$

In other words, the map $\chi_{\mathrm{loc}}^{h,\psi} : S_g^p \to \boldsymbol{Q}$ *is a local normal euler number.*

PROOF. Let $\{f_l, E_l, \Delta_l\}_{l=1}^n$ be representatives of the fiber germs in the fibered 4-manifold $f : E \to B$. We may assume that Δ_l are mutually disjoint in B. Let E_0 and B_0 denote the spaces $E - \amalg_{l=1}^n \operatorname{Int} E_l$ and $B - \amalg_{l=1}^n \operatorname{Int} \Delta_l$, respectively. Let S denote a horizontal component of the fixed point set E^h for $h \neq 1 \in G$, and let S_0 denote the intersection $S \cap E_0$. Denote by H the subgroup of G consisting of elements which fix S pointwise, and r_S the order of the subgroup H. Since the branching index of $q_H(S) \subset E/H$ with respect to the covering $E \to E/H$ is r_S, we have an isomorphism

$$N(S)^{\otimes r_S} \cong N(q_H(S)) .$$

Hence, we have

$$\chi(N(S)) = \frac{1}{r_S(m-1)(m-2)} \chi(N(q_H(S))^{\otimes(m-1)(m-2)}) .$$

Since $E_0/H \to B_0$ is an oriented sphere bundle, we can constructed the nonzero section s_{S_0} of the bundle $N(q_H(S))^{\otimes(m-1)(m-2)}$ over $q_H(S_0)$ as in Subsection 4.1. Extend the section s_{S_0} to a section $\tilde{s}$ of $N(q_H(S))^{\otimes(m-1)(m-2)}$ over $q_H(S)$ transversely to the zero section $s_0 : S \to N(q_H(S))^{\otimes(m-1)(m-2)}$.

In terms of the numbers $n(s_{S_0}|_{q_H(S_0)\cap(\partial(E_l/H))}, N(q_H(S))^{\otimes(m-1)(m-2)}|_{q_H(S)\cap E_l/H})$, the euler number of $N(q_H(S))^{\otimes(m-1)(m-2)}$ is described as

$$\begin{aligned}\chi(N(q_H(S))^{\otimes(m-1)(m-2)}) &= \tilde{s} \cdot s_0 \\ &= \sum_{l=1}^{n} n(s_{S_0}|_{q_H(S_0)\cap(\partial E_l/H)}, N(q_H(S))^{\otimes(m-1)(m-2)}|_{q_H(S)\cap(E_l/H)}) \\ &= \sum_{l=1}^{n} r_S(m-1)(m-2)\chi_{\mathrm{loc}}^{h,\psi}([f_l, E_l, \Delta_l]) .\end{aligned}$$

Hence we have

$$\sum_{S \subset I(h,\psi)} \chi(N(S)) = \sum_{l=1}^{n} \chi_{\mathrm{loc}}^{h,\psi}([f_l, E_l, \Delta_l]) .$$

□

5. The Meyer cocycle and symmetric mapping class groups. Let $\mathrm{Diff}_+ \Sigma_g$ denote the orientation-preserving diffeomorphism group of the closed surface Σ_g of genus g. The mapping class group $\mathcal{M}_g$ of the surface Σ_g is defined by the path-connected component $\pi_0 \mathrm{Diff}_+ \Sigma_g$ of this topological group with C^∞ topology. For a finite regular covering $p : \Sigma_g \to \Sigma$ on a compact surface Σ, Birman and Hilden [7] defined a group $\mathcal{M}_g(p)$ called the symmetric mapping class group. We restrict ourselves to the case $\Sigma = S^2$. Recall that we denote by $C(p)$ the centralizer of the deck transformation group of the G-covering p.

DEFINITION 5.1. The symmetric mapping class group of the G-covering $p : \Sigma_g \to S^2$ is defined by

$$\mathcal{M}_g(p) := \pi_0 C(p)\,.$$

Let T be a finite set in Σ_g. Denote by $\mathrm{Diff}_+(\Sigma_g, T)$ the group of orientation-preserving diffeomorphisms on the surface Σ_g which fixes the set T pointwise. Denote by $A = \{\alpha_1, \alpha_2, \dots, \alpha_m\} \subset S^2$ the branch set of p. Pick a point $*$ in $S^2 - A$. In the same way, the symmetric mapping class group of the pointed surface $(\Sigma_g, p^{-1}(*))$ is defined by $\mathcal{M}_g^{(*)}(p) := \pi_0(\mathrm{Diff}_+(\Sigma_g, p^{-1}(*)) \cap C(p))$.

Mapping a path-connected component of $C(p)$ to the corresponding component of $\mathrm{Diff}_+ \Sigma_g$, we obtain the natural homomorphism

$$\Phi : \mathcal{M}_g(p) \to \mathcal{M}_g\,.$$

Assume that the G-covering $p : \Sigma_g \to S^2$ has at least 3 branch points as in Section 4. Meyer [20] introduced a 2-cocycle of the mapping class group $\mathcal{M}_g$, called the Meyer cocycle. We construct a cobounding function of the pullback of the Meyer cocycle by Φ in a subgroup $\mathcal{M}_{\mathrm{mon}}(p) \subset \mathcal{M}_g(p)$ in Theorem 5.5, when $p : \Sigma_g \to S^2$ has at least 3 branch points. When this subgroup coincides with the symmetric mapping class group $\mathcal{M}_g(p)$, we can construct a local signature for fibered 4-manifolds of the G-covering p in the broad sense, using this cobounding function (Proposition 5.6).

The symmetric mapping class group $\mathcal{M}_g(p)$ arises as the monodromy group of the Σ_g-bundles whose structure groups are contained in $C(p)$. Let us recall the monodromy homomorphisms of Σ_g-bundles. Let $f : E \to B$ be a Σ_g-bundle on a manifold B. Fix a base point b in B, and an identification $\Psi_0 : \Sigma_g \to f^{-1}(b)$, which is called the reference fiber. For a homotopy class $\gamma \in \pi_1(B, b)$, choose a based loop $l : [0,1] \to B$ which represents γ. Since the pullback $q : l^*E \to [0,1]$ is the trivial Σ_g-bundle, we can choose a trivialization $\tilde{\Psi} : \Sigma_g \times [0,1] \to l^*E$ such that $\tilde{\Psi}(x, 0) = \Psi_0(x)$. Define the diffeomorphism $\Psi_1 : \Sigma_g \to f^{-1}(b)$ by $\Psi_1(x) = \tilde{\Psi}(x, 1)$. The isotopy class of the diffeomorphism $\Psi_1^{-1}\Psi_0$ is called the monodromy of the Σ_g-bundle $f : E \to B$ along the loop l. It does not depend on the choice of $\tilde{\Psi}$ and l. Thus, we can define a homomorphism $\pi_1(B, b) \to \mathcal{M}_g$, called the monodromy homomorphism. If the structure group of the Σ_g-bundle $f : E \to B$ is in $C(p)$, the diffeomorphism $\Psi_1^{-1}\Psi_0$ is also contained in $C(p)$. Similarly, we have the homomorphism $\pi_1(B, b) \to \mathcal{M}_g(p)$. We also call it the monodromy homomorphism.

Define the subgroup $\mathcal{M}_{\mathrm{mon}}(p)$ of the symmetric mapping class group $\mathcal{M}_g(p)$ as follows.

DEFINITION 5.2. Denote by $\mathcal{M}_{\mathrm{mon}}(p)$ the subgroup of the symmetric mapping class group $\mathcal{M}_g(p)$ normally generated by the monodromies which arise in the set of fiber germs S_g^p along the boundary circles.

Let us review the definition of the Meyer cocycle. For $i = 1, 2, 3$, let D_i be disjoint closed disks in a 2-sphere. Denote by P a pair of pants $S^2 - \amalg_{i=1}^{3} \operatorname{Int} D_i$. Let $E_g^{\varphi,\psi}$ be the total space of a Σ_g-bundle on P whose monodromies around the boundary circles ∂D_i are given by φ, ψ, $(\varphi\psi)^{-1} \in \mathcal{M}_g$. This Σ_g-bundle is unique up to isomorphism.

DEFINITION 5.3 (Meyer [20]). The map

$$\begin{array}{rccccc} \tau_g : & \mathcal{M}_g & \times & \mathcal{M}_g & \to & \boldsymbol{Z}, \\ & (\varphi & , & \psi) & \mapsto & \operatorname{Sign} E_g^{\varphi,\psi} \end{array}$$

is called the Meyer cocycle.

Meyer proved that the map τ is a 2-cocycle on the mapping class group. Moreover, he showed that this cocycle represents a nontrivial 2-cohomology class of $\mathcal{M}_g$ when $g \geq 3$. Later, it is rediscovered by Turaev [26].

Birman and Hilden proved that if the deck transformation group fixes the branch set pointwise, then Φ is injective [7, Theorem 1]. Let $p' : \Sigma_g \to S^2$ be a $\boldsymbol{Z}_2$-covering on S^2 for $g \geq 2$. Especially, the symmetric mapping class group for p' is isomorphic to a subgroup of the mapping class group called the hyperelliptic mapping class group.

To construct a cobounding function of $\Phi^*\tau_g$, we need a following lemma.

LEMMA 5.4. *For a mapping class $\hat{\varphi} \in \mathcal{M}_{\mathrm{mon}}(p)$, there exists a fibered 4-manifold $f : E \to D^2$ of the G-covering p whose monodromy along the boundary circle ∂D^2 is $\hat{\varphi}$.*

PROOF. Let $f : E \to \Delta$ be a representative of a fiber germ. Let t be a point in $\partial\Delta$, and $\Psi_0 : \Sigma_g \to f^{-1}(t)$ a reference fiber. Assume that the monodromy of f along the boundary circle is given by $\hat{\varphi}_0 \in \mathcal{M}_{\mathrm{mon}}(p)$. First, we construct fiber germs whose monodromies are $\hat{\psi}\hat{\varphi}_0\hat{\psi}^{-1}$ and $\hat{\varphi}_0^{-1}$ for $\hat{\psi} \in \mathcal{M}_g(p)$.

Let $\hat{h}$ be a diffeomorphism of Σ_g which represents $\hat{\psi} \in \mathcal{M}_g(p)$. If we change the reference fiber by $\Psi_0\hat{h}^{-1} : \Sigma_g \to f^{-1}(t)$, the monodromy is given by $\hat{\psi}\hat{\varphi}_0\hat{\psi}^{-1}$. Choose an orientation-reversing diffeomorphism $\iota : \Delta \to \Delta$. If we endow the other orientation on E, the smooth map $\iota f : E \to \Delta$ is also a fibered 4-manifold of the G-covering p. This fibered 4-manifold has $\hat{\varphi}_0^{-1}$ as its monodromy along the boundary circle.

Let $f_i : E_i \to \Delta_i$ be representatives of fiber germs for $i = 1, 2, \ldots, n$. Choose a point t_i in each boundary, and a reference fiber $\Psi_i : \Sigma_g \to f_i^{-1}(t_i)$. Assume that the monodromies of f_i along the boundary circles are $\hat{\varphi}_i$ for $i = 1, 2, \ldots, n$. It suffices to construct a fibered 4-manifold whose monodromy along the boundary circle is given by $\prod_{i=1}^{n} \hat{\varphi}_i$. If we glue together all reference fibers $f_i^{-1}(t_i) \subset E_i$ by $\Psi_{i'}\Psi_i^{-1} : f_i^{-1}(t_i) \to f_{i'}^{-1}(t_{i'})$ for $1 \leq i, i' \leq n$, we obtain the space $\bigcup_{i=1}^{n} E_i$. Denote by $\bigvee_{i=1}^{n} \Delta_i$ the wedge sum obtained by identifying each $t_i \in \Delta_i$. The maps f_i induce the map $F : \bigcup_{i=1}^{n} E_i \to \bigvee_{i=1}^{n} \Delta_i$. Embed the wedge sum $\bigvee_{i=1}^{n} \Delta_i$ in D^2 as in Figure 1. Then, there exists a deformation retraction $r : D^2 \to \bigvee_{i=1}^{n} \Delta_i$. Denote by $b_i \in \Delta_i$ the critical value of f_i. By taking the pullback of the Σ_g-bundle $\bigcup_{i=1}^{n}(E_i - f_i^{-1}(b_i)) \to \bigvee_{i=1}^{n}(\Delta_i - \{b_i\})$ by r, we obtain the fibered 4-manifold on D^2 whose

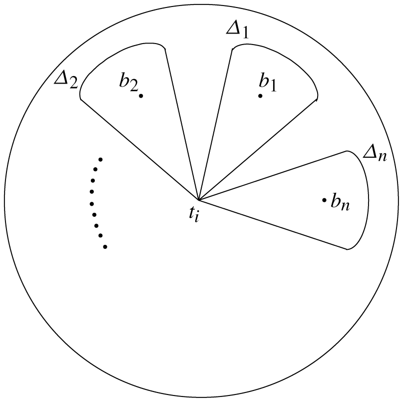

FIGURE 1. The wedge sum $\bigvee_{i=1}^{n} \Delta_i$ in D^2.

monodromy along the boundary circle is $\prod_{i=1}^{n} \hat{\varphi}_i$. There exists a (smooth) fibered 4-manifold topologically isomorphic to this fibered 4-manifold. □

THEOREM 5.5. *Let G be a finite group, and $p : \Sigma_g \to S^2$ a G-covering with at least* 3 *branch points. For a mapping class $\hat{\varphi} \in \mathcal{M}_{\mathrm{mon}}(p)$, there exists a fibered* 4*-manifold $f : X \to D^2$ of the G-covering p on a closed* 2*-disk whose monodromy along the boundary circle ∂D^2 is $\hat{\varphi}$. Denote by $\{[f_l, E_l, \Delta_l]\}_{l=1}^{n}$ the fiber germs arise in the fibered* 4*-manifold. Define a map $\phi : \mathcal{M}_{\mathrm{mon}}(p) \to \boldsymbol{Q}$ by*

$$\phi(\hat{\varphi}) := \sum_{l=1}^{n} \sigma_{\mathrm{loc}}([f_l, E_l, \Delta_l]) - \operatorname{Sign} X ,$$

where σ_{loc} is the local signature described in Theorem 1.1. *This is well-defined, and cobounds the* 2*-cocycle $\Phi^*\tau_g$ in $\mathcal{M}_{\mathrm{mon}}(p)$. That is to say, it satisfies*

$$\phi(\hat{\varphi}) + \phi(\hat{\psi}) + \phi((\hat{\varphi}\hat{\psi})^{-1}) = \Phi^*\tau_g(\hat{\varphi}, \hat{\psi})$$

for $\hat{\varphi}, \hat{\psi} \in \mathcal{M}_{\mathrm{mon}}(p)$.

PROOF. Choose a Σ_g-bundle $E \to P$ on the pair of pants whose monodromies along the boundary circles are $\hat{\varphi}$, $\hat{\psi}$, and $(\hat{\varphi}\hat{\psi})^{-1}$, respectively. Since these mapping classes lie in $\mathcal{M}_{\mathrm{mon}}(p)$, there exist fibered 4-manifolds $X_i \to D^2$ $(i = 1, 2, 3)$ of the G-covering p whose monodromies along the boundary circles are $\hat{\varphi}$, $\hat{\psi}$, and $(\hat{\varphi}\hat{\psi})^{-1}$, respectively. Let $\{f_l^i, E_l^i, \Delta_l^i\}_{l=1}^{n_i}$ denote representatives of the fiber germs arise in the fibered 4-manifold X_i. By the definition of the Meyer cocycle, we have

$$\Phi^*\tau_g(\hat{\varphi}, \hat{\psi}) = \operatorname{Sign} E .$$

By the definition of the local signature, we have

$$\operatorname{Sign} E = \sum_{i=1}^{3} \left(\sum_{l=1}^{n_i} \sigma_{\mathrm{loc}}([f_l^i, E_l^i, \Delta_l^i]) - \operatorname{Sign} X_i \right) = \phi(\hat{\varphi}) + \phi(\hat{\psi}) + \phi((\hat{\varphi}\hat{\psi})^{-1})\,.$$

Even if we substitute another fibered 4-manifold $X_1' \to D^2$ whose monodromy along the boundary circle is $\hat{\varphi}$ for $X_1 \to D^2$, the left-hand side of the equation does not change. Hence the value $\phi(\hat{\varphi})$ does not depend on the choice of X_1 either. Moreover, the equation shows that the map ϕ cobounds the Meyer cocycle. □

In terms of the cobounding function, we obtain a local signature for fibered 4-manifolds of the G-covering p in the broad sense, if the subgroup $\mathcal{M}_{\mathrm{mon}}(p)$ coincides with the whole group $\mathcal{M}_g(p)$. This local signature is defined as a function on another kind of fiber germs. Denote by Δ a closed 2-disk. Consider fibered 4-manifolds (f, E, Δ) of the G-covering p in the broad sense with unique critical values $b \in \Delta$. Let (f_1, E_1, Δ_1) and (f_2, E_2, Δ_2) be such fibered 4-manifolds which have unique critical values b_1 and b_2, respectively. These fibered 4-manifolds are equivalent if and only if there exist

(i) closed 2-disks $\Delta_1' \subset \Delta_1$ and $\Delta_2' \subset \Delta_2$ including the critical values,
(ii) an orientation-preserving diffeomorphism $\varphi : (\Delta_1', b_1) \to (\Delta_2', b_2)$,
(iii) an orientation-preserving diffeomorphism $\tilde{\varphi} : f_1^{-1}(\Delta_1) \to f_2^{-1}(\Delta_2)$ such that $\varphi f_1 = f_2 \tilde{\varphi}$ and it restricts to a G-equivariant diffeomorphism $f_1^{-1}(\Delta_1 - b_1) \to f_2^{-1}(\Delta_2 - b_2)$.

We denote this set of equivalent classes by $\tilde{S}_g^p$.

PROPOSITION 5.6. *Let G be a finite group, and $p : \Sigma_g \to S^2$ a G-covering with at least* 3 *branch points. Assume that the group $\mathcal{M}_{\mathrm{mon}}(p)$ coincides with the whole group $\mathcal{M}_g(p)$. Let $\phi : \mathcal{M}_g(p) \to \boldsymbol{Q}$ be the cobounding function of the pullback $\Phi^*\tau_g$ of the Meyer cocycle in Theorem* 5.5. *For a fiber germ $[f, E, \Delta] \in \tilde{S}_g^p$, denote by $\hat{\varphi}$ the monodromy along the boundary curve $\partial\Delta$. The map $\sigma'_{\mathrm{loc}} : \tilde{S}_g^p \to \boldsymbol{Q}$ defined by*

$$\sigma'_{\mathrm{loc}}([f, E, \Delta]) := \phi(\hat{\varphi}) + \operatorname{Sign} E$$

is a local signature for fibered 4*-manifolds of the G-covering p in the broad sense.*

The proof is the same as that of Endo [9, Theorem 4.4].

REMARK 5.7. In general, a cobounding function of the pullback $\Phi^*\tau_g$ of the Meyer cocycle in $\mathcal{M}_{\mathrm{mon}}(p)$ is not unique.

6. Generators of symmetric mapping class groups. In this section, we describe a generating set of the symmetric mapping class group $\mathcal{M}_g(p)$ of the G-covering p, assuming that the finite group G is abelian. Let $d \geq 2$ be an integer. In Subsection 7.1, we will construct fiber germs whose monodromies are inverses of the standard generator system $\{\sigma_{ij}\}$, and prove that $\mathcal{M}_g(p_1) = \mathcal{M}_{\mathrm{mon}}(p_1)$.

Let $\hat{f}$ be in the centralizer $C(p)$. The diffeomorphism $\hat{f}$ induces a diffeomorphism f of S^2 satisfying a commutative diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{\hat{f}} & \Sigma_g \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ S^2 & \xrightarrow{f} & S^2 . \end{array}$$

We call the diffeomorphism f the projection of $\hat{f}$. Recall that A is the branch set in S^2 of the G-covering p. Let $k : \pi_1(S^2 - A) \to G$ be the monodromy homomorphism of the G-covering p. Since G is abelian, this induces the homomorphism $\bar{k} : H_1(S^2 - A) \to G$. Choose a base point $*$ in $S^2 - A$. Denote by $\gamma_{\alpha_i} : [0, 1] \to S^2$ a based loop which rotates around a point α_i counterclockwise once. For $h \in G - \{1\}$, define a subset A_h of the branch set A in S^2 by $A_h = \{\alpha \in A\,;\ \bar{k}_*[\gamma_\alpha] = h\}$, where $[\gamma_\alpha]$ is the homology class of γ_α. Let $\mathcal{M}_0^A$ denote the mapping class group of S^2 which preserves the set A_h setwise for any $h \in G - \{1\}$. We also denote by $\mathcal{M}_0^{A,*}$ the mapping class group which preserves the base point $*$ and each A_h setwise. The projection f of $\hat{f} \in C(p)$ preserves each branch set A_h. For the details, see [25, Proposition 1.2]. Thus, we have homomorphisms $\Phi' : \mathcal{M}_g^{(*)}(p) \to \mathcal{M}_0^{A,*}$ and $\Phi : \mathcal{M}_g(p) \to \mathcal{M}_0^A$ defined by $[\hat{f}] \mapsto [f]$.

LEMMA 6.1. *Assume that the finite group G is abelian.*

(i) *The homomorphism* $\Phi' : \mathcal{M}_g^{(*)}(p) \to \mathcal{M}_0^{A,*}$ *is isomorphic.*

(ii) *The homomorphism* $\Phi : \mathcal{M}_g(p) \to \mathcal{M}_0^A$ *is surjective, and the kernel is generated by* $\mathrm{Deck}(p)$.

PROOF. The surjectivity of Φ is a special case of [25, Proposition 1.2]. We can show that the homomorphism Φ' is surjective in the same way.

We compute the kernels of Φ and Φ'. Let f be the projection of $\hat{f} \in C(p)$. If the mapping class $[\hat{f}] \in \mathcal{M}_g(p)$ is in the kernel of Φ, there exists an isotopy $\{f_s\}_{0 \le s \le 1}$ satisfying $f_0 = f$ and $f_1 = \mathrm{id}$. Choose the lift of this isotopy $\{\hat{f}_s\}_{0 \le s \le 1}$ such that $\hat{f}_0 = \hat{f}$. Since $\hat{f}_1$ is a lift of the identity map, it is a deck transformation. Hence, the kernel is generated by $\mathrm{Deck}(p)$. By the same argument, we can show that the kernel of Φ' is also generated by $\mathrm{Deck}(p) \cap \mathrm{Diff}_+(\Sigma_g, p^{-1}(*))$, which is the trivial group. □

For mutually distinct integers i, j, choose a simple closed curve C_{ij} as in Figure 2. When there exist mutually distinct $h, h' \in G - \{1\}$ which satisfy $\alpha_i \in A_h$ and $\alpha_j \in A_{h'}$, denote by $\tau_{ij} \in \mathcal{M}_0^{A,*}$ the full Dehn twist along C_{ij}. When there exists $h \in G - \{1\}$ which satisfies $\alpha_i, \alpha_j \in A_h$, denote by $\sigma_{ij} \in \mathcal{M}_0^{A,*}$ the half Dehn twist along C_{ij}. This is the mapping class which exchanges the points α_i and α_j and whose square is the full Dehn twist along C_{ij}. We denote by $\hat{\sigma}_{ij}$ and $\hat{\tau}_{ij}$ the lifts of σ_{ij} and τ_{ij} in $\mathcal{M}_g^{(*)}(p)$ which preserve the fiber $p^{-1}(*)$ pointwise, respectively. The inclusion $\mathrm{Diff}_+(\Sigma_g, p^{-1}(*)) \cap C(p) \to C(p)$ induces a homomorphism $\mathcal{M}_g^{(*)}(p) \to \mathcal{M}_g(p)$. We also denote by the same symbol the images of $\hat{\sigma}_{ij}$ and $\hat{\tau}_{ij}$ under this homomorphism.

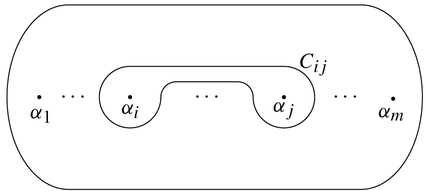

FIGURE 2. The simple closed curve C_{ij}.

LEMMA 6.2. *If the finite group G is abelian, then both of the groups $\mathcal{M}_g^{(*)}(p)$ and $\mathcal{M}_g(p)$ are generated by $\{\hat{\sigma}_{ij}\}_{1\le i<j\le m} \cup \{\hat{\tau}_{ij}\}_{1\le i<j\le m}$.*

PROOF. Since we have the isomorphism $\Phi' : \mathcal{M}_g^{(*)}(p) \cong \mathcal{M}_0^{A,*}$, it suffices to show that

(i) the mapping classes σ_{ij}, τ_{ij} generates $\mathcal{M}_0^{A,*}$,
(ii) the homomorphism $\mathcal{M}_g^{(*)}(p) \to \mathcal{M}_g(p)$ is surjective.

First, we show (i). Let $\mathrm{Diff}_+(S^2, A, *)$ be the diffeomorphism group which preserves the base point $*$ and the set A pointwise. Denote by $n(h)$ the order of A_h, and by $S_{n(h)}$ the symmetric group of degree $n(h)$. Since $\mathcal{M}_0^{A,*}$ permutes the elements of each set A_h, we have the homomorphism $\eta : \mathcal{M}_0^{A,*} \to \prod_{h\in G-\{1\}} S_{n(h)}$. Since the group $\prod_{h\in G-\{1\}} S_{n(h)}$ is generated by the images of $\hat{\sigma}_{ij}$ under η, we have the exact sequence

$$1 \longrightarrow \pi_0 \mathrm{Diff}_+(S^2, A, *) \longrightarrow \mathcal{M}_0^{A,*} \xrightarrow{\ \eta\ } \prod_{h\in G-\{1\}} S_{n(h)} \longrightarrow 1\,.$$

It is known that $\pi_0 \mathrm{Diff}_+(S^2, A, *)$ is generated by σ_{ij}^2 and τ_{ij}. For example, see Birman [6, 1.5]. Thus, we have proved (i).

Next, we show (ii). Let $\hat{*}$ be a point in $p^{-1}(*)$. Let $\hat{f} \in C(p)$ be a diffeomorphism, and let $f \in \mathrm{Diff}_+ S^2$ denote the projection of $\hat{f}$. The map $\rho : \mathrm{Diff}_+(S^2, A) \to S^2 - A$ defined by $h \mapsto h(*)$ is a fiber bundle with fiber $\mathrm{Diff}_+(S^2, A, *)$ as Birman [6, Theorem 4.1]. Pick a path $\hat{\gamma} : [0, 1] \to \Sigma_g - p^{-1}(A)$ such that $\hat{\gamma}(0) = \hat{*}$ and $\hat{\gamma}(1) = \hat{f}(\hat{*})$. Denote by $\Psi : [0, 1] \to \mathrm{Diff}_+(S^2, A)$ a lift of $p\hat{\gamma} : [0, 1] \to S^2 - A$ with respect to the fiber bundle $\rho : \mathrm{Diff}_+(S^2, A) \to S^2 - A$ such that $\Psi(0)$ is the identity map. By the lifting property of the G-covering $p : \Sigma_g \to S^2$, this can be lifted to the map $\hat{\Psi} : [0, 1] \to C(p)$ such that $\hat{\Psi}(0)$ is the identity map. Then, the composite $\hat{\Psi}(1)^{-1}\hat{f}$ preserves the point $\hat{*}$. Moreover, since $\hat{\Psi}(1)^{-1}\hat{f}$ commutes with the deck transformations, it preserves each point of $p^{-1}(*)$. Hence $\hat{\Psi}(1)^{-1}\hat{f}$ represents a mapping class in $\mathcal{M}_g^{(*)}(p)$. By the isotopy $\hat{\Psi}$, we have $[\hat{\Psi}(1)^{-1}\hat{f}] = [\hat{f}] \in \mathcal{M}_g(p)$. This shows that $\mathcal{M}_g^{(*)}(p) \to \mathcal{M}_g(p)$ is surjective. □

7. Proof of Proposition 1.2. Let $d \ge 2$ and $m \ge 3$ be integers such that m is divided by d. Let A be a set of m distinguished points $\{\alpha_i\}_{i=1}^m$ in S^2. For each $i = 1, 2, \ldots, m$, choose a loop γ_{α_i} which rotates around a point α_i counterclockwise once. Define a surjective homomorphism $k : H_1(S^2 - A) \to \mathbf{Z}_d$ by mapping each homology class $[\gamma_{\alpha_i}]$ to $1 \in \mathbf{Z}_d$.

Since $m \equiv 0 \bmod d$, this is well-defined. Let $p_1 : \Sigma_g \to S^2$ be the $\boldsymbol{Z}_d$-covering on S^2 which has the branch set A in S^2 and the monodromy homomorphism k.

In Subsection 7.1, we will construct a fiber germ of $\boldsymbol{Z}_d$-covering p_1 whose monodromy is the inverse $\hat{\sigma}_{12}^{-1}$ of the generator of $\mathcal{M}_g(p_1)$ introduced in Section 6. This proves $\mathcal{M}_g(p_1) = \mathcal{M}_{\mathrm{mon}}(p_1)$ (Lemma 7.1). Especially, we obtain a homogeneous quasi-morphism on the mapping class group $\mathcal{M}_0^m$ of the m-pointed sphere (Corollary 7.3). In Subsection 7.2, we will also calculate the local signature of this fiber germ and the value $\phi(\hat{\sigma}_{12})$ of the cobounding function of the pullback of the Meyer cocycle τ_g by the homomorphism $\Phi : \mathcal{M}_g(p) \to \mathcal{M}_g$.

7.1. The construction of a fiber germ.

LEMMA 7.1. *The subgroup $\mathcal{M}_{\mathrm{mon}}(p_1)$ coincides with the whole symmetric mapping class group $\mathcal{M}_g(p_1)$.*

PROOF. We need to show that the generating set $\{\hat{\sigma}_{ij}\}_{i,j\in A}$ of the symmetric mapping class group $\mathcal{M}_g(p_1)$ is contained in the subgroup $\mathcal{M}_{\mathrm{mon}}(p_1)$. Since $\{\hat{\sigma}_{ij}\}_{i,j\in A}$ are in the same conjugacy class in $\mathcal{M}_g(p_1)$, it suffices to construct a fiber germ in $S_g^{p_1}$ whose monodromy is $\hat{\sigma}_{12}$.

Let $\bar{E}$ be the product space $\Delta_1 \times \boldsymbol{CP}^1$, where $\Delta_1 = \{b \in \boldsymbol{C}\,;\, |b| \leq 1/2\}$ is a closed 2-disk. Let $[x_1 : x_2]$ be the homogeneous coordinate of $\boldsymbol{CP}^1$, and $m' = m - 2$. Denote by $\bar{J}$ a submanifold of $\bar{E}$ defined by the equation $(x_1^{m'} - x_2^{m'})(x_1^2 - bx_2^2) = 0$. For $b = re^{\sqrt{-1}\phi} \in \Delta_1$ where $r \geq 0$ and $0 \leq \phi < 2\pi$, define the root by $\sqrt{b} = \sqrt{r}e^{\sqrt{-1}\phi/2}$. Pick a diffeomorphism $T : S^2 \to \{1/2\} \times \boldsymbol{CP}^1$ which maps α_1, α_2, and α_i to $(1/2, [\sqrt{1/2} : 1])$, $(1/2, [-\sqrt{1/2} : 1])$, and $(1/2, [1 : e^{2\pi\sqrt{-1}(i-2)/m'}])$, for $i = 3, 4, \ldots, m$, respectively. Then, we have

$$H_1(\bar{E} - \bar{J}) = \bigoplus_{j=1}^{m} \boldsymbol{Z}e_j / (\boldsymbol{Z}(e_1 - e_2) \oplus \boldsymbol{Z}(e_1 + e_2 + \cdots + e_m)),$$

where e_j is the homology class represented by the loop $T \circ \gamma_{\alpha_j}$. Define a homomorphism $l : H_1(\bar{E} - \bar{J}) \to \boldsymbol{Z}_d$ by mapping each e_j to 1. This is well-defined since $m \equiv 0 \bmod d$. Hence there exists a $\boldsymbol{Z}_d$-covering $q : E \to \bar{E}$ branched along $\bar{J}$ whose monodromy homomorphism is l.

Denote by $\bar{f} : \bar{E} \to \Delta_1$ the projection to the first factor, and by $f : E \to \Delta_1$ the composite of $q : E \to \bar{E}$ and $\bar{f}$. The map f is a fibered 4-manifold of the $\boldsymbol{Z}_d$-covering p_1 in the narrow sense with the unique singular value $b = 0$. We only have to check that it has the monodromy $\hat{\sigma}_{12}$. The restriction of $\bar{f}$

$$(\bar{E} - (\{0\} \times \boldsymbol{CP}^1)\,, \quad \bar{J} - \{(0, [0 : 1])\}) \to \Delta_1 - 0$$

can be considered as a m-pointed sphere bundle. Consider $T : S^2 \to \{1/2\} \times \boldsymbol{CP}^1$ as a reference fiber of this bundle. Then, the monodromy of this bundle along $\partial\Delta_1$ is $\sigma_{12}^{-1} \in \mathcal{M}_0^A$. Moreover, since there is a section $s : \Delta_1 \to \bar{E}$ defined by $s(b) = (b, [1 : 0])$, the monodromy can be considered as $\sigma_{12} \in \mathcal{M}_0^{A,*}$. Since the isomorphism $\Phi' : \mathcal{M}_g^{(*)}(p) \cong \mathcal{M}_0^{A,*}$ maps $\hat{\sigma}_{12}$ to σ_{12}, the fibered 4-manifold $f : E \to \Delta_1$ has the monodromy $\hat{\sigma}_{12}^{-1} \in \mathcal{M}_g(p)$. $\square$

LEMMA 7.2. *In the symmetric mapping class group $\mathcal{M}_g(p_1)$, the cobounding function of the pullback $\Phi^*\tau_g$ of the Meyer cocycle under $\Phi : \mathcal{M}_g(p_1) \to \mathcal{M}_g$ is unique.*

PROOF. If there exist two cobounding functions ϕ and ϕ', the map $\phi - \phi' : \mathcal{M}_g(p_1) \to \boldsymbol{Q}$ is a 1-cocycle. Hence it suffices to show that $H^1(\mathcal{M}_g(p_1); \boldsymbol{Q}) = 0$. By Lemma 6.1, we have the exact sequence

$$\mathrm{Deck}(p_1) \longrightarrow \mathcal{M}_g(p_1) \longrightarrow \mathcal{M}_0^m \longrightarrow 1\,. \tag{4}$$

Since the deck transformation group $\mathrm{Deck}(p_1)$ is finite, we have the isomorphism

$$H^1(\mathcal{M}_0^m; \boldsymbol{Q}) \cong H^1(\mathcal{M}_g(p_1); \boldsymbol{Q})\,.$$

It is known that $H^1(\mathcal{M}_0^m; \boldsymbol{Q}) = 0$, for example, this follows from the presentation of $\mathcal{M}_0^m$ obtained by Birman [6, Theorem 4.5]. Hence we have $H^1(\mathcal{M}_g(p_1); \boldsymbol{Q}) = 0$. □

Recall that the Meyer cocycle is a bounded 2-cocycle. By Theorem 5.5 and the exact sequence (4), we obtain the following corollary.

COROLLARY 7.3. *The function $\phi : \mathcal{M}_g(p_1) \to \boldsymbol{Q}$ is a quasimorphism. Especially, the homogenization $\tilde{\phi} : \mathcal{M}_g(p_1) \to \boldsymbol{Q}$ defined by*

$$\tilde{\phi}(\hat{\varphi}) = \lim_{n\to\infty} \frac{\phi(\hat{\varphi}^m)}{n}$$

induces a homogeneous quasimorphism on the mapping class group $\mathcal{M}_0^m$ of the m-pointed sphere.

7.2. The calculation of the local signature for the fiber germ. To calculate the local euler number of the horizontal components, we need Lemma 7.4. Let $D(\Delta) \to \Delta$ be a D^2-bundle on a closed 2-disk Δ, and $S(\Delta)$ its sphere bundle. The manifold $S(\Delta)$ induces the orientation on the boundary $S(\Delta)|_{\partial\Delta}$. Let $s : \Delta \to S(\Delta)$ be a section. For a section $s' : \partial\Delta \to S(\Delta)|_{\partial\Delta}$, take an extension $\tilde{s'} : \Delta \to D(\Delta)$. Then we can consider the following two intersection forms $H_2(D(\Delta), S(\Delta)) \times H_2(D(\Delta), D(\Delta)|_{\partial\Delta}) \to \boldsymbol{Z}$ and $H_1(S(\Delta)|_{\partial\Delta}) \times H_1(S(\Delta)|_{\partial\Delta}) \to \boldsymbol{Z}$. We denote by $\tilde{s'} \cdot s$ the former one, by $s' \cdot s|_{\partial\Delta}$ the latter one. Then we have:

LEMMA 7.4. *For the sections $s : \Delta \to S(\Delta)$ and $s' : \partial\Delta \to S(\Delta)|_{\partial\Delta}$ as above, we have*

$$\tilde{s'} \cdot s = -s' \cdot s|_{\partial\Delta}\,.$$

PROOF. We may identify the disk Δ with the embedded 2-disk $D^2 = \{b \in \boldsymbol{C}\,;\ |b| \le 1\}$ in $\boldsymbol{C}$. The section s gives a trivialization $D(\Delta) \cong D^2 \times D^2$.

For some integer k, the section s' represents the same class in $H_1(S(D^2)|_{\partial D^2})$ as the curve $\{(z, z^k)\,;\ z \in S^1\}$ in $D^2 \times D^2$. Then, we have $s' \cdot s|_{\partial D^2} = -k$. Since the homomorphism $H_2(D(D^2), S(D^2)) \to H_1(S(D^2)|_{\partial D^2})$ is injective, the homology class in $H_2(D(D^2), S(D^2))$ of $\tilde{s'}$ is represented by the surface $\{(z, z^k)\,;\ z \in D^2\}$. Hence we have $\tilde{s'} \cdot s = k = -s' \cdot s|_{\partial D^2}$. □

In the following, we prove Proposition 1.2. Let $E \to \Delta_1$ denote the fibered 4-manifold of the $\mathbf{Z}_d$-covering p_1 in Lemma 7.1. Let $\bar{S}_{12}$ and $\bar{S}_i$ be the submanifolds of $\bar{E}$ defined by $x_1^2 = bx_2^2$ and by $e^{2\pi\sqrt{-1}(i-2)/m'}x_1 = x_2$ for $i = 3, \dots, m$, respectively. Denote by S_{12} and $\{S_i\}_{i=3}^m$ their inverse images $q^{-1}(\bar{S}_{12})$ and $\{q^{-1}(\bar{S}_i)\}_{i=3}^m$ under the quotient map $q : E \to \bar{E}$.

Choose a Riemannian metric g_r of $T(\bar{E})$ whose restriction to $T(\bar{E})|_{\bar{S}_i}$ makes $T(\bar{E}/\Delta_1)|_{\bar{S}_i}$ and $T\bar{S}_i$ orthogonal, and restriction to $T(\bar{E})|_{\partial\bar{S}_{12}}$ makes $T(\bar{E}/\Delta_1)|_{\partial\bar{S}_{12}}$ and $T\bar{S}_{12}|_{\partial\bar{S}_{12}}$ orthogonal. Then we have

$$N(\bar{S}_i)|_{\bar{S}_i} = T(\bar{E}/\Delta_1)|_{\bar{S}_i} \text{ and } N(\bar{S}_{12})|_{\partial\bar{S}_{12}} = T(\bar{E}/\Delta_1)|_{\partial\bar{S}_{12}}\,.$$

In Section 4.1, we constructed the sections of $N(\bar{S}_i)|_{\partial\bar{S}_i}^{\otimes(m-1)(m-2)}$ and $N(\bar{S}_{12})|_{\partial\bar{S}_{12}}^{\otimes(m-1)(m-2)}$, named $s_{\partial\bar{S}_i}$ and $s_{\partial\bar{S}_{12}}$, respectively. We first review the definitions of these sections. Define maps $\alpha_i : \Delta_1 \to \bar{E}$ by $\alpha_1(b) = (b, [1 : \sqrt{b}])$, $\alpha_2(b) = (b, [1 : -\sqrt{b}])$, and $\alpha_i(b) = (b, [1, e^{2\pi\sqrt{-1}(i-2)/m'}])$ for $i = 3, \dots, m$. Let j and k be integers such that $1 \le j \le m$, $1 \le k \le m$, and i, j, k are mutually distinct. For such j and k, define a not necessarily continuous section $s_i(j,k) : \partial\bar{S}_i \to T(\bar{E}/\Delta_1)|_{\partial\bar{S}_i}$ by

$$s_i(j,k)(\alpha_i(b)) = (t_b^{ijk})_*\left(\frac{d}{dz}\right)$$

as in Subsection 4.1. Note that, if $\{j,k\} \cap \{1,2\} \neq \emptyset$, the section $s_i(j,k)$ is not continuous since the root of b is not continuous. The (continuous) section $s_{\partial\bar{S}_i}$ is defined by

$$s_{\partial\bar{S}_i} = \bigotimes_{j,k} s_i(j,k),$$

where j and k run through integers such that $1 \le j \le m$, $1 \le k \le m$, and i, j, k are distinct. In the same way, for integers j and k such that $3 \le j \le m$, $3 \le k \le m$, and i, j, k are distinct, define sections of the bundle $T(\bar{E}/\Delta_1)|_{\partial\bar{S}_{12}} \to \bar{S}_{12}$ by

$$s_{12}(j,k)(\alpha_i(b)) = (t_b^{ijk})_*\left(\frac{d}{dz}\right), \quad \text{for } i = 1, 2\,,$$

$$s_{12}^+(j)(\alpha_i(b)) = \begin{cases} (t_b^{12j})_*\left(\dfrac{d}{dz}\right), & \text{if } i = 1\,, \\ (t_b^{21j})_*\left(\dfrac{d}{dz}\right), & \text{if } i = 2\,, \end{cases}$$

$$s_{12}^-(j)(\alpha_i(b)) = \begin{cases} (t_b^{1j2})_*\left(\dfrac{d}{dz}\right), & \text{if } i = 1\,, \\ (t_b^{2j1})_*\left(\dfrac{d}{dz}\right), & \text{if } i = 2\,. \end{cases}$$

The section $s_{\partial\bar{S}_{12}}$ is defined by

$$s_{\partial\bar{S}_{12}} = \bigotimes_{j,k} s_{12}(j,k) \otimes \bigotimes_{j=3}^{m} (s_{12}^+(j) \otimes s_{12}^-(j))\,.$$

For any $h \neq 0 \in \boldsymbol{Z}_d$, the connected components of the fixed point set E^h are S_{12} and $\{S_i\}_{i=3}^m$. Since $h \in \boldsymbol{Z}_d$ rotates the normal bundle of these components $2h\pi/d$, we have isomorphisms

$$N(S_{12})^{\otimes d} \cong N(\bar{S}_{12}), \quad \text{and} \quad N(S_i)^{\otimes d} \cong N(\bar{S}_i).$$

The local normal euler number is described as

$$\begin{aligned}&\chi_{\mathrm{loc}}^{h,2h\pi/d}([f, E, \Delta_1]) \\ &= \frac{1}{d(m-1)(m-2)}\left\{\sum_{i=3}^m n(s_{\partial\bar{S}_i}, N(\bar{S}_i)^{\otimes(m-1)(m-2)}) + n(s_{\partial\bar{S}_{12}}, N(\bar{S}_{12})^{\otimes(m-1)(m-2)})\right\}.\end{aligned}$$

We will calculate $n(s_{\partial\bar{S}_i}, N(\bar{S}_i)^{\otimes(m-1)(m-2)})$ and $n(s_{\partial\bar{S}_{12}}, N(\bar{S}_{12})^{\otimes(m-1)(m-2)})$ in Lemmas 7.5 and 7.6, respectively.

LEMMA 7.5. *For the sections* $s_{\partial\bar{S}_i} : \partial\bar{S}_i \to N(\bar{S}_i)|_{\partial\bar{S}_i}^{\otimes(m-1)(m-2)}$, *we have*

$$n(s_{\partial\bar{S}_i}, N(\bar{S}_i)^{\otimes(m-1)(m-2)}) = -1,$$

for $3 \leq i \leq m$.

PROOF. Let $w = x_2/x_1$ be the inhomogeneous coordinate of the second factor of $\bar{E} = \Delta_1 \times \boldsymbol{CP}^1$. Since the map $t_b^{ijk} : \boldsymbol{CP}^1 \to \bar{f}_0^{-1}(b)$ is written as

$$t_b^{ijk}(z) = \frac{\alpha_k(b)(\alpha_i(b) - \alpha_j(b))z + \alpha_i(b)(\alpha_j(b) - \alpha_k(b))}{(\alpha_i(b) - \alpha_j(b))z + (\alpha_j(b) - \alpha_k(b))},$$

the vector $(t_b^{ijk})_*(d/dz)$ is described as

$$(5) \qquad (t_b^{ijk})_*\left(\frac{d}{dz}\right) = \frac{(\alpha_i(b) - \alpha_j(b))(\alpha_k(b) - \alpha_i(b))}{\alpha_j(b) - \alpha_k(b)}\left(\frac{d}{dw}\right).$$

Suppose $j \geq 3$ and $k \geq 3$. Let j', k' also be integers such that $3 \leq j' \leq m$, $3 \leq k' \leq m$, and i, j', k' are mutually distinct. Let $s_0 : \partial\bar{S}_i \to T(\bar{E}/\Delta_i)|_{\partial\bar{S}_i}$ be the zero section. The intersection form on the first homology group of the sphere bundle of $T(\bar{E}/\Delta_1)|_{\partial\bar{S}_i}$ induces that of the $(\boldsymbol{C} - 0)$-bundle $T(\bar{E}/\Delta_1)|_{\partial\bar{S}_i} - s_0(\partial\bar{S}_i)$. By the explicit description (5), we can calculate the intersection numbers

$$(6) \qquad s_i(j,k) \cdot s_i(j',k') = 0,$$

$$(7) \qquad (s_i(1,k) \otimes s_i(2,k)) \cdot s_i(j',k')^{\otimes 2} = (s_i(k,1) \otimes s_i(k,2)) \cdot s_i(j',k')^{\otimes 2} = 0,$$

$$(8) \qquad (s_i(1,2) \otimes s_i(2,1)) \cdot s_i(j',k')^{\otimes 2} = 1.$$

Since we have the isomorphism $T(\bar{E}/\Delta_1)|_{\bar{S}_i} \cong N\bar{S}_i$, it follows that

$$n(s_{\partial\bar{S}_i}, N(\bar{S}_i)^{\otimes(m-1)(m-2)}) = n(s_{\partial\bar{S}_i}, T(\bar{E}/\Delta_1)|_{\bar{S}_i}^{\otimes(m-1)(m-2)}).$$

The section $s_i(j',k')$ of $T(\bar{E}/\Delta_1)|_{\partial \bar{S}_i}$ can be extended to the nonzero section $\tilde{s}_i(j',k')$ of $T(\bar{E}/\Delta_1)|_{\bar{S}_i}$ defined by $\tilde{s}_i(j',k')(\alpha_i(b)) = (t_b^{ij'k'})_*(d/dz)$ for $3 \le i \le m$. Hence a trivialization of the bundle $T(\bar{E}/\Delta_1)|_{\bar{S}_i}$ is given by $s_i(j',k')$. By Lemma 7.4, we have

$$n(s_{\partial \bar{S}_i}, T(\bar{E}/\Delta_1)|_{\bar{S}_i}^{\otimes(m-1)(m-2)}) = -\left(\bigotimes_{j,k} s_i(j,k)\right) \cdot s_i(j',k')^{\otimes(m-1)(m-2)} .$$

The calculations (6), (7), and (8) show that this is equal to -1. □

LEMMA 7.6. *For the section* $s_{\partial \bar{S}_{12}} : \partial \bar{S}_{12} \to N(\bar{S}_{12})|_{\partial \bar{S}_{12}}^{\otimes(m-1)(m-2)}$, *we have*

$$n(s_{\partial \bar{S}_{12}}, N(\bar{S}_{12})^{\otimes(m-1)(m-2)}) = (m+1)(m-2) .$$

PROOF. Suppose $j \ge 3$ and $k \ge 3$. The section $s_{12}(j,k)$ of $T(\bar{E}/\Delta_1)|_{\partial \bar{S}_{12}}$ can be extended to a nonzero section $\tilde{s}_{12}(j,k)$ of $T(\bar{E}/\Delta_1)|_{\bar{S}_{12}}$ defined by $\tilde{s}_{12}(j,k)(\alpha_i(b)) = (t_b^{ijk})_*(d/dz)$ for $i = 1, 2$. Let $r_{\bar{S}_{12}} : T\bar{E}|_{\bar{S}_{12}} \to N\bar{S}_{12}$ be the projection. The section $r_{\bar{S}_{12}}\tilde{s}_{12}(j,k)$ of $N(\bar{S}_{12})$ intersects the zero section $s'_0 : \bar{S}_{12} \to N(\bar{S}_{12})$ transversely in one point $s'_0(0, [0:1])$. Hence we have $n(s_{12}(j,k), N(\bar{S}_{12})) = 1$. By the explicit description (5), we can also calculate the intersection number

$$s_{12}^{\varepsilon}(j) \cdot s_{12}(j,k) = -1$$

of the $(\boldsymbol{C} - 0)$-bundle $N(\bar{S}_{12})|_{\partial \bar{S}_{12}} - s'_0(\partial \bar{S}_{12})$ for $\varepsilon = +, -$. Lemma 7.4 shows

$$n(s_{12}^{\varepsilon}(j), N(\bar{S}_{12})) = -s_{12}^{\varepsilon}(j) \cdot s_{12}(j,k) + n(s_{12}(j,k), N(\bar{S}_{12})) = 2 . \tag{9}$$

Thus, we have

$$\begin{aligned} &n(s_{\partial \bar{S}_{12}}, N(\bar{S}_{12})^{\otimes(m-1)(m-2)}) \\ &= \sum_{\substack{3 \le j \le m \\ 3 \le k \le m \\ j \ne k}} n(s_{12}(j,k), N(\bar{S}_{12})) + \sum_{j=3}^{m} n(s_{12}^{+}(j), N(\bar{S}_{12})) \\ &\quad + \sum_{j=3}^{m} n(s_{12}^{-}(j), N(\bar{S}_{12})) \\ &= (m+1)(m-2) . \end{aligned}$$

□

By Lemmas 7.5 and 7.6, the local euler number is calculated as

$$\chi_{\mathrm{loc}}^{2h\pi/d,h}([f, E, \Delta_1]) = \frac{m}{d(m-1)} .$$

Since there is no vertical component, by Theorem 1.1, we have

$$\sigma_{\mathrm{loc}}([f, E, \Delta_1]) = -\sum_{h=1}^{d-1} \chi_{\mathrm{loc}}^{2h\pi/d,h}([f, E, \Delta_1]) \operatorname{cosec}^2\left(\frac{h\pi}{d}\right) + \operatorname{Sign} \bar{E} .$$

It is known that (for example, see Hirzebruch-Zagier [13, p. 178])

$$\sum_{h=1}^{d-1} \operatorname{cosec}^2\left(\frac{h\pi}{d}\right) = \frac{d^2-1}{3}.$$

There is a deformation retraction of $\bar{E}$ onto $\{0\} \times \boldsymbol{CP}^1$ and its self-intersection number is 0. Hence we have $\operatorname{Sign} \bar{E} = 0$. Thus the local signature and the cobounding function of the pullback of the Meyer cocycle is

$$\sigma_{\mathrm{loc}}([f, E, \Delta_1]) = -\phi(\hat{\sigma}_{ij}) = -\frac{(d-1)(d+1)m}{3d(m-1)}.$$

REFERENCES

[1] T. ARAKAWA AND T. ASHIKAGA, Local splitting families of hyperelliptic pencils, I, Tohoku Math. J. 53 (2001), no. 3, 369–394.

[2] T. ASHIKAGA AND H. ENDO, Various aspects of degenerate families of Riemann surfaces, Sugaku Expositions 19 (2006), no. 2, 171–196.

[3] T. ASHIKAGA AND K. KONNO, Global and local properties of pencils of algebraic curves, Adv. Stud. Pure Math. 36, Algebraic Geometry 2000 Azumino, 1–49, Math. Soc. Japan, Tokyo, 2002.

[4] M. F. ATIYAH, The logarithm of the Dedekind η-function, Math. Ann. 278 (1987), no. 1–4, 335–380.

[5] M. F. ATIYAH AND I. M. SINGER, The index of elliptic operators: III, Ann. of Math. (2) 87 (1968), 546–604.

[6] J. S. BIRMAN, Braids, links, and mapping class groups, Ann. of Math. Studies, No. 82, Princeton Univ. Press, Princeton, N.J., 1974.

[7] J. S. BIRMAN AND H. M. HILDEN, On isotopies of homeomorphisms of Riemann surfaces, Ann. of Math. (2) 97 (1973), 424–439.

[8] P. E. CONNER, Differentiable periodic maps. Second edition, Lecture Notes in Math. 738, Springer, Berlin, 1979.

[9] H. ENDO, Meyer's signature cocycle and hyperelliptic fibrations, Math. Ann. 316 (2000), no. 2, 237–257.

[10] M. FURUTA, Surface bundles and local signatures, Topological Studies around Riemann Surfaces (1999), 47–53. (Japanese)

[11] C. M. GORDON, On the G-signature theorem in dimension four, À la recherche de la topologie perdue, 159–180, Progr. Math. 62, Birkhäuser Boston, Boston, MA, 1986.

[12] F. HIRZEBRUCH, The signature of ramified coverings, Global Analysis, 253–265, Univ. Tokyo Press and Princeton Univ. Press, Tokyo and Princeton, 1969.

[13] F. HIRZEBRUCH AND D. ZAGIER, The Atiyah-Singer theorem and elementary number theory, Mathematics Lecture Series, No. 3, Publish or Perish, Inc., Boston, Mass., 1974.

[14] S. IIDA, Adiabatic limit of η-invariants and the Meyer function of genus two, Master's thesis, University of Tokyo, 2004.

[15] N. KAWAZUMI AND T. UEMURA, Riemann–Hurwitz formula for Morita–Mumford classes and surface symmetries, Kodai Math. J. 21 (1998), 372–380.

[16] Y. KUNO, The mapping class group and the Meyer function for plane curves, Math. Ann. 342 (2008), no. 4, 923–949.

[17] Y. KUNO, The Meyer functions for projective varieties and their application to local signatures for fibered 4-manifolds, Algebr. Geom. Topol. 11 (2011), 145–195.

[18] Y. MATSUMOTO, On 4-manifolds fibered by tori II, Proc. Japan Acad. 59 (1983), 100–103.

[19] Y. MATSUMOTO, Lefschetz fibrations of genus two—a topological approach, Topology and Teichmüller spaces (Katinkulta, 1995), 123–148, World Sci. Publ., River Edge, NJ, 1996.

[20] W. MEYER, Die Signatur von Flächenbündeln, Math. Ann. 201 (1973), no. 3, 239–264.
[21] T. MORIFUJI, On Meyer's function of hyperelliptic mapping class groups, J. Math. Soc. Japan 55 (2003), no. 1, 117–129.
[22] S. MORITA, Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I, Topology 28 (1989), no. 3, 305–323.
[23] T. NAKATA, On a local signature of Lefschetz singular fibers in hyperelliptic fibrations, Master's thesis, University of Tokyo, 2005. (Japanese)
[24] M. NAMBA AND M. TAKAI, Degenerating families of finite branched coverings, Osaka J. Math. 40 (2003), no. 1, 139–170.
[25] M. SATO, The abelianization of a symmetric mapping class group, Math. Proc. Cambridge Philos. Soc. 147 (2009), no. 2, 369–388.
[26] V. G. TURAEV, First symplectic Chern class and Maslov indices, J. Math. Sci. 37 (1987), no. 3, 1115–1127.
[27] K. UENO, Discriminants of curves of genus 2 and arithmetic surfaces, Algebraic geometry and commutative algebra, Vol. II, 749–770, Kinokuniya, Tokyo, 1988.
[28] K. YOSHIKAWA, A local signature for generic 1-parameter deformation germs of a complex curve, Algebraic Geometry and Topology of Degenerations, Coverings and Singularities (2000), 188–200. (Japanese)

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