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A Local Small Gain Theorem and Its Use for Robust Stability of Uncertain Feedback Volterra Systems

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Abstract

The requirement to evaluate a gain over the whole signal space is one of the restrictions in the well-known small gain theorem. Using the concepts of local gain and strict causality a local form of small gain theorem is proposed, which can be used to analyze input magnitude dependent stability problems of feedback nonlinear systems, such as a Volterra system. Since only finite order Volterra series can be handled in practice, an uncertainty model is derived to address the robustness issue of approximating a nonlinear system by a finite Volterra series in the context of closed-loop control. The local small gain theorem is then used to analyze the feedback properties of the uncertain Volterra system and a sufficient condition for robust stability is obtained.

1. Introduction

The small gain theorem plays a fundamental role in the analysis of nonlinear feedback systems using input-output notations. It was first proposed by Sandberg [1] and Zames [2], and comprehensively discussed in [3,4]. It has found wide applications in analysis where it is desired to show bounded-input bounded-output stability of a nonlinear feedback system, such as in adaptive control [5], nonlinear internal model control [6] and robust nonlinear control [7].

The small gain theorem, in its traditional form, has some restrictions on its application. One of the restrictions, as pointed out by Hill [8], is that the affine gain formulation can inhibit adoption of input-output stability methods, and so a generalization form of the small gain theorem was proposed there. Another restriction lies in that the gain defined as an operator norm over the entire input space may not exist; even if exists in theory, its calculation may be too difficult to carry out. Several chemical process control examples were given to reveal this restriction in [9], and a so called set gain was proposed (but not used for feedback analysis).

Volterra series are an important input-output representation of nonlinear dynamic systems [10, 11]. During the past decades, its representation properties have received intensive investigation (e.g. [12, 13]). For chemical process systems, in particular, strong industrial interest has been expressed recently [14]. Unfortunately, very little progress has been made on investigating the feedback properties of Volterra systems. One possible explanation of this situation might be that there were not appropriate tools available to systematically solve the problem. Indeed, the closed-loop stability problem of a Volterra system is intertwined with the convergence problem, and the latter is extremely difficult to analyze quantitatively in general.

In this paper, a novel form of the small gain theorem is given and then used to address the feedback properties of uncertain Volterra systems.

2. A Local Form of Small Gain Theorem

The basic formulation for a feedback system can be expressed in the functional form

$$\begin{aligned} y &= H(e) \\ e &= u - y \end{aligned} \tag{1}$$

where u denotes the external input, y the output and e the actual input to the operator H. u, y and e are functions of time t; usually they are defined for $t \in R_+$ (nonnegative real number) and take values in R. For simplicity of notation, the systems discussed hereafter are assumed to be single-input single-output systems.

Let P_T be the linear map $P_T : R_+ \to R_+$ such that with $f_T(t) \triangleq P_T f(t)$ we mean that,

$$f_T(t) = \begin{cases} f(t), t \leq T \\ 0, t > T \end{cases} \quad \forall t, T \in R_+ \tag{2}$$

An extended normed space L_{pe} is defined as

$$L_{pe} = \{f(t) | f_T(t) \in L_p,\ \forall T \geq 0\},\ \ 1 \leq p \leq \infty \tag{3}$$

where $L_p = \{f : [0, \infty) \to R \mid \int_0^\infty |f(t)|^p dt < \infty\}$ for $1 \leq p < \infty$ and $L_\infty = \{f : [0,\infty) \to R \,| ess \sup |f(t)| < \infty\}$. For simplicity of notation, we use L_e to represent an extended normed space for some value $1 \leq p \leq \infty$.

Let $H : L_e \to L_e$. H is said to be causal if and only if $P_T H P_T = P_T H$, $\forall T \in R_+$. It is said to be L-stable if and only if there exists a finite constant γ such that

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$$||(Hu)_T|| \leq \gamma||u_T||, \ \forall u \in L_e \ and \ T \in R_+ \quad (4)$$

If H is causal and L-stable, then $||Hu|| \leq \gamma||u||, \ \forall u \in L_p$.

The system gain of H, as defined in the small gain theorem, is

$$\gamma_H = \sup_{T\geq 0, u\neq 0} \frac{||(Hu)_T||}{||u_T||} \quad (5)$$

The small gain theorem states that if the system gain $\gamma_H < 1$, then the closed-loop system is L-stable with $||e_T|| \leq \frac{1}{1-\gamma_H}||u_T||$ and $||y_T|| \leq \frac{\gamma_H}{1-\gamma_H}||u_T||$.

The system gain γ_H as defined in (5) should be evaluated along all possible inputs $u \in L_e$. As such, there are two restrictions in the small gain theorem:

(a) Many systems of practical interest do not possess such system gains. For example, if we consider a nth-order Volterra operator, say $H = H_n$, $n > 1$, then we know from the property of Volterra operators that, for any $u \in L_e$ and any constant $c > 0$, the following equality holds

$$\frac{||(H(c \cdot u))_T||}{||(c \cdot u)_T||} = c^{n-1} \cdot \frac{||(Hu)_T||}{||u_T||} \quad (6)$$

Suppose there exist a system gain γ_n, then

$$||(Hu)_T|| \leq \gamma_n||u_T||, \ \forall u \in L_e \quad (7)$$

It must be true that

$$||(Hu)_T|| \leq c^{n-1} \cdot \gamma_n||u_T||, \ \forall u \in L_e \quad (8)$$

This suggests that $(c^{n-1} \cdot \gamma_n)$ is also a system gain. Since c can be any positive constant, the only possibility for this to be true is $\gamma_n = \infty$ or $\gamma_n = 0$ (this is a trivial case). Hence, this system does not have a system gain.

(b) Even if such a system gain exists, its determination is quite difficult for a general nonlinear system. In fact, a systematic method for evaluating a system gain using input-output notations is available only for linear and memoryless nonlinear systems [4].

The above suggests the necessity for deriving a local form of the small gain theorem, which only utilizes the system properties for those input signals that belong to some subset of L_e, not the complete set of L_e.

Let H be a causal map $H : L_e \longrightarrow L_e$. For a given constant $d > 0$, let S_d be a subset of L_e defined by $S_d = \{u(t) : ||u_T|| \leq d, \forall t, T \in R_+\}$.

Definition 1 *H is said to be locally bounded if there exists a finite constant $\gamma > 0$ such that $||(Hu)_T|| \leq \gamma||u_T||$, $\forall u_T \in S_d$ and $T \geq 0$. The lowest upper bound for γ is said to be a local gain of H with respect to S_d, denoted by γ_{Hd}, i.e.,*

$$\gamma_{Hd} = \sup_{T\geq 0, u\in S_d\setminus\{0\}} \frac{||(Hu)_T||}{||u_T||} \quad (9)$$

The significance in establishing the concept "local gain" lies in the followings: for many nonlinear system models only local gains exist; most system models are valid only for a limited magnitude of input signals; when the system model is given as a truncated Volterra series, an approximation of the local gain can be obtained as discussed later in this paper.

Definition 2 *The closed-loop system (1) is said to be locally stable if and only if there exist a positive constant M and finite positive constants k_1, k_2 such that*

$$||e_T|| \leq k_1||u_T|| \ and \ ||y_T|| \leq k_2||u_T||, \forall ||u_T|| \leq M, \ T \geq 0 . \quad (10)$$

Remark 1 *If the system is locally stable for any $M > 0$, then clearly, it is L-stable.*

In the traditional small gain theorem, the information used for the analysis of closed-loop stability is the gains of each elements along the loop. One can derive the theorem readily through some algebra, i.e., manipulations without being concerned with the well-posedness issue of the closed-loop solution. This owes to the fact that all signals and quantities along the loop are defined in normed linear spaces. But in the present definitions, they do not belong to linear spaces anymore. So the argument used in the derivation of the small gain theorem does not hold under present settings. For this reason, we need put some additional conditions on H.

Definition 3 *Let $H : L_e \longrightarrow L_e$. If for all $u_1, u_2 \in L_e$ with $P_T u_1(t) = P_T u_2(t)$, there exist constants $\beta < 1$ and $\sigma > 0$ such that*

$$||P_{T\sigma}(Hu_1 - Hu_2)(t)|| \leq \beta||P_{T\sigma}(u_1 - u_2)(t)||, \forall t, T \in R_+ \quad (11)$$

where $P_{T\sigma} \triangleq P_{T+\sigma} - P_T$, then, we say that H is strictly causal.

Some obvious examples of strictly causal systems are: linear systems with strictly proper transfer functions and systems with time delays.

Proposition 1 *If H is strictly causal, then it must be causal. But the reverse may not be true.*

Remark 2 *For strict causality, we only require that there exists some $\beta < 1$ such that the inequality (11) holds. But for strong causality [3], the same inequality should hold for all $\beta > 0$. Therefore, the strict causality defined here is a weaker condition than the strong causality. For example, if $Hu = k \cdot u$, $k < 1$, then H is strictly causal but not strongly causal. It should be mentioned that although not formally defined as "strict causality," the same condition had been used in [4] to address the well-posedness issue of feedback systems.*

Notice that the strict causality of H was defined on L_e, but the gain of H was defined on S_d. This is an important difference from the traditional scheme of stability analysis in the literature. Generally speaking, all physically realizable systems possess the property of strict causality for any inputs (a global property), but the gains may exist only for a limited magnitude of inputs (a local property). The concept of "physically realizable systems" is discussed systematically in [15].

Theorem 1 *Assume H is strictly causal and has a local gain $\gamma_{Hd} < 1$ with respect to some S_d. If the external input satisfies $\|u_T\|_\infty \le (1-\gamma_{Hd})d, \forall T \ge 0$, then the closed-loop system (1) is locally stable and has an unique solution for each u. Moreover,*

$$\begin{aligned}\|e_T\|_\infty &\le \frac{1}{1-\gamma_{Hd}}\|u_T\|_\infty \\ \|y_T\|_\infty &\le \frac{\gamma_{Hd}}{1-\gamma_{Hd}}\|u_T\|_\infty\end{aligned}$$

When H is a finite Volterra series model , we can derive the corresponding results about the strict causality and the local gain. Consider

$$H = H_1 + H_2 + \ldots + H_N \tag{12}$$

where, $H_n u(t) = \int_0^t \ldots \int_0^t h_n(\tau_1, \ldots, \tau_n)u(t-\tau_1)\ldots u(t-\tau_n)d\tau_1 d\ldots d\tau_n$. A sufficient condition for strict causality can be stated as follows.

(1) First order operator ($n = 1$): If there exists $\theta > 0$ such that $\int_0^\theta |h_1(\tau)|d\tau < 1$, then the operator is strictly causal;
(2) n-th order operator ($n \ge 2$): The n-th order operator is strictly causal if it satisfies (a) there exists $\theta > 0$ such that $|h(\tau_1, \ldots, \tau_n)| < \infty, \quad \forall \tau_i \in [0, \theta], i = 1, 2, \ldots n$; (b) $\int_0^\infty \cdots \int_0^\infty |h(\tau_1, \ldots, \tau_i, \tau_{i+1}, \ldots \tau_n)|d\tau_1 \ldots d\tau_i < \infty$, $i = 1, 2, \ldots, n-1$ and $\forall \tau_j \ge 0, j = i+1, \ldots, n$. Under these conditions, it can be shown that there exists $0 < \lambda < \infty$ such that the following holds,

$$\|P_{T\sigma}(Hu_1 - Hu_2)\| \le \lambda\sigma\|P_{T\sigma}(u_1 - u_2)\|, \ \forall \sigma \le \theta \tag{13}$$

An approximation of the local gain for a finite Volterra series can be obtained as follows. Define

$$|h_n| \triangleq \int_0^\infty \cdots \int_0^\infty |h_n(\tau_1, \ldots, \tau_n)|d\tau_1 \ldots d\tau_n \tag{14}$$

Assume that $|h_n| < \infty$ for all $n = 1, 2, \ldots, N$, and the input is bounded by $\|P_T u\|_\infty \le d$ for some constant d. Then

$$\|P_T H u\|_\infty \le \sum_{n=1}^{N} \|P_T H_n u\|_\infty \le (\sum_{n=1}^{N} |h_n| d^{n-1})\|P_T u\|_\infty \tag{15}$$

Let

$$\tilde{\gamma_d} = \sum_{n=1}^{N} |h_n| d^{n-1} \tag{16}$$

Then $\|P_T H u\|_\infty \le \tilde{\gamma_d}\|P_T u\|_\infty$. Hence, $\tilde{\gamma_d}$ is an upper bound of the local gain of H and can be used in its place to obtain the sufficient stability condition of a feedback Volterra system.

3. Uncertainty description of Volterra systems

One of advantages of a Volterra series model is its property of universal approximation to nonlinear dynamic systems [13]. The modeling uncertainty may come from different sources, including the mismatch in using an ad hoc system structure, external disturbances and insufficient information in identification. In this paper, we will only consider those systems which possess Volterra series representations and the mismatch in kernels and the truncation error are their exclusive uncertainty sources.

Consider a time-invariant, fading memory operator H. Let the set of input signals be

$$\begin{aligned}K = \{u \in C(R) | \ \|u\|_\infty \le M_1, \\ |u(t_1) - u(t_2)| \le M_2(t_1 - t_2), \forall t_1 \le t_2\}\end{aligned} \tag{17}$$

for some $M_1 > 0$ and $M_2 > 0$. It has been shown in [13] that, for any given $\epsilon > 0$, there exists a finite Volterra series operator, say $\hat{H}$ such that

$$\|Hu - \hat{H}u\|_\infty < \epsilon, \ \ \forall u \in K \tag{18}$$

Assume $H0 = 0$. $\hat{H}$ can be expressed as

$$\begin{aligned}\hat{H}u &= \sum_{n=1}^{M} \int_0^t \cdots \int_0^t h_n(\tau_1, \ldots, \tau_n)u(t-\tau_1)\ldots \\ &\quad u(t-\tau_n)d\tau_1 \ldots d\tau_n\end{aligned} \tag{19}$$

where $h_n \in L_1$ and $M < \infty$. Of course, M depends on ϵ. Generally speaking, M could be too large to handle for a quite small ϵ. So, in applications, we will use a truncated Volterra series $\tilde{H}$ with a smaller M, say N, to approximate the original operator, namely,

$$\begin{aligned}\tilde{H}u &= \sum_{n=1}^{N} \int_0^t \cdots \int_0^t h_n(\tau_1, \ldots, \tau_n)u(t-\tau_1)\ldots \\ &\quad u(t-\tau_n)d\tau_1 \ldots d\tau_n\end{aligned} \tag{20}$$

Then the truncation error can be analyzed as follows.

Since

$$\begin{aligned}& \|(Hu)_t - (\hat{H}u)_t + (\hat{H}u)_t - (\tilde{H}u)_t\|_\infty \\ \le \ & \|(Hu)_t - (\hat{H}u)_t\|_\infty + \|(\hat{H}u)_t - (\tilde{H}u)_t\|_\infty \\ \le \ & \epsilon + \| \sum_{n=N+1}^{M} \int_0^t \cdots \int_0^t h_n(\tau_1, \ldots, \tau_n)u(t-\tau_1)\ldots \\ & u(t-\tau_n)d\tau_1 \ldots d\tau_n\|_\infty \\ \le \ & \epsilon + \alpha \cdot \|u_t\|_\infty^{N+1}\end{aligned}$$

where, $(\cdot)_t$ means the truncation up to t, α is some finite number because of the assumption $h_n \in L_1$. From this we can see that there exists a time-varying, causal operator δ on K, which is bounded by

$$\|\delta\|_\infty = \sup_{t,u \in K} |\delta(u, t)| \le 1 \tag{21}$$

such that

$$(Hu)_t = (\tilde{H}u)_t + (\alpha \cdot \|u_t\|_\infty^{N+1} + \epsilon) \cdot \delta_t \tag{22}$$

The above discussion only covers the case where $\tilde{H}$ is exactly equal to the up to Nth order operators in $\hat{H}$, and hence the truncation error is the only source of uncertainty. Usually, $\hat{H}$ itself is unknown. In this case, $\tilde{h}_n$ may differ from $\hat{h}_n$. To establish an uncertainty model for both the kernel mismatch uncertainty and the truncation error uncertainty, we first make the following assumptions:

(a) the original system H has a local gain $\gamma_H < \infty$ for $\|u\|_\infty \le d$;
(b) the nominal model $\tilde{H}$ also has a local gain $\gamma_{\tilde{H}} < \infty$

for $||u||_\infty \leq d$. For this to be true, we may assume that $\tilde{h}_n \in L_1,\ n = 1, 2, \ldots, N$.

Under these assumptions, we have

$$||(Hu)_t||_\infty \leq \gamma_H ||u_t||_\infty \text{ and } ||(\tilde{H}u)_t||_\infty \leq \gamma_{\tilde{H}} ||u_t||_\infty \quad (23)$$

for all $u \in S_d$. Then

$$||(Hu)_t - (\tilde{H}u)_t||_\infty \leq (\gamma_H + \gamma_{\tilde{H}})||u_t||_\infty$$

This suggests that there exist a finite number α and a time-varying, causal operator δ, which is bounded by

$$||\delta||_\infty = \sup_{t,u \in S_d} |\delta(u,t)| \leq 1 \quad (24)$$

such that

$$(Hu)_t = (\tilde{H}u)_t + \alpha \cdot ||u_t||_\infty \cdot \delta_t \quad (25)$$

Hence, we can define the uncertainty model R as

$$(R(u,\tau))_t = \alpha \cdot ||u_t||_\infty \cdot \delta_t,\ \forall u \in S_d \quad (26)$$

and the uncertainty family Π as

$$\Pi = \{H : H = \tilde{H} + R, \forall \delta \in \Delta\} \quad (27)$$

where, Δ is a family of time-varying, causal operators which are known only to satisfy the given norm bound

$$\Delta = \{\delta : \sup_{t,u \in S_d} |\delta(u,t)| \leq 1\} \quad (28)$$

and α will be called the uncertainty coefficient. Such an uncertainty family Π may be too general to have practical significance. This is because a loose description of uncertainty will produce a very conservative result in the robustness analysis. To tighten the "size" of the family Π, we will put an additional restriction on R.

Recall that almost all physically realizable systems are strictly causal and this is of crucial importance to the well-posedness of a feedback system. To assure a closed-loop system, which has an uncertainty model in the loop, having appropriate properties, we require R to satisfy the following condition: for any u_1, u_2 with $P_T u_1 = P_T u_2$, there exist finite numbers $\beta > 0$ and $\sigma > 0$ such that

$$||P_{T\sigma} R(u_1,t) - P_{T\sigma} R(u_2,t)||_\infty \leq \beta ||P_{T\sigma} u_1 - P_{T\sigma} u_2||_\infty \quad (29)$$

for all T. Accordingly, we use $\tilde{\Pi}$ to represent such an uncertainty family. It is clear that $\Pi \in \tilde{\Pi}$. To see if or not such a restriction on R is reasonable, we will consider two examples as follows.

Example 1 Let $H(s)$ be any stable and proper transfer function. If the nominal model $\tilde{H}(s)$ is also stable and proper, one can show that $(H - \tilde{H})$ satisfies (29), and hence $H \in \tilde{\Pi}$; if $\tilde{H}(s)$ is improper, then H not in $\tilde{\Pi}$.

Example 2 Let $H : L_e \to L_e$ and $\tilde{H} : L_e \to L_e$. If for all $u_1, u_2 \in L_e$ with $P_T u_1 = P_T u_2$ there exist finite numbers β_1, β_2 and σ_1, σ_2, such that

$$||P_{T\sigma_i}(Hu_1 - Hu_2)||_\infty \quad \leq \quad \beta_i ||P_{T\sigma_i}(u_1 - u_2||_\infty$$

for all T, $i = 1, 2$, then one can show that $(H - \tilde{H})$ satisfies (29), and hence $H \in \tilde{\Pi}$.

From these examples we can see that $\tilde{\Pi}$ contains a quite general class of uncertain systems.

4. Robustness analysis of uncertain feedback Volterra systems

We will investigate the nonlinear control problem by using an Internal Model Control structure (IMC). There are several advantages in using an IMC over classic feedback control[6, 16].

Consider an additive uncertainty family $\tilde{\Pi} = \{H : Hu(t) = \tilde{H}u(t) + (R(u,\tau))_t,\ u \in S_d\}$, where, the nominal model $\tilde{H}$ is assumed to be a time-invariant, stable and causal operator, and the unstructured uncertainty R is defined by (26) and (29). An IMC-based control system takes the form of figure 1. The disturbance z is assumed to be independent of the dynamics of the plant and hence, those disturbances which are not independent of the dynamics are assumed having been included in $\tilde{\Pi}$. In the diagram, $C_{\tilde{H}}$ is the nominal compensator which is designed based on $\tilde{H}$, and C_f is the generalized filter. As long as closed-loop stability is concerned, the above IMC structure can be rearranged as figure 2.

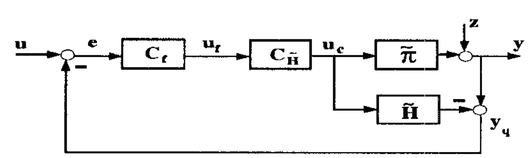

Figure 1: The IMC-based control system structure

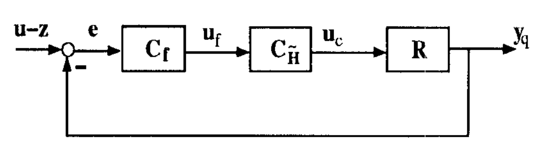

Figure 2: The closed-loop equivalent system for robust stability analysis

Assume that $C_{\tilde{H}}$ has a local gain γ_{CM} for $||u_f||_\infty \leq M$ and $\gamma_{CM} \cdot M \leq d$, which is needed to guarantee $u_c \in S_d$, and C_f has a system gain γ_f (not only a local gain). From the local small gain theorem, a sufficient condition for the closed-loop robust stability can be obtained as follows.

Theorem 2 *The IMC system is robustly stable if*
(a) $C_{\tilde{H}} C_f$ is strictly causal with the property $\lim_{\sigma \to 0} \beta = 0$;
(b) $\gamma_f \gamma_{CM} < \alpha^{-1}$;
(c) $||u - z||_\infty \leq M(\gamma_f^{-1} - \alpha\gamma_{CM})$.

Proof Since $u_c \in S_d$ is assumed by $\gamma_{CM} M \leq d$, what remains to be examined is under what condition the inequality $||u_f||_\infty \leq M$ holds true. From the local small gain theorem, a sufficient condition for this to be true is,
(a') $RC_{\tilde{H}} C_f$ be strictly causal;
(b') $\gamma_R \gamma_{CM} \gamma_f < 1$;

(c') $||u-z||_\infty \leq M(\gamma_f^{-1} - \gamma_R\gamma_{CM})$.
Since

$$\begin{aligned}\gamma_R &= \sup_{t,u\in S_d} \frac{||(R(u,t))_t||_\infty}{||u_t||_\infty} \\ &= \sup_{t,u\in S_d} \frac{\alpha||u_t||_\infty ||(\Delta(u,\tau))_t||_\infty}{||u_t||_\infty} \\ &\leq \alpha \end{aligned} \quad (30)$$

a sufficient condition for (b') and (c') to be true is that (b) and (c) hold true. Since R satisfies inequality (29), it can be shown that $RC_{\bar{H}}C_f$ is strictly causal if $C_{\bar{H}}C_f$ is strictly causal with the property $\lim_{\sigma\to 0}\beta = 0$. □

Remark 3 *(a) The requirement " $C_{\bar{H}}C_f$ is strictly causal with the property* $\lim_{\sigma\to 0}\beta = 0$*" is, in fact, quite weak. This is because C_f usually acts as a low-pass filter. For example, if C_f is a linear stable operator and has a strictly proper transfer function, then $C_{\bar{H}}C_f$ will satisfy the requirement; (b) Since we have assumed that $C_{\bar{H}}$ and C_f have local gains, the case where there is a pure integration element along the forward path will not satisfy this sufficient condition.*

5. Concluding Remarks

Using the concept of local gains and strict causality, a local form of small gain theorem has been derived. The proposed method can be used to address the input magnitude dependent stability issue, which is unavailable in traditional functional analysis methods. This new result is useful in analyzing the properties of feedback Volterra systems and the stability of inverse Volterra series [17].

Uncertainty modeling for a nonlinear system is a very challenging issue. In the case where a finite Volterra series is used as a nominal model for a nonlinear system, two different uncertainty models have been derived, one for the truncation error and the another for both the truncation error and the kernel mismatch. Both models can be estimated via identification procedures. A robust stability condition has been established through the use of the local small gain theorem. The work is aiming at establishing a control-relevant identification method by using these results.

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