

A Local-to-Global Theorem for Congested Shortest Paths

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Abstract

Amiri and Wargalla proved the following local-to-global theorem about shortest paths in directed acyclic graphs (DAGs): if G is a weighted DAG with the property that for each subset S of 3 nodes there is a shortest path containing every node in S , then there exists a pair (s, t) of nodes such that there is a shortest st -path containing every node in G . We extend this theorem to general graphs. For undirected graphs, we prove that the same theorem holds (up to a difference in the constant 3). For directed graphs, we provide a counterexample to the theorem (for any constant). However, we prove a *roundtrip* analogue of the theorem which guarantees there exists a pair (s, t) of nodes such that every node in G is contained in the union of a shortest st -path and a shortest ts -path.

The original local-to-global theorem for DAGs has an application to the k -Shortest Paths with Congestion c ((k, c) -SPC) problem. In this problem, we are given a weighted graph G , together with k node pairs $(s_1, t_1), \dots, (s_k, t_k)$, and a positive integer $c \leq k$, and tasked with finding a collection of paths P_1, \dots, P_k such that each P_i is a shortest path from s_i to t_i , and every node in the graph is on at most c paths P_i , or reporting that no such collection of paths exists. When $c = k$, there are no congestion constraints, and the problem can be solved easily by running a shortest path algorithm for each pair (s_i, t_i) independently. At the other extreme, when $c = 1$, the (k, c) -SPC problem is equivalent to the k -Disjoint Shortest Paths (k -DSP) problem, where the collection of shortest paths must be node-disjoint. For fixed k , k -DSP is polynomial-time solvable on DAGs and undirected graphs. Amiri and Wargalla interpolated between these two extreme values of c , to obtain an algorithm for (k, c) -SPC on DAGs that runs in polynomial time when $k - c$ is constant.

In the same way, we prove that (k, c) -SPC can be solved in polynomial time on undirected graphs whenever $k - c$ is constant. For directed graphs, because of our counterexample to the original theorem statement, our roundtrip local-to-global result does not imply such an algorithm (k, c) -SPC. Even without an algorithmic application, our proof for directed graphs may be of broader interest because it characterizes intriguing structural properties of shortest paths in directed graphs.

*MIT CSAIL, naysh@mit.edu. Supported in part by NSF grant CCF-2129139.

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1 Introduction

An intriguing question in graph theory and algorithms is: “can we understand the structure of shortest paths in (directed and undirected) graphs?” Or more specifically: “can we understand the structure of the *interactions* between shortest paths in graphs?” This question has been approached from various angles in the literature. For instance, Bodwin [Bod19] characterizes which sets of nodes can be realized as *unique* shortest paths in weighted graphs, and Cizma and Linial investigate the properties of graphs whose shortest paths satisfy or violate certain geometric properties [CL22a, CL22b]. Additionally, there is a large body of work on distance preservers, where the goal is to construct a subgraph that preserves distances in the original graph (see [Bod17] and the references therein). There are also numerous computational tasks in which structural results about shortest paths inform the creation and analysis of algorithms, including distance oracle construction [CZ22], the k -disjoint shortest paths problem, [BNRZ21], and the next-to-shortest path problem [Wu10] (the cited papers are the most recent publications in their respective topics).

The angle of our work is inspired by the following local-to-global theorem¹ of Amiri and Wargalla [AW20] concerning the structure of shortest paths in directed acyclic graphs (DAGs):

Theorem 1.1 ([AW20]). *Let G be a weighted DAG with the property that, for each set S of 3 nodes, there is a shortest path containing every node in S . Then, there exists a pair (s, t) of nodes such that there is a shortest st -path containing every node in G .*

Note that in the statement of Theorem 1.1, the condition that “for each set S of 3 nodes, there is a shortest path containing every node in S ” is equivalent to the condition that one of the 3 nodes is on a shortest path between the other two.

Theorem 1.1 is “local-to-global” in the sense that from a highly congested local structure (every small subset of nodes is contained in a shortest path) we deduce a global structure (all nodes in the graph live on a single shortest path).

At first glance, Theorem 1.1 may appear rather specialized, since the existence of shortest paths through *all* triples of nodes is a rather strong condition. However, Amiri and Wargalla [AW20] show that Theorem 1.1 has applications to the k -Shortest Paths with Congestion c ((k, c) -SPC) on problem on DAGs: in this problem we are given a DAG, and are tasked with finding a collection of shortest paths between k given source/sink pairs, such that each node in the graph is on at most c of the paths. We discuss this application in detail in Section 1.2.

Amiri and Wargalla [AW20] raised the question of whether their results can be extended to general graphs, both undirected and directed. Our work answers this question.

1.1 Structural Results

We ask the following question:

Is Theorem 1.1 true for general (undirected and directed) graphs?

Our first result answers this question affirmatively for undirected graphs (with constant 4 instead of 3).

Theorem 1.2 (Undirected graphs). *Let G be a weighted undirected graph with the property that, for each set S of 4 nodes, there is a shortest path containing every node in S . Then, there exists a pair (s, t) of nodes such that some shortest st -path contains every node in G .*

¹This is slightly different from the statement of the theorem in [AW20], where the authors write present their result in the context of the Shortest Paths with Congestion problem, which we discuss in detail later.

The constant 4 in the statement of Theorem 1.2 cannot be replaced with 3, as seen by considering an undirected cycle on four vertices.

Theorem 1.2 implies a faster algorithm for (k, c) -SPC on *undirected* graphs, answering an open question raised in [AW20, Section 4]. We discuss the background of this problem and our improvement in Section 1.2.

Our second result is for directed graphs. First, we observe that there is actually a *counterexample* to Theorem 1.1 for general directed graphs: let a be the constant for which we desire a counterexample (i.e., the constant that is 3 in Theorem 1.1, and 4 in Theorem 1.2). Let G be a cycle with bidirectional edges, where all clockwise-pointing edges have weight 1 and all counterclockwise-pointing edges have weight a . One can verify that the precondition of the theorem holds: for each set S of a nodes there is a shortest path containing every node in S , just by taking the shortest clockwise path through all nodes in S . However, no single shortest path contains every node in G , so the theorem does not hold.

Even though a direct attempt at generalizing Theorem 1.1 to all directed graphs fails, one might hope for some analogue of Theorem 1.1 that does hold. One interpretation of the above counterexample is that the exact statement of Theorem 1.1 is not the “right” framework for getting a local-to-global shortest path phenomenon in general directed graphs. To that end, we consider the *roundtrip* analogue of Theorem 1.1, where the final path through every node is a shortest roundtrip path, i.e., the union of a shortest st -path and a shortest ts -path (roundtrip distances are a common object of study in directed graphs, with there being much research, for example, in roundtrip routing [CW99], roundtrip spanners [RTZ08], and roundtrip diameter computation [AWW16]). Note that the above counterexample does not apply to the roundtrip analogue of Theorem 1.1 since there exists a pair (s, t) of nodes such that the union of a shortest st -path and a shortest ts -path are both in the clockwise direction and thus contain all nodes in the graph.

For our second result, we present a roundtrip analogue of Theorem 1.1 which holds true for general directed graphs (with the constant 11 instead of 3):

Theorem 1.3 (Directed graphs). *Let G be a weighted directed graph with the property that, for each set S of 11 nodes, there is a shortest path containing every node in S . Then, there exists a pair (s, t) of nodes such that the union of a shortest st -path and a shortest ts -path contains every node in G .*

Proving Theorem 1.3 requires overcoming a number of technical challenges involving the complex structure of shortest paths in directed graphs. Due to its roundtrip nature, unlike Theorem 1.2, Theorem 1.3 does not appear to have any immediate algorithmic applications.

1.2 Disjoint and Congested Shortest Path Problems

In this section we introduce the k -Shortest Paths with Congestion c ((k, c) -SPC) problem and state the implications of our work for this problem.

1.2.1 Background

We begin by discussing the related k -Disjoint Shortest Paths (k -DSP) problem. For more related work on disjoint path problems in general, see Appendix A.

Disjoint Shortest Paths Formally, the k -Disjoint Shortest Paths (k -DSP) problem is defined as follows:

k -Disjoint Shortest Paths (k -DSP): Given a graph G and k node pairs $(s_1, t_1), \dots, (s_k, t_k)$, find a collection of node-disjoint paths P_1, \dots, P_k such that each P_i is a shortest path from s_i to t_i , or report that no such collection of paths exists.

The k -DSP problem was introduced in the 90s by Eilam-Tzoreff [ET98], who gave a polynomial-time algorithm for undirected graphs when $k = 2$, and conjectured that there is a polynomial-time algorithm for any fixed k in both undirected and directed graphs. Recently, Lochet [Loc21] proved Eilam-Tzoreff’s conjecture for undirected graphs by showing that k -DSP can be solved in polynomial time for any fixed k . Subsequently, the dependence on k in the running time was improved by Bentert, Nichterlein, Renken, and Zschoche [BNRZ21]. It is known that k -DSP on undirected graphs is W[1]-hard [BNRZ21, Proposition 36], so this problem is unlikely to be fixed-parameter tractable.

For directed graphs, Bérczi and Kobayashi [BK17] showed that 2-DSP can be solved in polynomial time. For $k \geq 3$ however, determining the complexity of k -DSP on directed graphs remains a major open problem. This problem is only known to be polynomial-time solvable for special classes of directed graphs, such as DAGs and planar graphs [BK17].

Shortest Paths with Congestion The k -Shortest Paths with Congestion c ((k, c) -SPC) problem is the variant of k -DSP where some amount of *congestion* (paths overlapping at nodes) is allowed. In general, problems of finding paths with limited congestion are well-studied in both theory and practice. For instance, there is much work on the problem in undirected graphs where the goal is to find a maximum cardinality subset of node pairs (s_i, t_i) that admit (not necessarily shortest) paths with congestion at most c [RT87, KS98, BS00, AR06, AZ07, CKS09, And10, KK11, CE13, Chu16]. As another example, [KKK14] provides, for fixed k , a polynomial-time algorithm for the problem on directed graphs of determining that either there is no set of disjoint paths between the node pairs (s_i, t_i) , or finding a set of such paths with congestion at most 4. Another example for directed graphs is the problem of finding paths between the node pairs (s_i, t_i) where only some nodes in the graph have a congestion constraint [LS22].

Formally, the k -Shortest Paths with Congestion c ((k, c) -SPC) problem is defined as follows:

k -Shortest Paths with Congestion c ((k, c) -SPC): Given a graph G , along with k node pairs $(s_1, t_1), \dots, (s_k, t_k)$, and a positive integer $c \leq k$, find a collection of paths P_1, \dots, P_k such that each P_i is a shortest path from s_i to t_i , and every node in $V(G)$ is on at most c paths P_i , or report that no such collection of paths exists.

The (k, c) -SPC problem was introduced by Amiri and Wargalla [AW20]. Before that, the version of (k, c) -SPC where the paths are not restricted to be shortest paths was studied by Amiri, Kreuzer, Marx, and Rabinovich [AKMR19].

When $c = 1$, the (k, c) -SPC problem is equivalent to the k -DSP problem. At the other extreme, when $c = k$, there are no congestion constraints, so the problem can be easily solved in polynomial time by simply finding a shortest path for each pair (s_i, t_i) independently. Amiri and Wargalla [AW20] asked the following question: *can we interpolate between these two extremes?* In particular, can we get algorithms for (k, c) -SPC where the exponential dependence on k for k -DSP can be replaced with some dependence on $O(k - c)$ instead?

Amiri and Wargalla [AW20] achieved this goal for DAGs. In particular, they gave a reduction from (k, c) -SPC on DAGs to k -DSP on DAGs of the following form: letting $d = k - c$, if k -DSP on DAGs can be solved in time $f(n, k)$, then (k, c) -SPC on DAGs can be solved in time $O\left(\binom{k}{3d} \cdot f(2dn, 3d)\right)$. The essential aspect of this running time is that the second input to the function f is not k but rather an $O(d)$ term. A key tool in their reduction is Theorem 1.1 (stated in a different way).

Since k -DSP can be solved in $n^{O(k)}$ time on DAGs [BK17], the above result implies that (k, c) -SPC can be solved in $\binom{k}{3d} \cdot (2dn)^{O(d)}$ time on DAGs. That is, (k, c) -SPC on DAGs is polynomial-time solvable for arbitrary k whenever d is constant. We note that for every c , the (k, c) -SPC problem on DAGs is W[1]-hard with respect to d , so the problem is unlikely to be fixed-parameter tractable with respect to d [AW20, Proof of Theorem 3].

1.2.2 Algorithmic Results

Similarly to Amiri and Wargalla’s result for DAGs, our result for undirected graphs, Theorem 1.2, implies a reduction from (k, c) -SPC to k -DSP on undirected graphs.

Lemma 1.4. *If k -DSP can be solved in $f(n, k)$ time on undirected graphs, then (k, c) -SPC can be solved in $O\left(\binom{k}{4d} \cdot f(3dn, 4d)\right)$ time on undirected graphs.*

Lemma 1.4 follows from Theorem 1.2 using an argument nearly identical to the one presented for DAGs in [AW20] (up to a difference in constants). For completeness, we include a full proof of this result in Appendix C.

Since it is known that k -DSP can be solved in undirected graphs in time $n^{O(k \cdot k!)}$ [BNRZ21], applying Theorem 1.2 together with Lemma 1.4, we deduce the following result.

Corollary 1.5. *(k, c) -SPC on undirected graphs can be solved in $\binom{k}{4d} \cdot (3dn)^{O(d \cdot (4d)!)}$ time.*

Thus (k, c) -SPC on undirected graphs is in polynomial time whenever $d = k - c$ is constant; that is, it is in the complexity class XP with respect to the parameter d . Prior to our work, no polynomial-time algorithm for this problem appears to have been known in this regime, even for simple cases such as $(k, k - 1)$ -SPC on undirected graphs.

In contrast, our structural result for directed graphs, Theorem 1.3, does not imply a faster algorithm for (k, c) -SPC in directed graphs. This is because Theorem 1.3 does not appear to imply a reduction from (k, c) -SPC to k -DSP in the manner of Lemma 1.4. Moreover, even if such a reduction did exist, this would not imply an algorithm for directed graphs analogous to Corollary 1.5. This is because while k -DSP is polynomial-time solvable for constant k in undirected graphs, it remains open whether even 3-DSP over directed graphs can be solved in polynomial time.

1.3 The Structure of Shortest Paths in Directed Graphs

Although our result for directed graphs, Theorem 1.3, does not appear to have immediate algorithmic applications, we believe it is still interesting from a graph theoretic perspective, especially in light of the current scarcity of results for shortest disjoint path problems in directed graphs. In this section, we expand upon this idea with some remarks, and then state a lemma from our proof concerning the structure of shortest paths in directed graphs.

We currently have a poor understanding of the complexity of the k -DSP problem in directed graphs: for fixed $k \geq 3$, it is still not known if this problem is either polynomial-time solvable or NP-hard. In fact, even the complexity of the seemingly easier $(3, 2)$ -SPC problem on directed graphs is open. In this context, Theorem 1.3 is compelling because it presents an example of interesting behavior which holds for collections of shortest paths in DAGs, and then continues to hold, under suitable generalization, for systems of shortest paths in general directed graphs. This sort of characterization appears to be rare in the literature.

More generally, the methods we use to establish Theorem 1.3 involve combinatorial observations about the structure of shortest paths in directed graphs, and the interactions between them. We believe our analysis could offer more insight into resolving other problems that concern systems of shortest paths in directed graphs. There are many such problems, where the undirected case is well-understood, but in the directed case not much is known. This barrier is in-part due to the relatively complex patterns of shortest paths which can appear in directed graphs. We hope that our analysis of directed shortest paths may shed light on problems for which the structural complexity of directed shortest paths is the bottleneck towards progress.

One example of a problem where the undirected case is well-understood while the directed case remains poorly understood is, as discussed previously, the k -DSP problem. Another curious example

is the **Not-Shortest Path** problem, where the goal is to find an st -path that is not a shortest path. Although **Not-Shortest Path** can be solved over undirected graphs in polynomial time [KN04], no polynomial time algorithm is known for this problem in directed graphs. A third example is the problem of approximating the diameter of a graph. For undirected graphs there is an infinite hierarchy of algorithms that trade off between time and accuracy [CGR16], while only two points on the hierarchy are known for directed graphs. Additionally, the roundtrip variant of diameter is the least understood of any studied variant of the diameter problem [RW19]. Another example is the construction of approximate hopsets: for directed graphs there is there a polynomial gap between upper and lower bounds [HP21, BW22, BH23], while for undirected graphs the gap is subpolynomial [EN19, HP19]. The preponderance of such examples motivates proving results like Theorem 1.3, which characterize interesting behavior of shortest paths in directed graphs.

Structural Lemma for Directed Shortest Paths One of the lemmas we establish on the way to proving Theorem 1.3 is a general statement about the structure of shortest paths in directed graphs. It can be stated independently of the context of the proof of Theorem 1.3 and we highlight it here.

We categorize any shortest path P into one of two main types, based on the ways that other shortest paths intersect with it. The following simple definition will be useful for defining our path types.

Definition 1.6. For any nonnegative integer ℓ and set of ℓ nodes, v_1, v_2, \dots, v_ℓ , we say that the order $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_\ell$ is a *shortest-path ordering* if there is a shortest path containing all of the nodes v_1, v_2, \dots, v_ℓ in that order.

In addition to our two main path types, there is a third path type which we call a *trivial path* because it is easy to handle:

Definition 1.7 (Trivial Path). Given a directed graph, nodes a and b , and a shortest path P from a to b , we say P is a *trivial path* if P contains at least one node w such that $a \rightarrow w \rightarrow b$ is the *only* shortest-path ordering of a, w, b .

Now we are ready to state our two main types of shortest paths. The first type is a *reversing path*:

Definition 1.8 (Reversing path). Given a directed graph, nodes a and b , and a non-trivial shortest path P from a to b , P is *reversing* if P contains at least one node w such that w falls on some shortest path from b to a . A *non-reversing path* is a non-trivial path that is not reversing.

We prove a lemma that characterizes the structure of reversing and non-reversing paths in terms of the possible shortest-path orderings of each node on the path and the endpoints of the path. See Figures 1a and 1b for a depiction of the structure enforced by the lemma.

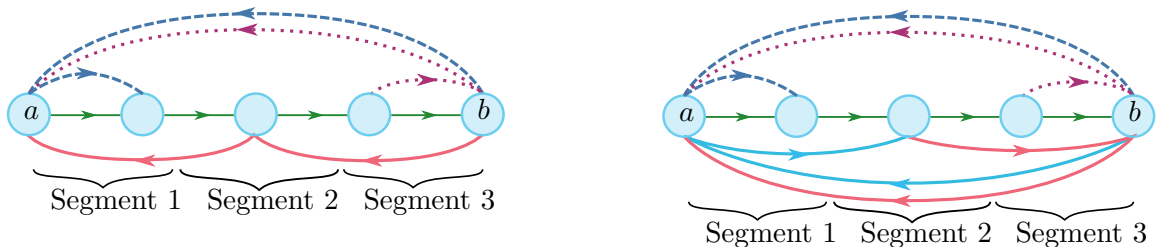
Lemma 1.9 (Reversing/Non-Reversing Lemma). *Let P be a non-trivial shortest path and let a and b be the first and last nodes of P respectively. Then P can be partitioned into three contiguous ordered segments with the following properties (where a is defined to be in Segment 1, b is defined to be in Segment 3, and Segment 2 could be empty).*

Segment 1 consists of nodes w such that the shortest-path orderings of a, w, b are precisely $a \rightarrow w \rightarrow b$, and $b \rightarrow a \rightarrow w$.

Segment 2 consists of nodes w such that the shortest-path orderings of a, w, b are precisely

$$\begin{cases} a \rightarrow w \rightarrow b, \text{ and } b \rightarrow w \rightarrow a & \text{if } P \text{ is a reversing path} \\ a \rightarrow w \rightarrow b, \text{ and } b \rightarrow a \rightarrow w, \text{ and } w \rightarrow b \rightarrow a & \text{if } P \text{ is a non-reversing path.} \end{cases}$$

Segment 3 consists nodes w such that the shortest-path orderings of a, w, b are precisely $a \rightarrow w \rightarrow b$, and $w \rightarrow b \rightarrow a$.



(a) **Reversing Path.** The structure of a reversing path, as given by Lemma 1.9. The blue dashed path, pink solid path, and purple dotted path are examples of the allowed orderings for nodes in segments 1, 2, and 3 respectively.

(b) **Non-Reversing Path.** The structure of a non-reversing path, as given by Lemma 1.9. The possible shortest-path orderings for nodes in segment 2 are represented by pink and light blue solid paths, while the orderings allowed for nodes in segments 1 and 3 are represented by blue dashed and purple dotted segments respectively.

Figure 1: Possible orderings of vertices on shortest paths in the reversing and non-reversing cases. The blue circles are representative examples of the types of nodes on the path from a to b (in general this path will contain more than just five nodes).

In the proof of Theorem 1.3 we employ the strategy of categorizing shortest paths as reversing or non-reversing (or trivial), and applying Lemma 1.9 to glean some structure. We note, however, that Lemma 1.9 itself is not the main technical piece of the proof.

2 Preliminaries

All graphs are assumed to have positive edge weights. Graphs are either undirected or directed, depending on the section. For any pair of nodes (u, v) , we use $\text{dist}(u, v)$ to denote the shortest path distance from u to v . Given a path P and two nodes u and v occurring on P in that order, we let $P[u, v]$ denote the subpath of P with u and v as endpoints.

For the (k, c) -SPC problem, we always let d denote the difference $d = k - c$. When considering a particular solution to a (k, c) -SPC instance, we refer to the paths P_1, \dots, P_k between $(s_1, t_1), \dots, (s_k, t_k)$ respectively as *solution paths*. Any node in the graph which lies in c of the solution paths is referred to as a *max-congestion* node.

2.1 Subpath Swapping

In our proofs, we will frequently modify collections of shortest paths by “swapping subpaths” between intersecting paths. This procedure is depicted in Figure 2, and we formally describe it below.

Definition 2.1 (Subpath Swap). Let \mathcal{R} be a collection of shortest paths in a directed graph. Let P and Q be two paths in \mathcal{R} . Let $a, b \in P \cap Q$ be nodes in these paths, such that a appears before b in both P and Q . We define *swapping* the subpaths of P and Q between a and b to be updating the set of paths \mathcal{R} by simultaneously replacing P with $(P \setminus P[a, b]) \cup Q[a, b]$ and Q with $(Q \setminus Q[a, b]) \cup P[a, b]$. We often refer to this process simply as “swapping $P[a, b]$ and $Q[a, b]$.”

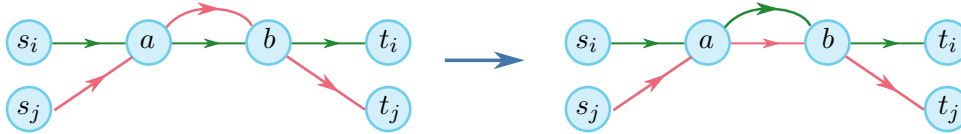


Figure 2: A simple subpath swap, where the subpaths from a to b of the green path (from s_i to t_i) and pink path (from s_j to t_j) are switched.

Observation 2.2 (Subpath Swap). *Let \mathcal{R} be the solution to some (k, c) -SPC problem. Then swapping subpaths in \mathcal{R} produces a new solution to the same (k, c) -SPC instance with the same set of max-congestion nodes.*

Proof. This observation holds because swapping subpaths does not change the number of solution paths any given node is contained in, does not change the endpoints of any solution path, and all solution paths remain shortest paths. ■

2.2 Correspondence Between Our Results and (k, c) -SPC

In this section we detail some nuances regarding the correspondence between the statement of our results and the (k, c) -SPC problem. For the sake of generality and simplicity, we stated Theorems 1.1-1.3 independently of the (k, c) -SPC problem. In contrast, the original result of Amiri and Wargalla, corresponding to Theorem 1.1, was stated as follows:

Lemma 2.3 ([AW20]). *If $k > 3d$, then any instance of (k, c) -SPC on DAGs either has no solution, or has a solution where some solution path P_i passes through all max-congestion nodes.*

Although the statement of Lemma 2.3 may initially seem unrelated to the statement of Theorem 1.1, their correspondence becomes clearer with the following observation, which is a simple generalization of an observation from [AW20]:

Observation 2.4. *Let N be a positive integer. Suppose $k > Nd$, and let \mathcal{R} be a solution to an arbitrary (k, c) -SPC instance. Then for any set S of N max-congestion nodes in \mathcal{R} , there exists some solution path in \mathcal{R} which contains every node in S .*

Proof. By assumption, each max-congestion node lies on exactly $c = k - d$ of the solution paths of \mathcal{R} , and does *not* lie on exactly d of these solution paths. This means that the total number of solution paths of \mathcal{R} avoided by at least one node in S is $|S|d = Nd < k$. Consequently there exists at least one solution path passing through every node in S . ■

To prove our results for the (k, c) -SPC problem in undirected graphs (Lemma 1.4 and Corollary 1.5), we need to prove a lemma analogous to Lemma 2.3 but for undirected graphs:

Lemma 2.5. *If $k > 4d$, then any instance of (k, c) -SPC either has no solution, or has a solution where some solution path P_i passes through all max-congestion nodes.*

The following generalizes Theorem 1.2 and Lemma 2.5.

Lemma 2.6 (General Undirected Result). *Given an undirected graph and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

$$\text{for every set } S \subseteq W \text{ of 4 nodes, some path in } \mathcal{R} \text{ contains every node in } S. \quad (\star)$$

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of shortest paths that still has property (\star) . Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which some path P passes through all nodes in W .

We note that the quantity $O(n^3)$ is an unimportant technicality used in the proof of the following observation, and is chosen as a loose upper bound on the number of subpath swaps we will perform.

Observation 2.7. *Lemma 2.6 generalizes both Theorem 1.2 and Lemma 2.5.*

Proof. We begin by showing that Lemma 2.6 generalizes Theorem 1.2. In the statement of Lemma 2.6, let W be the entire vertex set. To see that the precondition of Theorem 1.2 meets the precondition of Lemma 2.6, let \mathcal{R} be a multiset of shortest paths containing the (perhaps exponentially large) set of all shortest paths in the graph each with multiplicity Cn^3 for a sufficiently large constant C . Then, after any sequence of $O(n^3)$ subpath swaps, \mathcal{R} still contains at least one copy of every shortest path in the graph, so property (\star) still holds. Finally, it is clear that the conclusion of Lemma 2.6 implies the conclusion of Theorem 1.2.

Now we show that Lemma 2.6 generalizes Lemma 2.5. In the statement of Lemma 2.6, let \mathcal{R} be a solution (if one exists) to the (k, c) -SPC instance from the statement of Lemma 2.5 and let W be the set of max-congestion nodes. To see that the precondition of Lemma 2.5 meets the precondition of Lemma 2.6, notice that by Observation 2.2, after any sequence of subpaths swaps \mathcal{R} is still a solution to the (k, c) -SPC instance with the same set of max-congestion nodes. Thus, by Observation 2.4 property (\star) holds after any sequence of subpath swaps applied to \mathcal{R} . Finally, it is clear that the conclusion of Lemma 2.6 implies the conclusion of Lemma 2.5. \blacksquare

We also prove an analogue of Lemma 2.3 and Lemma 2.6 for directed graphs:

Lemma 2.8 (General Directed Result). *Given a directed graph and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

for every set $S \subseteq W$ of 11 nodes, some path in \mathcal{R} contains every node in S . (†)

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of paths that still has property (†). Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which either:

- (i) *some path P passes through all nodes in W , or*
- (ii) *the union of two paths P, P' contain all nodes in W , and the first and last nodes on P from W are the same as the last and first nodes on P' from W , respectively.*

Lemma 2.8 generalizes Theorem 1.3, in exactly the same way as Lemma 2.6 generalizes Theorem 1.2 for undirected graphs. In the same way that Lemma 2.6 generalizes Lemma 2.5 for undirected graphs, Lemma 2.8 implies the following lemma:

Lemma 2.9. *If $k > 11d$, then any instance of (k, c) -SPC on directed graphs either has no solution, or has a solution where the union of some two solution paths P_i and P_j contains all max-congestion nodes.*

Although Lemma 2.9 does not immediately lead to an algorithm for (k, c) -SPC on directed graphs, it specifies some structure which may be useful for future work on (k, c) -SPC and related problems.

One might wonder whether Lemma 2.9 can be modified to have only one solution path P_i that contains all max-congestion nodes, like for DAGs (Lemma 2.3) and undirected graphs (Lemma 2.5), since this would imply interesting algorithms for (k, c) -SPC on directed graphs. Unfortunately, the answer to this question turns out to be no. Similar to the counterexample against extending Theorem 1.2 to directed graphs, we present a counterexample in Appendix B which rules out replacing two solution paths with a single solution path in Lemma 2.9.

2.3 Organization

As suggested in the discussion from Section 2.2, the goal of this work is to establish our general undirected result (Lemma 2.6) and our general directed result (Lemma 2.8). To that end, in Section 3 we provide a technical overview of our results, then in Section 4 we prove Lemma 2.6, and in Section 5 we prove Lemma 2.8.

In Appendix A we provide background on related work on disjoint paths problems. In Appendix B we provide a counterexample against extending Lemma 2.5 to directed graphs. Finally, in Appendix C we prove our corollary for undirected graphs (Lemma 1.4) given Lemma 2.5, which is identical to Amiri and Wargalla’s argument for DAGs, except with different constants.

3 Technical Overview

3.1 Prior Work on DAGs

As a starting point for our proofs for general graphs, we use Amiri and Wargalla’s proof of Theorem 1.1 for DAGs [AW20]. We describe their proof differently than they do for the sake of comparison to our work. We actually describe their proof of the following lemma (which they do not explicitly state), which is analogous to the statements of our general results (Lemmas 2.6 and 2.8).

Lemma 3.1 ([AW20]). *Given a DAG and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

$$\text{for every set } S \subseteq W \text{ of 3 nodes, some path in } \mathcal{R} \text{ contains every node in } S. \quad (*)$$

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of paths that still has property (). Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which some path P passes through all nodes in W .*

The proof of Lemma 3.1 is as follows. Let a and b be the first and last nodes in W (respectively) in a topological ordering of the DAG. Let $v_1, \dots, v_{|W|-2}$ be the remaining nodes in W in order topologically. For ease of notation, we consider \mathcal{R} to be changing over time via subpath swaps, and we will let \mathcal{R} denote the current value of \mathcal{R} .

The argument is inductive. For the base case, by property (*) there is a path in \mathcal{R} that contains a and b . Suppose inductively that a path $P \in \mathcal{R}$ currently contains a , b , and v_1, \dots, v_ℓ for some ℓ . We would like to perform a subpath swap to augment P by adding $v_{\ell+1}$ to P . To do so, we consider a path $P' \in \mathcal{R}$ that contains v_ℓ , $v_{\ell+1}$, and b , where such a path exists by property (*). Importantly, because the graph is a DAG, v_ℓ , $v_{\ell+1}$, and b appear in that order on both P and P' . Thus, according to the definition of a subpath swap (Definition 2.1) we can swap $P[v_\ell, b]$ with $P'[v_\ell, b]$, as shown in Figure 3. As a result of this subpath swap, P now contains $v_{\ell+1}$ in addition to all of the nodes in W that P originally contained. By induction, this completes the proof.

3.2 Undirected Graphs

Recall that our goal for undirected graphs is to prove the following theorem:

Lemma 2.6 (General Undirected Result). *Given an undirected graph and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

$$\text{for every set } S \subseteq W \text{ of 4 nodes, some path in } \mathcal{R} \text{ contains every node in } S. \quad (*)$$

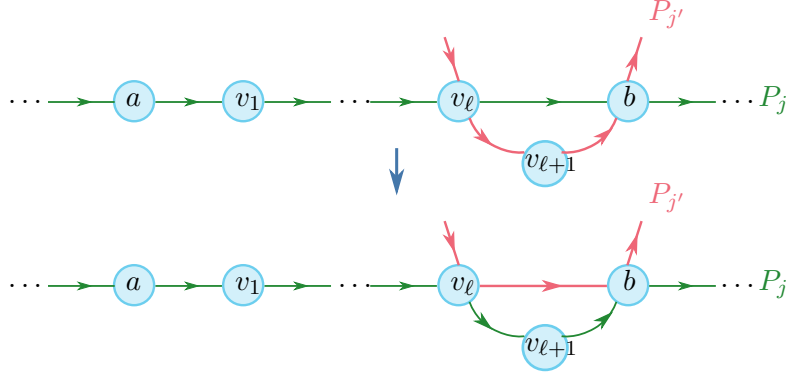


Figure 3: In a DAG, the topological ordering of the nodes allows us to perform a sequence of subpath swaps, each adding the next node in W in order to a path P .

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of shortest paths that still has property (\star) . Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which some path P passes through all nodes in W .

The essential property that enables the subpath swapping in the above argument for DAGs is the fact that for any triple of nodes in a DAG, there is *only one* possible order this triple can appear on any path. This property is not exactly true for general undirected graphs, but we observe that a similar “consistent ordering” property is true: if there is a *shortest* path containing nodes u, v, w in that order, then any shortest path containing these nodes, has them in that order (or in the reverse order w, v, u , but since the graph is undirected we can without loss of generality assume they are in the order u, v, w). This property is true simply because $\text{dist}(u, w)$ is larger than both $\text{dist}(u, v)$ and $\text{dist}(v, w)$, so v must appear between u and w on any shortest path.

To perform subpath swapping on undirected graphs, we need an initial pair of nodes in W such that the rest of the nodes in W will be inserted between this initial pair (in the DAG algorithm, this initial pair a, b was the first and last nodes in W in the topological order). For undirected graphs, our initial pair is the pair a, b of nodes in W whose distance is maximum. We order the rest of the nodes in W by their distance from a , to form $v_1, \dots, v_{|W|-2}$.

Now, our consistent ordering property from above implies the following: for any shortest path P containing a , all nodes in $W \cap P$ are ordered as a subsequence of $a, v_1, v_2, \dots, v_m, b$ on P . As a result, we can perform the same type of iterative subpath swapping argument as the DAG algorithm.

3.3 Roundtrip Paths in Directed Graphs

The situation for directed graphs is significantly more involved than the previous cases. There are several challenges that are present for directed graphs that were not present for either undirected graphs or DAGs. These difficulties stem from the fact that the *interactions* between shortest paths is much more complicated in directed graphs than in DAGs or undirected graphs.

We first outline these challenges, and then provide an overview of how we address them. Our techniques for addressing these issues exemplify that despite the possibly complex arrangement of shortest paths in directed graphs, there still exists an underlying structure to extract. We hope that our methods might illuminate some structural properties of shortest paths in directed graphs in a way that could apply to other directed-shortest-path problems.

Recall that our goal is to prove the following theorem:

Lemma 2.8 (General Directed Result). *Given a directed graph and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

for every set $S \subseteq W$ of 11 nodes, some path in \mathcal{R} contains every node in S . (†)

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of paths that still has property (†). Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which either:

- (i) *some path P passes through all nodes in W , or*
- (ii) *the union of two paths P, P' contain all nodes in W , and the first and last nodes on P from W are the same as the last and first nodes on P' from W , respectively.*

Challenges for directed graphs

Challenge 1: No “extremal” nodes. In the proofs for DAGs and undirected graphs, to begin building the path P containing all nodes in W , we chose two initial extremal nodes $a, b \in W$ and added the rest of the nodes of W between a and b . These nodes a, b were straightforward to choose because there was only one pair of nodes in W that could possibly appear first and last on a path containing all nodes in W (for DAGs a and b were the first and last nodes in a topological ordering of W , and for undirected graphs a and b were the pair of nodes in W with largest distance).

For directed graphs however, it is entirely unclear how to pick these two initial extremal nodes. For instance, choosing the pair of nodes $a, b \in W$ with largest directed distance $\text{dist}(a, b)$ does not work because there could be a shortest path Q containing a and b , where a and b are not the first and last nodes in W on Q (in particular, if b appears before a on the shortest path).

To circumvent this issue for directed graphs, we avoid selecting a pair of initial nodes at all. Without initial nodes as an anchor, we cannot build our path P in order from beginning to end as we did for DAGs and undirected graphs. Instead, our goal is to ensure the following weaker property: as we iteratively transform the overall collection of shortest paths \mathcal{R} , the set of nodes in W on the path in \mathcal{R} containing the most nodes in W grows over time. That is, the path of \mathcal{R} containing the most nodes in W might currently be P , but at the previous iteration, the path of \mathcal{R} with the most nodes in W might have been a different path P' . In this case, our weaker property ensures that P currently contains a *superset* of the nodes in W that P' contained at the previous iteration.

To perform a single iteration with this guarantee, we may need to significantly reconfigure *many* different paths of \mathcal{R} via many subpath swaps. As a result, our path building procedure is much more intricate than the procedures employed for DAGs and undirected graphs.

Challenge 2: No consistent ordering of nodes on shortest paths. In the proof for DAGs and undirected graphs, we were able to perform subpath swaps due to the following crucial property: consider an arbitrary set of nodes v_1, \dots, v_ℓ (for any ℓ) in a DAG or an undirected graph. If there is a shortest path containing the nodes v_1, \dots, v_ℓ in that order, then *every* shortest path containing these nodes has them in that same order.

For directed graphs, however, this property is not even close to being true. In fact, given that the nodes v_1, \dots, v_ℓ appear in that order on some shortest path, there are *exponentially many* other possible orderings of these nodes on other shortest paths. For instance, when $\ell = 4$, given that the nodes v_1, v_2, v_3, v_4 appear in that order on some shortest path, there are eight possible orderings of these nodes on shortest paths, as depicted in Figure 4 (note that despite the large number of possible orderings, not all orderings are possible; for instance the ordering v_1, v_2, v_4, v_3 is not possible, as this

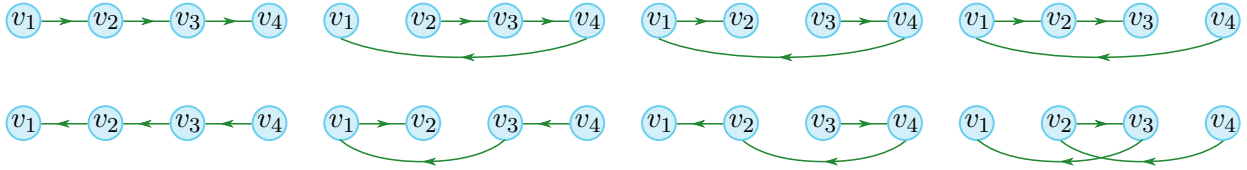


Figure 4: If nodes v_1, v_2, v_3, v_4 appear in that order on some shortest path, they can still appear on different shortest paths in up to seven other distinct possible orders.

would imply that $\text{dist}(v_1, v_4) < \text{dist}(v_1, v_3)$, which contradicts our assumption that some shortest path contains v_1, v_2, v_3, v_4 in that order).

Because directed graphs have no consistent ordering of nodes on shortest paths, it becomes much more difficult to perform subpath swaps like those in the algorithms for DAGs and undirected graphs.

To address this challenge, we provide a structural analysis of the ways in which shortest paths in directed graphs can interact with each other. First, as introduced in Section 1.3, we categorize shortest paths into two main types, *reversing* paths and *non-reversing* paths, and we prove the Reversing/Non-Reversing Path Lemma (Lemma 1.9). Roughly speaking, this lemma is useful because it helps us construct sets of nodes that exhibit some sort of consistent ordering property. This, in turn, enables us to perform sequences of subpath swaps. Defining these consistently ordered sets of nodes and the corresponding subpath swaps is the most involved part of the proof, and works differently for each of the two path types.

Proof structure

Our proof is structured as follows. Initially, we define P to be the path in \mathcal{R} that contains the most nodes in W (breaking ties arbitrarily). Then we proceed with the following two cases:

- (Case 1) We first check whether, roughly speaking, P is contained in a *cycle* that contains *all* nodes in W . In this case, we can use a sequence of subpath swaps to build a second path P' so that the union of P and P' contains all nodes in W and have the “roundtrip” structure specified in the theorem statement, in which case we are done.
- (Case 2) If we are not in Case 1, our goal is to augment some path in \mathcal{R} so that the set of nodes of W on the path in \mathcal{R} with the most nodes in W grows (the goal introduced in the discussion of Challenge 1).

After going through these cases, if we are not done we redefine P as the path in \mathcal{R} containing the most nodes in W and repeatedly apply the appropriate case, until we are done. Most technical details of our proof are in the path augmentation procedure of Case 2. We elaborate on the main ideas for this procedure next.

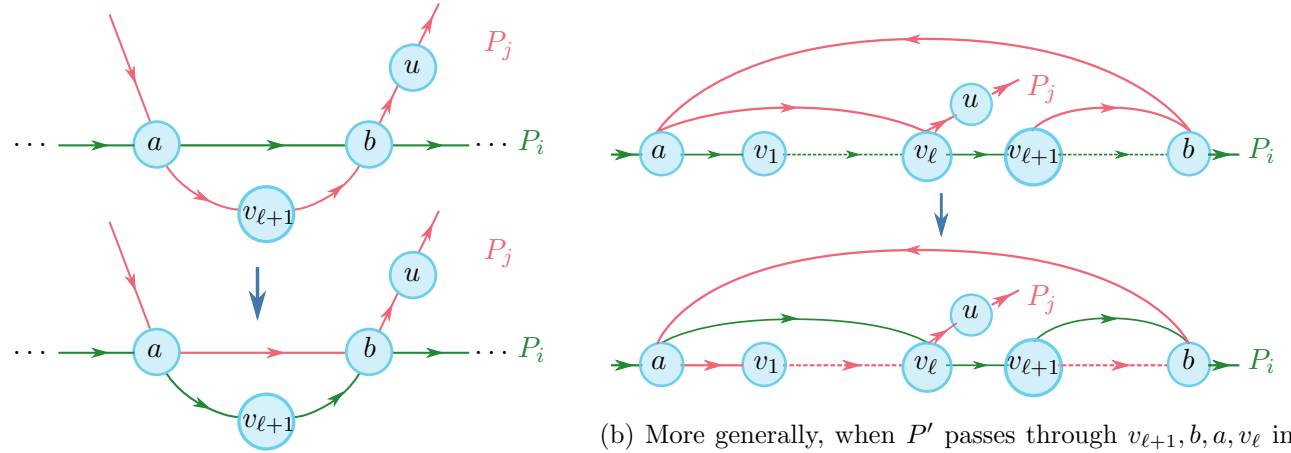
Handling Case 2: Path Augmentation

Recall that our goal is the following: Let P be the path in \mathcal{R} containing the most nodes in W , and let $W' = W \cap P$. Fix a node $u \in W \setminus W'$. We wish to perform subpath swaps to yield a path P' that contains every node in $W' \cup \{u\}$.

We begin with a few warm-up cases to motivate our general approach.

Warm-up cases Let a and b be the first and last nodes on P that are in W , respectively. By property (†) there exists a path in \mathcal{R} containing a , b , and u . We will suppose in all of the subsequent examples, that for *all* paths in \mathcal{R} containing the nodes a , b , and u , the node u is always the last node in W on the path; this is not a conceptually important assumption and it makes the description simpler.

We claim that if there is a path $P' \in \mathcal{R}$ containing a , b , and u , such that a appears before b , then we are done. Indeed, as depicted in Figure 5a, in this case we can simply swap the subpath $P[a, b]$ with $P'[a, b]$, to form a new solution where P' contains $W' \cup \{u\}$.



(a) If a appears before b on P' , a subpath swap lets P' pass through u together with all of the nodes in W' .

(b) More generally, when P' passes through v_{l+1}, b, a, v_l in that order, we can perform two subpath swaps to get P' to pass through the rest of W' . Here, the dotted segments indicate portions of the paths which pass through the nodes of W' that are not labeled in the figure.

Figure 5: The two warm-up cases.

Now we will slightly generalize this warm-up case. Let a, v_1, v_2, \dots, b be the nodes in W' in the order they appear on P . Consider v_ℓ and $v_{\ell+1}$ for any ℓ . By property (†) there exists a path in \mathcal{R} containing a, b, u, v_ℓ , and $v_{\ell+1}$. We know from the previous warm-up case that if there is a path in \mathcal{R} containing these nodes such that a appears before b , then we are done. We also claim that if there exists a path $P' \in \mathcal{R}$ containing these nodes such that $v_{\ell+1}, b, a, v_\ell$ appear in that order, then we are done. This is because as depicted in Figure 5b, we can swap the subpath $P[a, v_\ell]$ with $P'[a, v_\ell]$, and swap the subpath $P[v_{\ell+1}, b]$ with $P'[v_{\ell+1}, b]$. Now, P' contains $W' \cup \{u\}$.

General Approach: “Critical nodes” We will motivate our general approach in the context of the above warm-up cases. In the second warm-up case, we considered 5 nodes (a, b, u, v_ℓ , and $v_{\ell+1}$) in W' , and argued that if there is a path $P' \in \mathcal{R}$ containing these 5 nodes in one of several “good” orders, then we are done because we can perform subpath swaps to reroute P' through all of $W' \cup \{u\}$. Thus, our goal is to choose these 5 (or in general, at most 11) nodes carefully, to guarantee that they indeed fall into a “good” order on some path in \mathcal{R} . We refer to these at most 11 nodes as *critical nodes*:

Definition 3.2 (Critical Nodes (Informal)). Given a path $P \in \mathcal{R}$, a set of nodes $T \subseteq W$ of size $|T| \leq 11$ are *critical nodes of P* if there exists a path $P' \in \mathcal{R}$ containing the nodes of T in an order that allows us to perform subpath swaps to reroute some path in \mathcal{R} through all of $W' \cup \{u\}$ (where $W' = W \cap P$).

Our general approach is to show that any path $P \in \mathcal{R}$ contains a set of at most 11 critical nodes. We remind the reader that this section only concerns Case 2, so our goal is to show that P contains a set of critical nodes only if P does not already fall into Case 1. After showing that P contains a set of critical nodes, we are done, because performing subpath swaps to yield a path containing $W' \cup \{u\}$ was our stated goal.

It is not at all clear a priori that any P should contain a set of critical nodes. Indeed, the critical nodes need to be chosen very carefully. Specifically, they need to be chosen based on the structure of the path P .

This is where the definitions from Section 1.3 come into play. We categorize P based on whether it is *reversing* or *non-reversing*, (or trivial). Then we construct the critical nodes of P using a different procedure specialized for which type of path P is.

If P is a reversing path, then we argue that a valid choice of critical nodes are a, b , and u , along with the two nodes at the two boundaries between the segments defined in Lemma 1.9 (and a few other nodes for technical reasons). To make this argument, which we will not detail here, we construct an involved series of subpath swaps to reroute some path through all of $W' \cup \{u\}$.

On the other hand, if P is a non-reversing path, the critical nodes are less straightforward to define than if P is a reversing path. Indeed, defining the critical nodes for non-reversing paths is the most conceptually difficult part of our proof. The high-level reason for this difficulty is the fact that the nodes of Segment 2 of a non-reversing path are quite unconstrained because they admit 3 possible shortest-path orderings instead of only 2.

To be more concrete, suppose every node on P falls into Segment 2. For simplicity, suppose we were to choose critical nodes a, b, w , where a and b are the endpoints of P and also happen to be in W' , and w is any other node in W' . Consider the path $P' \in \mathcal{R}$ containing a, b , and w , which exists by property (†). By the definition of Segment 2, there are 3 possible orderings of a, b, w on P' ($a \rightarrow w \rightarrow b$, or $b \rightarrow a \rightarrow w$, or $w \rightarrow b \rightarrow a$). Note that these 3 orderings are cyclic shifts of one another. Suppose, as an illustrative example, that P' has the ordering $b \rightarrow a \rightarrow w$. We would like to reroute P' through all of $W' \cup \{u\}$, but we have a problem. For any node $s \in W'$ that appears between a and w on P , we can reroute P' through s by swapping the subpath $P[a, w]$ with $P'[a, w]$; however, for a node $s \in W'$ that appears between w and b on P , we cannot do this because there is no path segment from w to b on P' , since b appears before w on P' . Thus, we cannot reroute P' through s . That is, no matter how we choose the critical nodes, if we do not impose extra structural constraints, we can always identify a segment of the path that we cannot reroute P' through. Overcoming this issue is our main technical challenge, and we defer it to the full proof.

4 Undirected Graphs

As detailed in Section 2, our goal for undirected graphs is to prove the following lemma:

Lemma 2.6 (General Undirected Result). *Given an undirected graph and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

$$\text{for every set } S \subseteq W \text{ of 4 nodes, some path in } \mathcal{R} \text{ contains every node in } S. \quad (\star)$$

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of shortest paths that still has property (\star) . Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which some path P passes through all nodes in W .

Proof. For ease of notation, we consider \mathcal{R} to be changing over time via subpath swaps, and we will let \mathcal{R} denote the current value of \mathcal{R} .

Among all nodes in W , select two nodes a, b which maximize $\text{dist}(a, b)$. Let $v_1, v_2, \dots, v_{|W|-2}$ be the remaining nodes of W , ordered by increasing distance from the node a . (For the sake of this definition we break ties arbitrarily, but in reality, there cannot be any ties.)

We now inductively show that for any positive integer ℓ , there is a sequence of subpath swaps that yields a path that passes through the nodes a, v_1, \dots, v_ℓ , and b , in that order.

For the base case where $\ell = 1$, \mathcal{R} contains a path passing through a, b , and v_1 by property (\star) . In this path, v_1 must be between a and b since a and b have maximum distance among all pairs of nodes in W . (Since the graph is undirected we can define the “beginning” of the path to be a to get a path through a, v_1 , and b , in that order.)

For the inductive step, let ℓ be a fixed positive integer with $1 < \ell < |W|$. By the inductive hypothesis, we may assume there is a path $P \in \mathcal{R}$ that passes through a, v_1, \dots, v_ℓ , and b in that order, for some fixed positive integer ℓ . Our goal is to show that we can perform some subpath swaps to \mathcal{R} , to produce a path which passes through $a, v_1, \dots, v_{\ell+1}$, and b in that order.

By property (\star) , some path $Q \in \mathcal{R}$ must pass through a, b, v_ℓ , and $v_{\ell+1}$.

By our choice of a and b , we know that v_ℓ and $v_{\ell+1}$ must be between a and b on the path Q . Since we ordered the v_i nodes by distance from a , it follows that P contains $a, v_\ell, v_{\ell+1}$, and b in that order. Now we swap the subpath $P[v_\ell, b]$ with $Q[v_\ell, b]$. After the swap, path P passes through the node $v_{\ell+1}$ in addition to a, v_1, \dots, v_ℓ , and b , in the desired order. This completes the induction, and proves the result. \blacksquare

5 Directed Graphs

As detailed in Section 2, our goal for directed graphs is to prove the following lemma:

Lemma 2.8 (General Directed Result). *Given a directed graph and subset W of nodes, let \mathcal{R} be a collection of shortest paths with the following property:*

for every set $S \subseteq W$ of 11 nodes, some path in \mathcal{R} contains every node in S . (†)

Further suppose that applying any sequence of $O(n^3)$ subpath swaps to \mathcal{R} yields a collection of paths that still has property (†). Then starting from \mathcal{R} , there exists a sequence of subpath swaps that results in a collection of shortest paths in which either:

- (i) *some path P passes through all nodes in W , or*
- (ii) *the union of two paths P, P' contain all nodes in W , and the first and last nodes on P from W are the same as the last and first nodes on P' from W , respectively.*

The section is organized as follows. In Section 5.1, we prove the Reversing/Non-Reversing Path Lemma introduced in the introduction (Lemma 1.9). In Section 5.2 we prove a “critical node” lemma that specifies conditions under which we can perform subpath swaps on directed graphs. In Section 5.3, we complete the proof of Lemma 2.8.

5.1 Structural Lemma for Reversing/Non-Reversing Paths

In this section, we will prove Lemma 1.9, which was introduced in the introduction. We actually prove something slightly more general than Lemma 1.9, so the lemma statement and some of the definitions are slightly different here than in the introduction.

First, we recall the definition of a shortest-path ordering (which is the same as before):

Definition 1.6. For any nonnegative integer ℓ and set of ℓ nodes, v_1, v_2, \dots, v_ℓ , we say that the order $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_\ell$ is a *shortest-path ordering* if there is a shortest path containing all of the nodes v_1, v_2, \dots, v_ℓ in that order.

The following simple lemma about shortest-path orderings will be useful.

Lemma 5.1. *For any triple $\{u, v, w\}$ of nodes, if $u \rightarrow v \rightarrow w$ is a shortest-path ordering, then neither $v \rightarrow u \rightarrow w$ nor $u \rightarrow w \rightarrow v$ is a shortest-path ordering.*

Proof. Suppose $u \rightarrow v \rightarrow w$ is a shortest-path ordering. This means that $\text{dist}(u, v) < \text{dist}(u, w)$ and $\text{dist}(v, w) < \text{dist}(u, w)$. If $v \rightarrow u \rightarrow w$ were a shortest-path ordering, this would imply that $\text{dist}(v, w) > \text{dist}(u, w)$, a contradiction. If $u \rightarrow w \rightarrow v$ were a shortest-path ordering, this would imply that $\text{dist}(u, v) > \text{dist}(u, w)$, a contradiction. ■

Now we define trivial paths, reversing paths, and non-reversing paths in a slightly more general way than in the introduction.

Definition 5.2 (Trivial Path). Given a directed graph, nodes a, b , and a shortest path P from a to b , P is a *trivial path* with respect to a node w on P if $a \rightarrow w \rightarrow b$ is the *only* shortest-path ordering of a, w, b .

Definition 5.3 (Reversing path). Suppose we are given a directed graph, nodes $\{a, b\}$, a shortest path P from a to b , and a subset W' of the nodes on P such that P is not a trivial path with respect to any node in W' . Then, P is a *reversing path* with respect to W' if P contains at least one node $w \in W'$ such that w falls on some b -to- a shortest path. Otherwise, P is a *non-reversing path* with respect to S .

We call the subset of nodes W' intentionally, to hint that when we apply this definition to prove Lemma 2.8, W' will be set to $P \cap W$.

Now, we state our structural lemma for reversing and non-reversing paths, which is slightly more general than Lemma 1.9:

Lemma 5.4 (Reversing/Non-Reversing Path Lemma). *Suppose we are given a directed graph, nodes $\{a, b\}$, a shortest path P from a to b , and a subset W' of the nodes on P such that P is not a trivial path with respect to any node in W' . Then, P can be partitioned into three contiguous segments with the following properties (where a is defined to be in Segment 1, b is defined to be in Segment 3, and Segment 2 could be empty).*

Segment 1: nodes w such that if $w \in W'$ then the shortest-path orderings of a, w, b are precisely $a \rightarrow w \rightarrow b$, and $b \rightarrow a \rightarrow w$.

Segment 2: nodes w such that if $w \in W'$ then the shortest-path orderings of a, w, b are precisely

$$\begin{cases} a \rightarrow w \rightarrow b, \text{ and } b \rightarrow w \rightarrow a & \text{if } P \text{ is a reversing path wrt } W' \\ a \rightarrow w \rightarrow b, \text{ and } b \rightarrow a \rightarrow w, \text{ and } w \rightarrow b \rightarrow a & \text{if } P \text{ is a non-reversing path wrt } W'. \end{cases}$$

Segment 3: nodes w such that if $w \in W'$ then the shortest-path orderings of a, w, b are precisely $a \rightarrow w \rightarrow b$, and $w \rightarrow b \rightarrow a$.

Proof. We first note that for any node w on P , $a \rightarrow w \rightarrow b$ is a shortest-path ordering, since P is a shortest path from a to b .

Given an arbitrary node $w \in W'$, we will enumerate the possible shortest-path orderings for a, w, b besides $a \rightarrow w \rightarrow b$. First, $a \rightarrow w \rightarrow b$ is the only shortest-path ordering of a, w, b that starts with

a , simply because $\text{dist}(a, w) < \text{dist}(a, b)$. Similarly, $a \rightarrow w \rightarrow b$ is the only shortest-path ordering of a, w, b that ends with b , simply because $\text{dist}(w, b) < \text{dist}(a, b)$. The remaining possible orderings are (1) $b \rightarrow a \rightarrow w$, (2) $w \rightarrow b \rightarrow a$, and (3) $b \rightarrow w \rightarrow a$.

By Lemma 5.1, if w admits ordering 3, then w cannot admit either of the first two orderings.

Let w_1 be the last node on P that admits the ordering 1 ($b \rightarrow a \rightarrow w_1$) and *not* ordering 2 ($w_1 \rightarrow b \rightarrow a$), if such a node exists. Define Segment 1 as the prefix of P up to and including w_1 . We claim that if w is in Segment 1 then w also admits ordering 1 and *not* ordering 2. Since P is a shortest path, $a \rightarrow w \rightarrow w_1 \rightarrow b$ is a shortest-path ordering. Combining this with $b \rightarrow a \rightarrow w_1$, we have that $b \rightarrow a \rightarrow w \rightarrow w_1$ is a shortest-path ordering. Thus, $b \rightarrow a \rightarrow w$ is a shortest-path ordering, so w admits ordering 1, as desired. Now for the contrapositive of the second part of our claim, we will show that if w admits ordering 2 ($w \rightarrow b \rightarrow a$), then so does w_1 . This is by a similar argument: Combining $w \rightarrow b \rightarrow a$ with $a \rightarrow w \rightarrow w_1 \rightarrow b$ yields $w \rightarrow w_1 \rightarrow b \rightarrow a$, and thus $w_1 \rightarrow b \rightarrow a$.

Now we claim that if w is in Segment 1 then a, w, b have no other shortest-path orderings besides $a \rightarrow w \rightarrow b$ and $b \rightarrow a \rightarrow w$. This is simply due to the above enumeration of possible shortest-path orderings: We know that w admits ordering 1, we have already ruled out ordering 2, and we have already shown that ordering 3 cannot coexist with ordering 1. Thus, we have shown that the shortest-path orderings of a, w, b are precisely $a \rightarrow w \rightarrow b$, and $b \rightarrow a \rightarrow w$, as desired.

Let w_2 be the first node on P that admits ordering 2 ($w_2 \rightarrow b \rightarrow a$) and *not* ordering 1 ($b \rightarrow a \rightarrow w_2$), if such a node exists. Define Segment 3 as the suffix of P starting at w_2 . An argument that is exactly symmetric to the above argument for Segment 1 proves that every node w in Segment 3 has precisely the shortest-path orderings $a \rightarrow w \rightarrow b$, and $w \rightarrow b \rightarrow a$, as desired.

Let w_2 be the first node on P that admits ordering 2 ($w_2 \rightarrow b \rightarrow a$) and *not* ordering 1 ($b \rightarrow a \rightarrow w_2$), if such a node exists. Define Segment 3 as the suffix of P starting at w_2 . An argument that is exactly symmetric to the above argument for Segment 1 proves that every node w in Segment 3 has precisely the shortest-path orderings $a \rightarrow w \rightarrow b$, and $w \rightarrow b \rightarrow a$, as desired.

Now, suppose w is in Segment 2 i.e. w lies between w_1 and w_2 . By the definition of w_1 , we know that w either admits both orderings 1 and 2, or w does not admit ordering 1. By the definition of w_2 , we know that w either admits both orderings 1 and 2, or w does not admit ordering 2. Putting these together, w either admits *both* or *neither* of orderings 1 or 2. If w admits both orderings, then we have shown that w cannot admit ordering 3, and thus w does not admit any additional shortest-path orderings besides $a \rightarrow w \rightarrow b$. In this case, w satisfies the non-reversing path case of Segment 2. On the other hand, if w admits neither ordering 1 nor 2, then because P is not a trivial path with respect to W' , w must admit ordering 3. In this case, w satisfies the reversing path case of Segment 2.

It remains to show that P cannot contain both a node w that satisfies the reversing path case of Segment 2, and a node w' that satisfies the non-reversing path case of Segment 2. This is simple to show: If w' appears before w on P , then $w' \rightarrow w \rightarrow b$ is a shortest-path ordering. By definition, $w' \rightarrow b \rightarrow a$ is also a shortest-path ordering. Combining these, we get the shortest-path ordering $w' \rightarrow w \rightarrow b \rightarrow a$, which implies the shortest-path ordering $w \rightarrow b \rightarrow a$, a contradiction to the definition of w . If w' appears after w on P , then a symmetric argument applies. This completes the proof. ■

5.2 Critical Node Lemma

In this section, we establish a lemma that specifies conditions under which we can perform subpath swaps on directed graphs. The intuition behind the lemma is motivated by our general approach of identifying “critical nodes,” outlined previously in the technical overview.

In the following lemma, the nodes T are the critical nodes. Although in the technical overview the critical nodes were said to come from a specific path P , no such path is mentioned in the statement of the lemma – instead, such a path will be involved when the lemma is applied.

Lemma 5.5 (Critical Node Lemma). *Assume the premise of Lemma 2.8. Suppose additionally we are given nodes $a_1, \dots, a_r, b_1, \dots, b_r$, a subset of nodes $W^* \subseteq W$, and a further subset of nodes*

$$T = (\{a_1, \dots, a_r, b_1, \dots, b_r\} \cup U) \subseteq W^*$$

satisfying $|T| \leq 9$, such that:

- (i) for each path $P \in \mathcal{R}$ that contains every node in T , for all i the node a_i appears before b_i on P , and
- (ii) for all $w \in W^*$, w is on a shortest path from a_i to b_i for exactly one i .

Then, given these conditions, we can perform a sequence of subpath swaps so that in the resulting path collection \mathcal{S}' , some path P' passes through all nodes in W^* , and all paths in \mathcal{R} that do not contain any nodes in $W^* \setminus T$ appear in \mathcal{R}' .

Proof. Partition the set $W^* \setminus T$ into W_1, \dots, W_r where $w \in W_i$ if $a_i \rightarrow w \rightarrow b_i$ is a shortest-path ordering. For all $u \in U$, if u is on a shortest path from a_i to b_i for some i , then u is also included in W_i . By condition 2 of the lemma statement, every node $w \in W^* \setminus T$ is in exactly one set W_i , and every node $u \in U$ is in at most one set W_i . Now, label all of the nodes in $\cup_i (W_i \cup \{a_i, b_i\})$ by $v_1, v_2, v_3 \dots$ according to the following order: $a_1, W_1, b_1, a_2, W_2, b_2, \dots, a_r, W_r, b_r$, where the nodes within each W_i are ordered by distance from a_i . (For the sake of this definition we break ties arbitrarily, but in reality, there cannot be any ties.)

For ease of notation, we consider \mathcal{R} to be changing over time via subpath swaps, and we will let \mathcal{R} denote the current value of \mathcal{R} .

We proceed by induction. Suppose \mathcal{R} contains a path P containing $\{v_1, \dots, v_\ell\} \cup T$ for some ℓ . Further suppose assumption 1 of the lemma statement: for each path in \mathcal{R} that contains every node in T , for all i the node a_i appears before b_i . We will show that we can perform subpath swaps so that the resulting path collection \mathcal{R} has a path that contains $\{v_1, \dots, v_{\ell+1}\} \cup T$, and assumption 1 of the lemma statement still holds.

If $v_{\ell+1} \in T$, then we are already done, so suppose $v_{\ell+1} \notin T$. Let j be such that $v_{\ell+1} \in W_j$. By assumption 1 of the lemma statement, a_i appears before b_i on P , for all i . By Lemma 5.1, for all i , every node $s \in W_i \cap P$ appears between a_i and b_i on P .

We also claim that every node in $\{v_1, \dots, v_{\ell-1}\} \cap W_j$ appears between a_j and v_ℓ on P . From the previous paragraph, we know that every node in $\{v_1, \dots, v_\ell\} \cap W_j$ appears between a_j and b_j on P . Along the subpath of P from a_j to b_j , the nodes of $\{v_1, \dots, v_\ell\} \cap W_j$ appear in that order, simply because the v_i 's were defined to be ordered by distance from a_j . Thus, every node in $\{v_1, \dots, v_{\ell-1}\} \cap W_j$ appears between a_j and v_ℓ on P .

Consider the set $\{v_1, v_\ell, v_{\ell+1}\} \cup T$, which is of size at most 11 since $|T| \leq 9$ and $v_1 \in T$. By property (†) there exists a path $P' \in \mathcal{R}$ that contains every node in $\{v_1, v_\ell, v_{\ell+1}\} \cup T$. By assumption 1 of the lemma statement, a_i appears before b_i on P' for all i .

Now we can define our sequence of subpath swaps. For all $i < j$, we swap the subpath $P[a_i, b_i]$ with $P'[a_i, b_i]$. We also swap the subpath $P[a_j, v_\ell]$ with $P'[a_j, v_\ell]$. From above we know that before these swaps, for all $i < j$ every node $w \in W_i$ appears between a_i and b_i on P . Thus, after these swaps, for all $i < j$ every node $w \in W_i$ appears between a_i and b_i on P' . Similarly, before these swaps, every node in $\{v_1, \dots, v_{\ell-1}\} \cap W_j$ appears between a_j and v_ℓ on P , so after these swaps every node in $\{v_1, \dots, v_{\ell-1}\} \cap W_j$ appears between a_j and v_ℓ on P' .

Now, we consider the set U . Before these swaps, P' contains all of U , and we claim that after the swap P' still contains all of U . Fix $u \in U$. If $u \in \{v_1, \dots, v_{\ell-1}\} \cap W_j$, then P' contains u after the swaps, from the previous paragraph. Otherwise, before the swaps, u is not on a path segment of P'

that got swapped, so u remains on P' . Thus, we have shown that after the swaps, P' contains every node in $\{v_1, \dots, v_{\ell+1}\} \cup T$ as desired.

For the induction, we need to verify that after the swap, assumption 1 still holds. This is simply because we only changed paths P and P' , and after the swap, both of them still have a_i appearing before b_i for all i .

Finally, we need to verify that all paths in \mathcal{R} that do not contain any nodes in $W^* \setminus T$ are also contained in \mathcal{S}' . This is true simply because the path P' is the only other path besides P that is affected by the swaps and by definition P' contains nodes in $W^* \setminus T$. \blacksquare

5.3 Proof of Lemma 2.8

We now use the lemmas proved in the previous sections to complete the proof of Lemma 2.8.

Proof Structure The structure of the proof is as follows. Let P be the path of \mathcal{R} passing through the most nodes in W (with ties are broken arbitrarily). If P passes through every node in W , then we are done.

Otherwise, as stated in the technical overview, we consider two cases: in case 1 we perform subpath swaps to construct a path P' so that P and P' together satisfy condition 2 of Lemma 2.8, in which case we are done, and in case 2 we augment one path.

More, specifically, for case 2, let $W' = P \cap W$. We construct a sequence of subpath swaps that yields a path P' passing through W' and at least one node in $W \setminus W'$.

After executing case 2, if we are not done we redefine P' to be the path P and repeatedly apply the two cases until we are done. Since each iteration of case 2 causes the number of nodes of W on the path P to grow, repeated application of these cases completes the proof. In the full proof below, case 2 is actually broken into 4 cases labeled cases 2-5.

To begin the description of the cases, fix an iteration, and fix the current value of \mathcal{R} . For ease of notation, we consider \mathcal{R} to be changing over time via subpath swaps, and we will always let \mathcal{R} denote the current value of \mathcal{R} . Fix the current path P . Let $W' = P \cap W$. Let a and b be the first and last nodes on P from W .

For all of the cases, it will help to have the following definition:

Definition 5.6 (Trapping nodes). For any node $v \in W$, we say that the *trapping nodes* of v , denoted $\text{trap}(v)$ is the pair $\{\text{trap}_1(v), \text{trap}_2(v)\}$ of nodes in W' such that $\text{dist}(a, \text{trap}_1(v))$ is maximized and $\text{dist}(a, \text{trap}_2(v))$ is minimized under the condition that $\text{dist}(a, \text{trap}_1(v)) \leq \text{dist}(a, v) \leq \text{dist}(a, \text{trap}_2(v))$.

Case 1: Create Two Paths Our assumption in case 1 is that for *all* $v \in W \setminus W'$, *all* paths in \mathcal{R} that contain $\{a, b, v\} \cup \text{trap}(v)$ have $b \rightarrow v \rightarrow a$ in that order (not necessarily consecutively). Then, letting v be an arbitrary node in $W \setminus W'$, we can apply Lemma 5.5 (the Critical Node Lemma) with $W^* = W \setminus W'$, and $T = \{a, b, v\} \cup \text{trap}(v)$, where $a_1 = a$, $b_1 = b$, and $U = \{v\} \cup \text{trap}(v)$. After the subpath swaps from Lemma 5.5, \mathcal{R} contains a path P' containing every node in $W \setminus W'$. Furthermore, Lemma 5.5 guarantees that P still appears in \mathcal{R} . Thus, \mathcal{R} contains two paths P and P' whose union contains every node in W .

To show that P and P' satisfy condition 2 of Lemma 2.8, we need to show that b is the first node on P' from W and a is the last. This is true simply because of the assumption that for all $v \in W \setminus W'$, all paths in \mathcal{R} that contain $\{a, b, v\} \cup \text{trap}(v)$ have $b \rightarrow v \rightarrow a$ in that order.

Case 2: Simple Insertion Our assumption in case 2 is that there exists a node $v \in W \setminus W'$ such that there exists a path $P' \in \mathcal{R}$ containing any one of the three following orderings of nodes (not necessarily consecutively):

- $a \rightarrow b \rightarrow v$, or
- $v \rightarrow a \rightarrow b$, or
- $\text{trap}_1(v) \rightarrow v \rightarrow \text{trap}_2(v)$.

For the first two orderings, we can simply swap the subpath $P[a, b]$ with $P'[a, b]$. As a result of this, the path P' contains $W' \cup \{v\}$, so we are done. For the third ordering, we can simply swap $P[\text{trap}_1(v), \text{trap}_2(v)]$ with $P'[\text{trap}_1(v), \text{trap}_2(v)]$. As a result, P contains $W' \cup \{v\}$, so we are done.

For the sake of analyzing future cases, we claim that we are in case 2 in the following situations:

- (i) there exists a node $v \in W \setminus W'$ such that there exists a path $P' \in \mathcal{R}$ containing $\{a, b, v\} \cup \text{trap}(v)$ that has a before b , or
- (ii) the subpath $P[a, b]$ is a trivial path with respect to some $w \in W'$. In this case, by the definition of a trivial path there exists a path $P' \in \mathcal{R}$ containing $\{a, b, v, w\} \cup \text{trap}(v)$ that has a before b by property (\dagger) .

For either situation, fix such a path P' . If P' has v before a or after b , then we fall into case 2 by definition. Otherwise, P' has v between a and b . By Lemma 5.1, P' also has both nodes in $\text{trap}(v)$ between a and b . Because $\{v\} \cup \text{trap}(v)$ all appear between a and b on P' , we know that since $\text{dist}(a, \text{trap}_1(v)) \leq \text{dist}(a, v) \leq \text{dist}(a, \text{trap}_2(v))$ they appear in the order $\text{trap}_1(v) \rightarrow v \rightarrow \text{trap}_2(v)$ on P' . Thus, we fall into case 2.

Therefore, for the remaining cases, we know that the subpath $P[a, b]$ is not a trivial path with respect to any node in W' . Thus, we have two cases to consider: $P[a, b]$ is a reversing path with respect to W' , or $P[a, b]$ is non-reversing with respect to W' .

Case 3: Reversing Path Suppose $P[a, b]$ is a reversing path with respect to W' . Let v be an arbitrary node in $W \setminus W'$. Our goal is to perform subpath swaps so that as a result, \mathcal{R} has a path that contains every node in $W' \cup \{v\}$.

We apply Lemma 5.4 (the Reversing/Non-Reversing Path Lemma) to $P[a, b]$, where W' in Lemma 5.4 is the same as our set W' , to get a decomposition of $P[a, b]$ into three segments P_1, P_2, P_3 .

Let w be the last node on P_1 from W' , and let z be the first node on P_3 from W' . That is, $P[a, w] \subseteq P_1$, and $P[z, b] \subseteq P_3$.

Let p and q be the nodes in $P_2 \cap W$ which are the closest and furthest from b in terms of distance respectively (such nodes exist because P_2 is non-empty by the definition of a reversing path). Define the set $K = \{a, b, w, z, p, q, v\} \cup \text{trap}(v)$. K will be our set of *critical nodes*. By property (\dagger) , there is a path $Q \in \mathcal{R}$ containing every node in K .

Next, we will analyze the order in which Q traverses the nodes of K . After establishing this structure, we will perform subpath swaps by applying Lemma 5.5 (the Critical Node Lemma).

Because we are not in **case 2**, we know that *all* paths in \mathcal{R} that contain $\{a, b, v\} \cup \text{trap}(v)$ have b before a , and in particular Q . Because p and q are in P_2 , by Lemma 5.4 p and q are both on a shortest path from b to a . Thus, by Lemma 5.1, p and q both appear between b and a on Q . Furthermore, by considering the properties of P_1 and P_3 from the statement of Lemma 5.4, we know that z occurs before b on Q , and w occurs after a on Q . By our choice of p and q as the closest and furthest nodes from b respectively, we know that p appears before q on Q .

In summary, Q is such that z appears before b , p appears before q , and a appears before w . With this structure, we are ready to apply Lemma 5.5 with the following choice of parameters: T is set to K , W^* is set to $W' \cup \{v\}$, $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are set to $(z, b), (p, q), (a, w)$, and U is set to $\{v\} \cup \text{trap}(v)$. We will now verify that the assumptions of Lemma 5.5 indeed hold. For the first assumption, Q was chosen to be an arbitrary path in \mathcal{R} containing every node in K , so the properties

we proved for Q are indeed true for every such path. For the second assumption, by definition, every node in $P_1 \cap W$ falls on a shortest path from a to w , every node in $P_2 \cap W$ falls on a shortest path from b to a , and every node in $P_3 \cap W$ falls on a shortest path from z to b . No node falls into multiple path segments P_i , so each node in W' is indeed on a shortest path from a_i to b_i for exactly one i . The consequence of Lemma 5.5 is that \mathcal{R} has a path containing every node in $W' \cup \{v\}$, as desired.

The remaining cases consider the situation where $P[a, b]$ is a non-reversing path with respect to W' .

Non-Reversing Path Lemma When $P[a, b]$ is a non-reversing path with respect to W' , we will split into two cases. Before presenting either of the cases, we prove a lemma that will be useful for both cases.

Lemma 5.7. *Suppose $u, w \in W'$ and u is before w on P . Let u' be any node in W' that appears before u on P , and let w' be any node in W' that appears after w on P . Then, any path $Q \in \mathcal{R}$ containing $\{u, w, u', w', a, b\}$ that has w before u , has the order $w \rightarrow w' \rightarrow b \rightarrow a \rightarrow u' \rightarrow u$.*

Proof. Let $Q \in \mathcal{R}$ be a path containing $\{u, w, u', w', a, b\}$ that has w before u . Node a must occur after w on Q , as otherwise we would have the shortest-path ordering $a \rightarrow w \rightarrow u$, which by Lemma 5.1 contradicts the shortest-path ordering $a \rightarrow u \rightarrow w$, which we have by assumption. Symmetric reasoning shows that node b must occur before u on Q . Finally, b must appear before a on Q , because otherwise we would have the shortest-path ordering $w \rightarrow a \rightarrow b \rightarrow u$, which by Lemma 5.1 contradicts the shortest-path ordering $a \rightarrow u \rightarrow w \rightarrow b$, which we have by assumption.

So far we have simply used distance considerations to restrict the order these nodes appear on Q . We now use the fact that $P[a, b]$ is non-reversing (with respect to W') to pin down the precise order that Q passes through these nodes.

We claim that b appears after w on Q . Indeed, suppose to the contrary that b appeared before w . Then since w appears before u , which in turn appears before a , on Q , we would have the shortest-path ordering $b \rightarrow w \rightarrow a$. However, this contradicts the fact that $P[a, b]$ is non-reversing with respect to W' . Symmetric reasoning shows that a appears after u on Q .

Combining all these observations together, we see that Q traverses these nodes in the order $w \rightarrow b \rightarrow a \rightarrow u$.

Since w' falls on a shortest path from w to b , by Lemma 5.1 w' falls between w and b on Q . Similarly, since u' is on a shortest path from a to u , by Lemma 5.1 u' falls between a and u on Q . Thus, Q has the order $w \rightarrow w' \rightarrow b \rightarrow a \rightarrow u' \rightarrow u$, as desired. \blacksquare

Case 4: Non-Reversing Path Easy Case Suppose $P[a, b]$ is a non-reversing path with respect to W' . Our further assumption in case 4 is that there exists a node $v \in W \setminus W'$ with the following property: there exist nodes $u, w \in W'$ where u is before w on P and no node of W' falls between u and w on P , such that every path in \mathcal{R} containing $\{u, w, v\}$, has w before u . Fix such a node v . Our goal is to perform subpath swaps so that as a result, \mathcal{R} has a path that contains every node in $W' \cup \{v\}$.

Define the set $K = \{v, a, b, u, w\}$, which will be our set of *critical nodes*. By property (\dagger) there is a path $Q \in \mathcal{R}$ passing through all the nodes in K . By assumption, w occurs before u on Q . By Lemma 5.7, Q traverses these nodes in the order $w \rightarrow b \rightarrow a \rightarrow u$.

Now we are ready to apply Lemma 5.5 (the Critical Node Lemma) with the following choice of parameters: T is set to K , W^* is set to $W' \cup \{v\}$, $(a_1, b_1), (a_2, b_2)$ are set to $(w, b), (a, u)$, and U is set to $\{v\}$. We will now verify that the assumptions of Lemma 5.5 indeed hold. For the first assumption, Q was chosen to be an arbitrary path in \mathcal{R} containing every node in K , so the properties we proved for Q are indeed true for every such path. For the second assumption, by definition, every node in W' either falls on exactly one of $P[w, b]$ or $P[a, u]$. The consequence of Lemma 5.5 is that \mathcal{R} has a path containing every node in $W' \cup \{v\}$, as desired.

Case 5: Non-Reversing Path Hard Case In case 5, like in case 4, we assume $P[a, b]$ is a non-reversing path with respect to W' .

Because we are not in **case 2**, we know that *every* node $v \in W \setminus W'$ has the property that *all* paths in \mathcal{R} which contain $\{a, b, v\} \cup \text{trap}(v)$ have b before a . Moreover, because we are not in **case 1**, we know there exists a node $v \in W \setminus W'$ such that there is a path in \mathcal{R} containing $\{a, b, v\} \cup \text{trap}(v)$ where a, b, v appear in one of the following two orders: $v \rightarrow b \rightarrow a$ or $b \rightarrow a \rightarrow v$. Fix such a v . Our goal is to perform subpath swaps such that afterwards \mathcal{R} has a path that contains every node in $W' \cup \{v\}$.

Since we are not in **case 4**, for every choice of $u, w \in W'$ where u is before w on P and no node of W' falls between u and w on P , there exists a path in \mathcal{R} through u, w , and v in which u appears before w . Moreover, because we are not in **case 2**, we can assume that v does not appear between u and w on such a path. It follows that for any choice of such u and w , there is a path in \mathcal{R} with the ordering $v \rightarrow u \rightarrow w$ or $u \rightarrow w \rightarrow v$.

Notice that we have reached the same conclusion for the nodes u, w as we have for the nodes b, a . Because of this, it will be convenient to imagine the path P as a cycle in the following sense. Enumerate the nodes in W' in the order that they appear on P as $a = x_1, x_2, \dots, x_{|W'|} = b$, and interpret the indices modulo $|W'|$, so that $x_i = x_{i \pmod{|W'|}}$ whenever $i > |W'|$. In the language of this notation, we have shown that for all i , there is a path in \mathcal{R} with the ordering $v \rightarrow x_i \rightarrow x_{i+1}$ or $x_i \rightarrow x_{i+1} \rightarrow v$. We call the former type a *beginning path* and the latter type an *ending path*, referring to the position of v relative to x_i and x_{i+1} on the respective paths.

In the next step of the argument, we consider for which values of i the nodes x_i, x_{i+1} admit a beginning path, for which values of i they admit an ending path, and for which values of i they admit both.

Let s be such that x_{s+1} is the node in W' with minimum distance from v . Then no path in \mathcal{R} through x_s, x_{s+1} , and v can be a beginning path, because this would mean that $\text{dist}(v, x_s) < \text{dist}(v, x_{s+1})$. Thus, \mathcal{R} must have an ending path through x_s, x_{s+1} , and v .

Symmetrically, by considering the node in W' with minimum distance to v , we conclude that there exists an i such that there is no ending path through x_i, x_{i+1} .

Let t be the integer greater than s that minimizes $t - s \pmod{|W'|}$ such that there is no ending path through x_t, x_{t+1} , and v (such a t exists due to the previous paragraph). Thus, \mathcal{R} must have a beginning path through v, x_t, x_{t+1} . Now, define our set K of *critical nodes* as

$$K = \{a, b, x_s, x_{s+1}, x_t, x_{t+1}, v\} \cup \text{trap}(v). \quad (1)$$

By property (\dagger) there exists a path $Q \in \mathcal{R}$ containing all of the nodes in K .

We now proceed with casework on the relative orderings of these nodes on Q .

Suppose that x_s occurs before x_{t+1} on Q . We begin by proving the following claim.

Claim 1. *Node v is between x_s and x_{t+1} on Q .*

Proof. Suppose for sake of contradiction that Q passes through the nodes from the claim statement either in the order $v \rightarrow x_s \rightarrow x_{t+1}$ or in the order $x_s \rightarrow x_{t+1} \rightarrow v$.

We claim that the nodes x_{s+1} and x_t must appear (in that order) between x_s and x_{t+1} on Q . To see this, consider the following cases: If x_s comes before x_{t+1} on P then the order of these nodes on P is $x_s \rightarrow x_{s+1} \rightarrow x_t \rightarrow x_{t+1}$, in which case apply Lemma 5.1. On the other hand, if x_{t+1} comes before x_s on P then there are two possible orderings of these nodes on P : $x_t \rightarrow x_{t+1} \rightarrow x_s \rightarrow x_{s+1}$, in which case apply Lemma 5.7, or $x_{t+1} \rightarrow x_s \rightarrow x_{s+1} \rightarrow x_t$, in which case apply Lemma 5.1.

Thus, in the first case where Q has the ordering $v \rightarrow x_s \rightarrow x_{t+1}$, Q must have the ordering $v \rightarrow x_s \rightarrow x_{s+1} \rightarrow x_{t+1}$, so there is a path in \mathcal{R} with the ordering $v \rightarrow x_s \rightarrow x_{s+1}$, which contradicts our choice of s . Similarly, in the second case where Q has $x_s \rightarrow x_{t+1} \rightarrow v$, Q must have $x_s \rightarrow x_t \rightarrow$

$x_{t+1} \rightarrow v$, so there is a path in \mathcal{R} with the ordering $x_t \rightarrow x_{t+1} \rightarrow v$, which contradicts our choice of t . ■

Next, we prove a claim that we can perform subpath swaps to obtain a path in \mathcal{R} through $x_s \rightarrow x_{s+1} \rightarrow \cdots \rightarrow x_t \rightarrow v$, where the dots indicate all x_i between x_{s+1} and x_t :

Claim 2. *For any index j such that $s < j \leq t$, we can perform subpath swaps so that as a result \mathcal{R} has a path with the ordering $x_s \rightarrow x_{s+1} \rightarrow \cdots \rightarrow x_j \rightarrow v$ (where the dots indicate all x_i between x_{s+1} and x_j).*

Proof. We proceed by induction on j .

The base case of $j = s + 1$ holds simply because \mathcal{R} has an ending path through x_s, x_{s+1} , and v (in that order) by choice of s .

For the inductive step, suppose that for some index j such that $s < j < t$, \mathcal{R} has a path P' with the ordering $x_s \rightarrow x_{s+1} \rightarrow \cdots \rightarrow x_j \rightarrow v$. By our choice of t , we know that \mathcal{R} has an ending path P'' passing through x_j, x_{j+1} , and v (in that order). Now, we simply swap $P'[x_j, v]$ with $P''[x_j, v]$. As a result, P' has the ordering $x_s \rightarrow x_{s+1} \rightarrow \cdots \rightarrow x_{j+1} \rightarrow v$.

This completes the induction. ■

Let P' be the path from Claim 2 with $j = t$. Swap $P'[x_s, v]$ with $Q[x_s, v]$; we can do so because x_s falls before v on Q by Claim 1. As a result, Q contains the ordering

$$x_s \rightarrow x_{s+1} \rightarrow \cdots \rightarrow x_t \rightarrow v \rightarrow x_{t+1}.$$

Thus, $x_t \rightarrow v \rightarrow x_{t+1}$ is a shortest-path ordering. However, from the definition of t , \mathcal{R} has a beginning path through v, x_t, x_{t+1} , so $v \rightarrow x_t \rightarrow x_{t+1}$ is also a shortest-path ordering. These two orderings contradict Lemma 5.1. Thus, case 5(a) is impossible in the first place.

Case 5(b): x_{t+1} before x_s

Since case 5(a) cannot happen, we know that x_{t+1} appears before x_s on Q .

First, we determine the relative ordering of x_{s+1} and x_t on Q with the following claim:

Claim 3. *Node x_{s+1} appears before x_t on Q .*

Proof. Suppose for the sake of contradiction that x_t appears before x_{s+1} on Q .

By Lemma 5.7 we know that the ordering of $x_s, x_{s+1}, x_t, x_{t+1}$ on any path in \mathcal{R} containing

$$\{x_s, x_{s+1}, x_t, x_{t+1}, a, b\}$$

must be equivalent to

$$x_s \rightarrow x_{s+1} \rightarrow x_t \rightarrow x_{t+1}$$

up to cyclic reordering.

Applying the assumption of case 5(b) that x_{t+1} appears before x_s on Q , and the assumption (for contradiction) that x_t appears before x_{s+1} on Q , we conclude that Q passes through these nodes in the order

$$x_t \rightarrow x_{t+1} \rightarrow x_s \rightarrow x_{s+1}.$$

Now, where can v be located on Q , relative to the above nodes? By our choice of s , we know that v cannot occur before x_s , because if it did then Q would go through $v \rightarrow x_s \rightarrow x_{s+1}$ and thus be a beginning path. Thus, v must occur after x_s . Thus, v occurs after x_{t+1} . But then Q goes through $x_t \rightarrow x_{t+1} \rightarrow v$ as an ending path. This contradicts our choice of t and completes the proof. ■

With Claim 3 established, we know that x_{s+1} occurs before x_t , and x_{t+1} occurs before x_s on Q . Additionally, recall from the first paragraph of case 5 that because we are not in **case 2**, b falls before a on Q .

Consider the position of b on the cyclic order given by the x_i 's. There are 3 cases:

- (i) $b = x_s$ (and thus $a = x_{s+1}$) or $b = x_t$ (and thus $a = x_{t+1}$),
- (ii) b is between x_{s+1} and x_{t-1} (inclusive), or
- (iii) b is between x_{t+1} and x_{s-1} (inclusive).

Each of these 3 cases are similar, and for each we will apply Lemma 5.5 (the Critical Node Lemma). For all 3 instantiations of Lemma 5.5, T will be set to K where K is defined in Equation (1), W^* will be set to $W' \cup \{v\}$, and U will be set to $\text{trap}(v) \cup \{v\}$. The pairs (a_i, b_i) will be different for each of the 3 cases. The consequence of Lemma 5.5 for all 3 cases is that \mathcal{R} has a path containing every node in $W' \cup \{v\}$, as desired. Thus, it remains to define the pairs (a_i, b_i) for each of the 3 cases and verify that the assumptions of Lemma 5.5 hold for each case.

In case 1, (a_1, b_1) and (a_2, b_2) are defined as (x_{t+1}, x_s) and (x_{s+1}, x_t) . We will now verify that the assumptions of Lemma 5.5 hold. For the first assumption, Q was chosen to be an arbitrary path in \mathcal{R} containing every node in K , so the properties we proved for Q are indeed true for every such path. For the second assumption, $P[x_{t+1}, x_s] \cup P[x_{s+1}, x_t]$ by definition contains every node in W' , and these subpaths are non-overlapping.

In case 2, (a_1, b_1) , (a_2, b_2) , and (a_3, b_3) are defined as (a, x_t) , (x_{t+1}, x_s) , and (x_{s+1}, b) (where the last pair is not included if $b = x_{s+1}$). We will now verify that the assumptions of Lemma 5.5 hold.

For the first assumption, a appears before x_t on Q because otherwise Q would have the ordering $x_{s+1} \rightarrow x_t \rightarrow a$, where b is between x_t and a because b cannot fall before x_{s+1} or x_t due to the orderings imposed by Lemma 5.4 for non-reversing paths. This contradicts the ordering on P of $a \rightarrow x_t \rightarrow x_{s+1} \rightarrow b$ by Lemma 5.1. Similarly, x_{s+1} appears before b for a symmetric reason: otherwise Q would have the ordering $b \rightarrow a \rightarrow x_{s+1} \rightarrow x_t$ which again contradicts the ordering of these nodes on P . Q was chosen to be an arbitrary path in \mathcal{R} containing every node in K , so the properties we proved for Q are true for every such path.

For the second assumption, $P[a, x_t] \cup P[x_{t+1}, x_s] \cup P[x_{s+1}, b]$ by definition contains every node in the set W' , and these subpaths are non-overlapping.

Case 3 is precisely the same as case 2 but with s and t switched.

This completes case 5, which completes the proof of Lemma 2.8.

Acknowledgements

We thank Virginia Vassilevska Williams for helpful discussions related to this work.

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A Related Work on Disjoint Paths Problems

The problem of finding disjoint paths in a graph is a classical problem, which has been studied extensively (e.g., for applications in VLSI layout [SLK⁺90]). In the standard formulation of the problem, we are given node pairs $(s_1, t_1), \dots, (s_k, t_k)$ and the goal is to find k node-disjoint paths between each s_i and its corresponding t_i . When k is part of the input, this problem is known to be NP-complete [EIS76]. For fixed k in undirected graphs, Robertson and Seymour [RS95] showed that this problem can be solved in $O(n^3)$ time. This running time has since been improved to $O(n^2)$ [KKR12].

For directed graphs, this problem is much harder. Even for $k = 2$, the problem is NP-complete [FHW80]. However, it can be solved in polynomial time for fixed k on certain special classes of directed graphs, such as DAGs [FHW80], planar graphs, [Sch94b, CMPP13], tournament graphs [CSS15], and graphs of bounded treewidth [Sch94a]. These results on restricted graph classes, besides being interesting in their own right, have also found applications in showing that certain “detour” problems are fixed-parameter tractable on those same restricted graph classes [FGL⁺22].

It is natural to try to *minimize the lengths* of the disjoint paths. There are several ways to define this, such as the *min-sum* version where the goal is to minimize the sum of the lengths of the disjoint

paths, or the *min-max* version where the goal is to minimize the maximum length of a path in the solution. Because these variants are generalizations of the original problem, they remain NP-complete for directed graphs even for $k = 2$. However, even for undirected graphs, these problems pose barriers. The min-max version is NP-hard even for $k = 2$ [LMS90]. The min-sum version is in polynomial time for $k = 2$ [BH14] and its complexity is open for $k \geq 3$. Considerable effort has been put into solving the min-sum version for $k \geq 3$, however the most general graph classes for which it is known to be in polynomial time are planar graphs with certain additional restrictions on the placement of the terminals relative to the faces [KS10, dVS11, BNZ15, DIKM18].

With the intractability of the above length minimization variants in mind, it is natural to study the k -disjoint shortest paths problem (k -DSP) which imposes that *all paths* in the solution must be shortest. See Section 1.2 for the known results on k -DSP.

B Counterexample for Generalizing Lemma 2.3 to Directed Graphs

Recall Lemma 2.3:

Lemma 2.3 ([AW20]). *If $k > 3d$, then any instance of (k, c) -SPC on DAGs either has no solution, or has a solution where some solution path P_i passes through all max-congestion nodes.*

We will show a counterexample for extending Lemma 2.3 to general directed graphs. The counterexample graph is simply a directed cycle on n nodes. Let $k = n$ and let $c = 3n/4 + 1$, so $d = n/4 - 1$, so indeed the assumption $k > 3d$ holds. The input pairs $(s_1, t_1), \dots, (s_k, t_k)$ for the (k, c) -SPC instance are defined as follows. Suppose the nodes are numbered from 1 to n around the cycle. Then, for all i from 1 to n , s_i is defined as node i in the cycle, and t_i is node $i + 3n/4 \pmod{n}$. Notice that there is only one path from each s_i to the corresponding t_i . Taking these paths together, for all i , forms a valid (k, c) -SPC solution where every node has congestion $c = 3n/4 + 1$; that is, every node is a max-congestion node. This is the only solution because there is a unique shortest path between each pair (s_i, t_i) . That is, there does not exist a solution where some solution path P_i passes through all max-congestion nodes. This completes the counterexample. The parameters can be adjusted to allow for a constant other than 3 in the assumption $k > 3d$.

C Lemma 2.5 implies Lemma 1.4

We recall the statements of Lemmas 1.4 and 2.5 below:

Lemma 2.5. *If $k > 4d$, then any instance of (k, c) -SPC either has no solution, or has a solution where some solution path P_i passes through all max-congestion nodes.*

Lemma 1.4. *If k -DSP can be solved in $f(n, k)$ time on undirected graphs, then (k, c) -SPC can be solved in $O\left(\binom{k}{4d} \cdot f(3dn, 4d)\right)$ time on undirected graphs.*

The proof that Lemma 2.5 implies Lemma 1.4 is almost the same as the analogous proof for DAGs in [AW20].

Proof of Lemma 1.4 given Lemma 2.5. The following lemma is a generalization of [AW20, Lemma 5] which applies to undirected graphs instead of DAGs. Its proof is identical to the proof in [AW20], except that one repeatedly applies Lemma 2.5 instead of applying an analogous claim for DAGs, and $(3d, 2d)$ is replaced with $(4d, 3d)$.

Lemma C.1. *Let $k > 4d$. Then any instance of (k, c) -SPC either has no solution, or has a solution consisting of paths P_1, \dots, P_k such that there exists a subset $I \subseteq \{1, \dots, k\}$ of size $4d$ so that the paths $\{P_i\}_{i \in I}$ solve the $(4d, 3d)$ -SPC problem with sources $\{s_i\}_{i \in I}$ and targets $\{t_i\}_{i \in I}$.*

We also need the following lemma from [AW20] which observes that (k, c) -SPC can be reduced to k -DSP on a slightly larger graph.

Lemma C.2 ([AW20, Corollary 2]). *If k -DSP can be solved in $f(n, k)$ time, then (k, c) -SPC can be solved in $f(cn, k)$ time.*

Now we will use these lemmas to prove the result.

If $k \leq 4d$, then $c \leq 4d$ as well. Thus, by Lemma C.2, we can solve the given (k, c) -SPC instance in $f(n, 4d)$ time, which certainly implies the desired result.

Otherwise, $k > 4d$. In this case, by Lemma C.1 we know that if the (k, c) -SPC instance is solvable, it must have a solution where some subset of $4d$ (source, target) pairs solve a $(4d, 3d)$ -SPC instance. If we find a set of $4d$ paths solving such an instance, then we can easily complete it to a solution to the original (k, c) -SPC problem simply by finding any shortest paths between the remaining source and target nodes. This works because the congestion of the resulting k paths is at most $3d + (k - 4d) = k - d = c$.

Thus, we can solve the (k, c) -SPC instance by trying out all $\binom{k}{4d}$ possible subsets of $4d$ (source, target) pairs, attempting to solve $(4d, 3d)$ -SPC with respect to these terminal nodes, and if we find a solution, conclude by finding $k - 4d$ shortest paths between the remaining terminal nodes (e.g., by Dijkstra's algorithm). By Lemma C.2, each of the $(4d, 3d)$ -SPC instances constructed in this way can be solved in $f(3dn, 4d)$ time. It follows that we can solve the original problem in $O(\binom{k}{4d} f(3dn, 4d))$ time as desired. ■