

**A LONGITUDINAL STABILITY CRITERION
FOR BUNCHED BEAMS**

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Summary

The unstable motion of a bunched beam consists of rigid-bunch (dipole) and higher bunch-shape oscillations of the individual bunches (individual-bunch modes), plus perhaps coupled motion of the different bunches (coupled-bunch modes). Stability is achieved either by decoupling the bunches or by a spread in synchrotron frequencies within a bunch. A stability criterion analogous to the Keil-Schnell criterion¹ for coasting beams is given which includes the effect of a beam interacting with perfectly conducting walls, resistive walls, and resonant structures. Some examples for the CERN accelerators are included.

Introduction

Most previous work has concentrated on rigid-bunch motion, driven by resonant elements in the vacuum chamber, cavities, or the wall resistance.²⁻⁹ However, the higher modes can also be excited.¹⁰⁻¹² For example, a resonator of sufficiently high frequency that one or more oscillations occur during the passage of a bunch will excite primarily the higher modes. Also, the space-charge force depends on the variation of the line density within a bunch, and therefore increases with mode number. On the other hand, the resistive-wall wake decays relatively little over one bunch and is insensitive to the density variations of the higher modes; it drives mostly the rigid-bunch mode, as in the transverse case.

This paper presents a general and hopefully easy to use stability criterion for the higher modes of oscillation as well as the rigid-bunch mode. Four ingredients are necessary:

- i) oscillation modes
- ii) coherent frequency shifts $\Delta\omega$
- iii) decoupling criterion
- iv) stability criterion.

The derivations are given in another paper,¹³ and only the results are presented here.

Classification of modes

For bunched beams, the dominate force is the external synchrotron force, and the particle motion is approximately circular in the normalized $z - z/\omega_s$ phase plane (Fig. 1). An exactly circular distribution $g_0(r)$ is stationary, while small oscillations about the stationary distribution have the form

$$g(r, \phi, t) = R_m(r) e^{-im\phi} e^{-i\omega t} \quad (1)$$

and oscillate with the frequency

$$\omega = m\omega_s + \Delta\omega_m, \quad (2)$$

where ω_s is the synchrotron frequency, and $m = 1$ for dipole modes, $m = 2$ for quadrupole modes, etc. If Eq. (1) is inserted into the linearized Vlasov equation, an integral equation results for the radial mode patterns $R_m(r)$ and the coherent frequency shifts $\Delta\omega_m$. For the cases considered here, and probably in general, the solutions have the form

$$R_m(r) \sim r^{m-1} \frac{dg_0}{dr} \quad (3)$$

provided frequency spreads are neglected. A few of the low-order oscillation modes are sketched in Fig. 1.

In addition, coupled motion of the different bunches occurs if their oscillation frequencies are nearly equal. For M identical bunches, there are M coupled-bunch modes of oscillation. These are designated by the index n , which specifies the phase difference $2\pi n/M$ between adjacent bunches.

Notice that two indices are necessary to describe the complete oscillation: m specifies the type of oscillation the individual bunches are undergoing, while n describes how these individual-bunch modes are linked together in the larger coupled-bunch pattern. This convention for the indices m and n will be observed in the following.

Given the shape of the modes and the beam-equipment coupling impedance $Z_{||}(\omega)$, a straightforward procedure exists for computing $\Delta\omega_m$.¹⁴ Results for perfectly conducting walls, resistive walls, and resonators are given in the next section.

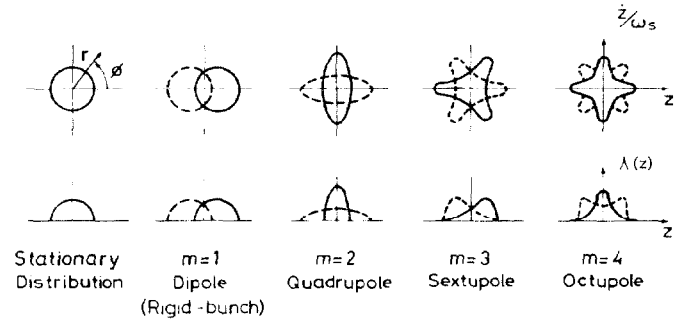


Fig. 1

Growth-rates in the absence of frequency spreads

The growth-rate is

$$\frac{1}{\tau} = \text{Im } \Delta\omega_m, \quad (4)$$

and the motion is unstable if $\text{Im } \Delta\omega_m$ is positive. It is convenient to write the expressions for the frequency shifts in the form

$$\frac{\Delta\omega_m}{\omega_s} = \frac{ZI}{V \cos \phi_s} \times \text{form factors}, \quad (5)$$

where Z is a characteristic impedance for the interaction in question; V is the peak accelerating voltage per turn; ϕ_s is the synchronous phase, with the convention that $V \cos \phi_s$ is positive below transition and negative above.

Perfectly conducting walls

For a bunch with approximately parabolic line density,

$$\Delta\omega_m = \sqrt{m} \Delta\omega_{sc}, \quad (6)$$

where

$$\frac{\Delta\omega_{sc}}{\omega_s} = 0.152 \frac{|Z_{sc}/k| I M^2}{V \cos \phi_s hB^3}, \quad (7)$$

$$|Z_{sc}/k| = \frac{g_c}{2\beta\gamma^2} Z_0,$$

and

$$g_c = 1 + 2 \ln (\text{vacuum chamber radius/beam radius}).$$

Z_{sc} is the usual longitudinal coupling impedance for mode k ; $Z_0 = 377 \text{ ohm}$; I is the total current in M bunches; h is the RF harmonic number (usually equal to M); B is the bunching factor (bunch length/bunch separation).

Note that the frequency shift $\Delta\omega_m$ is real, depends strongly on the bunching factor, and increases with the square root of the mode number m . (The analogous frequency shift for coasting beams increases linearly with mode number.) The square-root dependence on mode number is characteristic for bunched beams, and is apparently due to the r^m factor in the radial distribution (3), which constrains the motion more and more to the beam edge.

Resistive walls

The effect of a smooth round vacuum chamber on the dipole mode is

$$\frac{\Delta\omega_1}{\omega_s} = 0.0134 \frac{|Z_{\text{skin}}| I M^{3/2}}{V \cos \phi_s h} G \left(\frac{Q_s + n}{M} \right) \quad (8)$$

where

$$|Z_{\text{skin}}| = \frac{1}{\sqrt{2}} \beta \frac{c}{b} Z_c,$$

and c/b is the ratio of skin depth at the revolution frequency ω_0 to the vacuum chamber radius; Q_s is the number of synchrotron oscillations per revolution, $Q_s = \omega_s/\omega_0$; and G is a bunch function analogous to the one defined by Courant and Sessler for transverse motion:¹⁵

$$G(Q) = \sum_{s=1}^{\infty} \frac{e^{2\pi i s Q}}{s^{3/2}}. \quad (9)$$

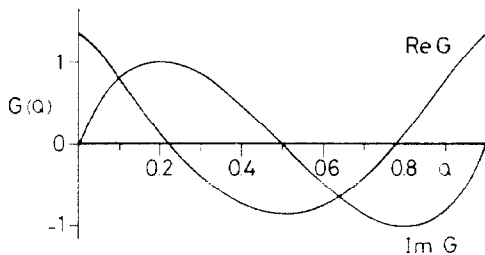


Fig. 2

Its maximum real or imaginary part is about unity, and $\text{Im } G$ is positive if its argument is less than one-half (Fig. 2). Thus for single-bunch motion ($M = 1$, $n = 0$, and $I \rightarrow I/M$), the bunches are unstable below transition and stable above, assuming that $Q_s < 1/2$. For coupled motion, about half the modes are unstable in all cases. Robinson³ has found a similar result, except with $s^{3/2}$ instead of $s^{5/2}$ in Eq. (9). I think Eq. (9) is correct, but in any case the difference is negligible.

The frequency shifts for the higher modes are estimated to be about

$$\Delta\omega_m \sim B^m \Delta\omega_1,$$

where B is the bunching factor.

Resonator

We assume a cavity or resonant element characterized by a shunt or parallel resistance R_s , resonant frequency f_{res} (or radian frequency ω_{res}), and quality factor Q . Transit time factors are ignored. Then

$$\frac{\Delta\omega_m}{\omega_s} = 0.159 \frac{R_s I}{V \cos \phi_s} \frac{M}{Bh} D F_m(\Delta\phi). \quad (10)$$

F_m is a form factor that specifies the efficiency with which the resonator can drive a given mode. It depends on the phase change $\Delta\phi$ that occurs during the passage of a bunch (see Fig. 3):

$$\Delta\phi = 2\pi f_{\text{res}} \times \text{bunch length in seconds}.$$

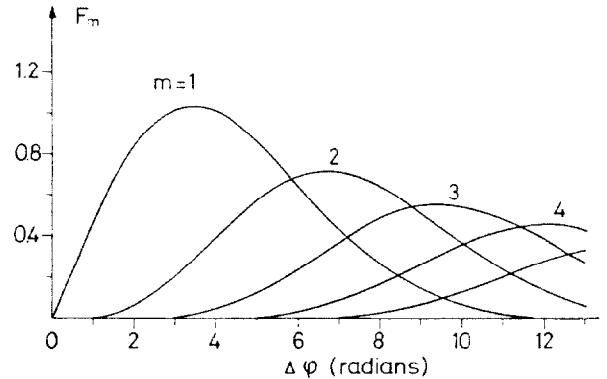


Fig. 3

The maximum value of F_1 for the dipole mode occurs when $\Delta\phi \approx \pi$ so that an approximately linear waveform acts on the bunch. Similarly, the quadrupole or breathing mode is most efficiently driven when $\Delta\phi$ is near 2π , and so on for the higher modes. In general, mode m is most efficiently driven when the resonator frequency is $f_{\text{res}} = m f_{\text{crit}}$, where

$$f_{\text{crit}} = \frac{M}{2B} f_0$$

is the most efficient frequency for driving dipole modes (f_0 is the revolution frequency in Hz). For these frequencies, the maximum value of F_m is approximately $1/\sqrt{m}$.

The factor D in Eq. (10) depends on the attenuation $e^{-\alpha}$ of the induced signal between bunches, where

$$\alpha = 2\pi \delta f \times \text{time between bunch centres} \\ = \frac{\Delta\phi}{2QB}$$

and $\delta f = f_{\text{res}}/2Q$ is the bandwidth of the resonator.

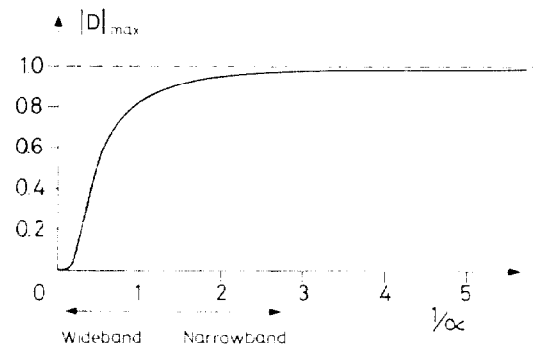


Fig. 4

The maximum value of $|D|$ is about unity for narrow-band resonators, with little attenuation between bunches, and approaches zero for wide-band resonators (Fig. 4).

Narrow band, $\alpha \ll 1$:

$$D \approx \frac{\delta f}{f_{\text{res}} - \text{integer} \times f_0 - i \delta f} \quad (11)$$

and the coupled-bunch mode n is excited when

$$f_{\text{res}} \approx \text{integer} \times M f_0 \pm n f_0 .$$

Wide band, $\alpha \gg 1$:

$$D \approx -2\alpha e^{-\alpha} e^{2\pi i n / M} \sin(2\pi f_{\text{res}} / M f_0) , \quad (12)$$

and coupled-bunch modes near $n = \pm M/4$ (phase difference between bunches of $\pm \pi/2$) are most strongly excited.

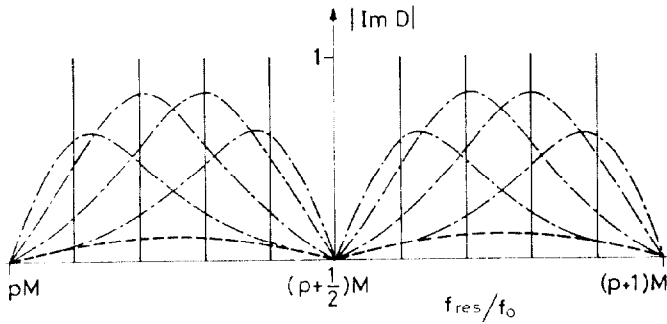
General case, any α :

$$D = i\alpha \left[\frac{1}{1 - e^{x_+}} - \frac{1}{1 - e^{x_-}} \right] , \quad (13)$$

where

$$x_{\pm} = \frac{2\pi i}{M} \left(n \pm \frac{f_{\text{res}}}{f_0} \right) - \alpha .$$

Examples are shown in Fig. 5.



$p =$ arbitrary integer

- wide-band resonance with attenuation e^{-4} between bunches
- · - · - intermediate case with attenuation e^{-1} between bunches
- narrow-band resonance with no attenuation between bunches

Fig. 5 $|\text{Im } D|$ versus frequency for the hypothetical case of 10 bunches ($M = 10$)

In all cases, modes $n = 0$ and $n = M/2$ (if M is even) are not excited. Also, if the bunches are decoupled from one another, the motion is stable because $n = 0$ for a single bunch. (Actually, the small phase change per revolution due to the synchrotron oscillation results in a weak instability for mode $n = 0$, but this is negligible compared with the other modes and has been neglected here.)

Decoupling criterion

The basic mechanism for decoupling the bunches is a spread in the frequencies of the individual bunches. A rule-of-thumb for decoupling is that the r.m.s. spread in individual bunch frequencies should exceed the frequency shift $\Delta\omega_m$ due to the coupling force:¹⁶

$$\text{spread} > \text{shift} . \quad (14)$$

If a beam control system is acting, somewhat larger spreads are required.⁸ The spread may be induced externally by modulating the RF voltage, or may arise naturally from a difference ΔN in bunch populations. In the last case, a convenient criterion for decoupling is

$$|\Delta\omega_{\text{sc}}| \left(\frac{\Delta N}{N} \right)_{\text{rms}} > |\Delta\omega'_m| , \quad (15)$$

where the prime indicates that the space-charge shift (6) should be omitted (it does not contribute to the coupling). Since $(\Delta N/N)_{\text{rms}}$ is usually less than 5% (or a full spread of 20%), large space-charge shifts are required for decoupling.

Stability criterion

If within-bunch frequency spreads are taken into account, the following dispersion relation can be derived for the different modes:

$$1 = \frac{\Delta\omega_m}{W_m} \int_0^\infty \frac{r^{2m}}{\omega - m\omega_s(r)} \frac{dg_0}{dr} dr , \quad (16)$$

where

$$W_m = \int_0^\infty r^{2m} \frac{dg_0}{dr} dr ,$$

and $\Delta\omega_m$ is the sum from all interactions, space charge, resistive wall, resonators, etc. We define S as the spread in ω_s between centre and edge of the bunch (full spread) due to the non-linearity of the synchrotron force; it is plotted in Fig. 6 as a function of the bunching factor and the parameter $\Gamma = \sin \phi_s$.

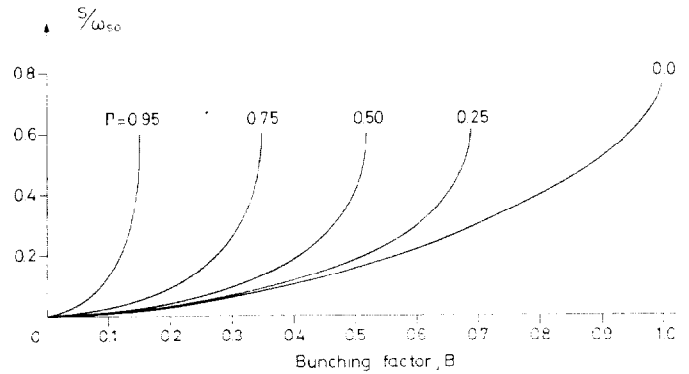


Fig. 6

Equation (16) specifies the stable regions in the complex $\Delta\omega_m/S$ plane, and these are plotted in Fig. 7 for the smooth distribution

$$g(r) \propto (1 - r^2)^2 \quad (17)$$

with zero slope at the beam edge. Following the example of Keil and Schnell¹ for coasting beams, we can approximate the stability boundary by semicircles to give

$$S > \frac{4}{\sqrt{m}} |\Delta\omega_m| \quad (18)$$

for stability. This is analogous to the coasting-beam criterion

$$(\text{full spread at base}) > \frac{4}{k} |\Delta\omega_k|$$

for mode k . The criterion (18) has been derived previously¹⁷ for the rigid-bunch mode ($m = 1$).

For space charge, $\Delta\omega_m$ increases as \sqrt{m} and therefore the threshold is the same for all modes, just as for coasting beams. For a resonator, $\Delta\omega_m$ decreases as $1/\sqrt{m}$ and therefore the higher modes require less frequency spread. Stated differently, m times more shunt resistance is required to excite mode m at its critical frequency $m f_{\text{crit}}$ than is necessary to excite the dipole mode at its critical frequency. However, once the threshold is exceeded, either by the action of space charge or the resonator itself, the growth rates for the higher modes decrease only as $1/\sqrt{m}$.

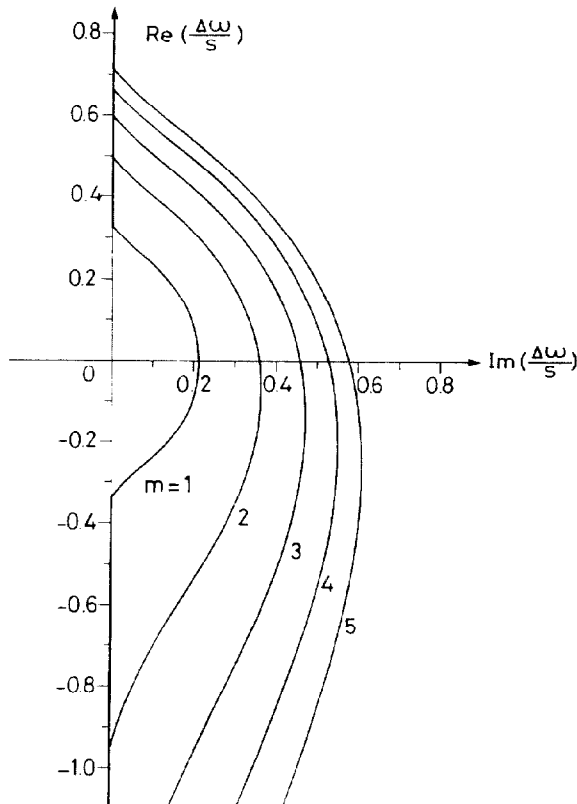


Fig. 7 Stability diagram. Regions to the left of the contour are stable for the indicated mode, regions to the right are unstable.

Examples

Some examples for the CERN machines are given in Table 1, including resonator shunt impedances necessary to drive dipole modes, namely

- R_∞ to reach threshold
- R_{50} for 50 msec growth time
- R_{10} for 10 msec growth time.

These are computed assuming that the quality factor is larger than $1/(\text{bunching factor})$ and that $f_{\text{res}} \approx f_{\text{crit}}$, so that the form factors D and F in Eq. (10) are about unity. R_∞ is determined by

$$|\Delta\omega_{\text{sc}}| + |\Delta\omega_1 \text{ due to resonator}| = \frac{1}{4} S,$$

and R_{50} by

$R_{50} = R_\infty + \delta R$ required to give $|\Delta\omega_1| = 20 \text{ rad/sec}$, and similarly for R_{10} . The resistive-wall growth rates are negligible ($Z_{\text{skin}} \sim 1 \text{ to } 10 \Omega$), and are not included.

PSB (Booster): Because of the large space-charge force, all modes are well outside the stable region during most of the acceleration cycle. Also, only small shunt resistances are required to drive dipole modes, 70Ω for 50 msec e-folding time and 350Ω for 10 msec. On the other hand, it is unlikely that elements exist in the machine with such small resonant frequencies. One expects frequencies of 30 MHz and above, which will drive the dipole modes with reduced efficiency, or drive higher modes. To drive mode $m = 4$ with an e-folding time of 10 msec requires $\sqrt{m} \times 350 = 700 \Omega$. Also the revolution frequency changes by a factor of $2\frac{1}{2}$ during the cycle so that many resonance regions $f_{\text{res}}/f_0 \approx \text{integer}$ are swept through.

In fact, dipole and higher bunch-shape oscillations occur, but it is too early to decide if they are due to bugs in the RF system or to beam-equipment interactions.

PS (1970): We concentrate on the region after transition when the revolution frequency is approximately constant. With a constant accelerating voltage of 115 kV, the RF bucket is large compared with the bunch, and the frequency spreads are insufficient to maintain stability. A strong dipole instability with growth time of 10 msec was observed when electrostatic septum tanks (2) were installed. One was measured (by H.H. Umstätter) and found to have $R_s = 18 \text{ k}\Omega$, $f_{\text{res}} \approx 60$ to 90 MHz depending on the position of the septa, and $Q \approx 700$. This was cured by means of damping resistors, but a slower dipole instability remained with growth times of about 50 msec. This is probably due to a parasitic resonance in the 14 RF cavities with $R_s = 14 \times 800 = 11.2 \text{ k}\Omega$, f_{res} spread between 46 and 51 MHz, and $Q \approx 20$. The computed e-folding time of 28 msec using Eq. (10) and assuming a single resonant frequency of 48 MHz is in reasonable agreement with observation. More exact computations using the measured impedance curves and including the effect of the beam control system are reported in Refs. 6, 8, and 9. The present cure is to reduce the size of the bucket by voltage reduction.

PS (1972): The threshold impedance increases from zero to $10 \text{ k}\Omega$ when the voltage is reduced until 85% of the bucket is filled. At present, no instabilities are observed.

Future PS: At 10^{13} particles, voltage reduction may not be sufficient, even allowing for a 50% increase in longitudinal emittance. A feedback system is being considered.⁹

ISR: At present, the bucket is very large compared with the bunch, so frequency spreads are small and space charge is sufficient to move all modes outside the stable region. A relatively large impedance of $15 \text{ k}\Omega$ is required for a 50 msec growth time, but when this is divided by the coasting-beam mode number $k = f_{\text{res}}/f_0 = 97$, we find

$$R_{50}/k = 155 \Omega,$$

which is in the range of possible impedances. This ratio is even smaller for the higher modes since R_s scales as \sqrt{m} while k scales as m :

$$R_{50}/k = 155 \Omega/\sqrt{m}.$$

In fact, higher bunch-shape oscillations are observed and lead to a doubling of the longitudinal emittance. Voltage reduction should cure this.

SPS: A threshold impedance of about $1 \text{ M}\Omega$ is required for instability.

Table 1: Computed parameters for dipole modes of instability

Machine	E (GeV)	V (kV)	$\sin \phi_s$	f_s (Hz)	g_c	B	$S/4\omega_s$	$\Delta\omega_{sc}/\omega_s$	f_{crit} (MHz)	R_{∞} (k Ω)	R_{s0} (k Ω)	R_{l0} (k Ω)
PSB	0.05	12	0.086	5470	2.3	0.78	0.102	0.041	1.9	15.0	15.0	15.0
	0.45	12	0.086	3190	3.5	0.51	0.044	0.113	6.9	0	0.068	0.34
	0.80	12	0.086	2200	3.9	0.41	0.028	0.150	9.8	0	0.069	0.35
PS (1970)	1.0	115	0.73	2500	3.7	0.46	0.117	0.007	9.0	250	250	270
	10.0	115	0.73	318	5.5	0.13	0.009	0.017	37.6	0	5.5	27
	24.0	115	0.73	237	6.4	0.11	0.006	0.005	43.8	0.51	6.9	32
PS (1972)	10	91.5	0.91	218	5.5	0.16	0.049	0.017	29.4	10	15	35
	24	89.9	0.93	156	6.4	0.14	0.048	0.006	33.0	11	16	38
Future PS	10	94.7	0.88	239	5.5	0.19	0.048	0.069	25.1	0	0.93	4.6
	24	92.4	0.91	170	6.4	0.17	0.048	0.020	28.1	1.5	2.6	6.7
ISR	10	16	0.05	65	4.8	0.16	0.004	0.011	30.7	0	15	75
	24	16	0.05	58	5.2	0.15	0.004	0.006	32.4	0	16	79
SPS	50	3380	0.77	287	4.6	0.31	0.063	-	321	4200	5000	8000
	200	3380	0.77	160	6.0	0.21	0.029	-	472	1300	2200	6000
	300	3380	0.77	131	6.4	0.19	0.023	-	526	960	2000	6000
	400	3380	0.77	114	6.7	0.18	0.020	-	567	770	1870	6000

Table 2: Fixed parameters used for computing Table 1

Machine	No. of particles $\times 10^{12}$	Bunch area eV \cdot sec	mrad	B Tesla/sec
PSB	2.5 per ring	0.156	10	0.80
PS	1.5	0.156	10	1.9
Future PS	10.0	0.234	15	1.9
ISR	1.5	0.156	10	0.01
SPS	10.0	0.100	130	0.75

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