A lower bound for sectional genus of quasi-polarized manifolds, II [*†]

Yoshiaki Fukuma

July 10, 2009 version

Abstract

Let (X, L) be a quasi-polarized variety of dimension n. In this paper we investigate a lower bound for the sectional genus $g(L)$ for the following types; (1) A lower bound for the sectional genus of the case in which (f, X, C, L) is a quasi-polarized fiber space, where C is a smooth curve. (2) Non-negativity of te sectional genus of the case where (X, L) is a quasi-polarized manifold or (X, L) is a polarized variety with some singularities. (3) A lower bound for the sectional genus of the case where $\dim X = 3$.

Introduction

Let X be a projective variety over the field of complex numbers $\mathbb{C}$ with $\dim X = n$, and L an ample (resp.a nef and big) line bundle on X. Then (X, L) is called a *polarized* (resp. a *quasi-polarized*) *variety*. Moreover if X is smooth, then (X, L) is called a polarized (resp. quasi-polarized) *manifold*.

When we study polarized varieties, it is useful to use their invariants. The following invariants are well-known.

(1) The degree L^n.

(2) The sectional geuns $g(L)$.

(3) The Δ-genus $\Delta(L)$.

Many authors studied polarized varieties by using these invariants. In particular, P. Ionescu classified polarized manifolds (X, L) for the case where L is very ample and $L^n \leq 8$, and T. Fujita classified polarized manifolds with low sectional genera and low Δ-genera.

[*] *Key words and phrases.* Quasi-polarized manifold, sectional genus.

[†] 2000 *Mathematics Subject Classification.* Primary 14C20; Secondary 14J99.

In this paper, we treat the sectional genus of (X, L). If X is smooth, then the sectional genus of L is defined to be a non negative integer valued function by the following formula ([7]):

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical divisor of X. Here we state some recent results about the sectional genus of quasi-polarized manifold, and propose some conjectures and problems. The following results are known for the fundamental properties of the sectional genus.

(A) The value of $g(L)$ is non-negative integer when L is ample. (Fujita [4], Ionescu [13])

(B) There exist a classification of polarized manifold (X, L) with sectional genus $g(L) \leq 2$. (For example see Fujita [4], [5], Ionescu [13], and Beltrametti-Lanteri-Palleschi [1].)

(C) Let (X, L) be a polarized manifold. Then there exist only finite deformation types of polarized masnifolds unless (X, L) is a scroll over a smooth curve. (For the definition of deformation type of polarized manifolds, see in [7, Chapter II, §13].)

On the other hand, there is the following conjecture which was proposed by T. Fujita.

Conjecture 1 *Let (X, L) be a quasi-polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = h^1(\mathcal{O}_X)$ (called the irregularity of X).*

In [8], we treat the case where $\dim X = 2$. But if $\dim X \geq 3$, the problem seems difficult. So in [9] we considered the following conjecture:

Conjecture 2 *Let (X, L) be a quasi-polarized manifold, Y a normal projective variety with $1 \leq \dim Y < \dim X$, and $f : X \to Y$ a surjective morphism with connected fibers. Then $g(L) \geq h^1(\mathcal{O}_{Y'})$, where Y' is a resolution of Y.*

Of course Conjecture 2 follows from Conjecture 1. The hypothesis of Conjecture 2 is natural because X has a fibration in many cases (Albanese fibration, Iitaka fibration e.t.c.).

In [9] we consider the case where $\dim Y = 1$ or some special cases when $\dim Y \geq 2$. In [9, Theorem 1.2.1] we proved that $g(L) \geq q(Y)$ if $\dim Y = 1$ and L is ample. Furthermore we proved that if $g(L) = q(Y)$, $\dim X \geq 3$, $\dim Y = 1$, and L is ample, then (f, X, Y, L) is a scroll (see [9, Theorem1.4.2]).

In this paper, we mainly consider the case where (X, L) is a polarized variety such that X has some singularities or the case where (X, L) is a quasi-polarized manifold. Concretely, we consider the following cases.

(1) A lower bound for the sectional genus of the case in which (f, X, C, L) is a quasi-polarized fiber space, where C is a smooth curve.

(2) Non-negativity of te sectional genus of the case where (X, L) is a quasi-polarized manifold or (X, L) is a polarized variety with some singularities.

(3) A lower bound for the sectional genus of the case where $\dim X = 3$.

First we study the case where (f, X, Y, L) is a quasi-polarized fiber space over a smooth curve Y, and we proved that $g(L) \geq q(Y)$ if (f, X, Y, L) is one of the following type.

(1.1) X is a normal projective variety with only Cohen-Macaulay singularities, L is ample, and $q(Y) \geq 1$.

(1.2) $\dim X = 3$, L is nef and big, and $\dim Y = 1$.

(1.3) $g(Y) \geq 1$, and there does not exist a birational morphism $\pi : F \to \mathbb{P}^{n-1}$ such that $L = \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for a general fiber F of f.

Second we investigate the non-negativity of $g(L)$ for quasi-polarized manifolds. In order to study the non-negativity of $g(L)$, we have only to investigate the case where $\kappa(X) = -\infty$. We note that some known facts about the non-negativity of $g(L)$ is the following.

(2.a) The case in which X has only rational normal Gorenstein singularities and L is ample [4, Corollary 1].

(2.b) The case in which (X, L) is a quasi-polarized manifold with $\dim X \leq 3$ [6, (4.8) Corollary].

In this paper, we proved the following.

(2.1) Let (X, L) be a quasi-polarized manifold with $\kappa(X) = -\infty$. Assume that $q(X) \geq 1$. Then

$$g(L) \geq 1 + \left\lceil \frac{m-2}{2} L^n \right\rceil ,$$

where m is the dimension of the image of the Albanese map (see Proposition 2.2).

(2.2) Let X be a normal projective variety with only rational singularities, $\kappa(X) = -\infty$, and $\dim H^1(\mathcal{O}_X) \geq 1$, and let L be an ample Cartier divisor on X. Then $g(L) \geq 1$.

Finally we consider the case where $\dim X = 3$, and we obtain some results about Conjecture 1.

In Appendix, we state a theorem (Theorem A) which appears in [18, p.319]. Theorem A is used in the proof of Lemma 0.1 and Lemma 0.2.

Finally we note that the most part of this paper was written up to 1995. After that we revised this paper several times.

Notation and Convention

We say that X is a *variety* if X is an integral separated scheme of finite type. In particular X is irreducible and reduced if X is a variety.

In this paper we shall study mainly a smooth projective variety X over the complex number field $\mathbb{C}$. The words "line bundles" and "Cartier divisors" are used interchangeably. The tensor products of line bundles are denoted additively.

$\mathcal{O}_X$: the structure sheaf of X.

$\chi(\mathcal{F})$: the Euler-Poincaré characteristic of a coherent sheaf $\mathcal{F}$.

$h^i(\mathcal{F}) := \dim H^i(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on X.

$h^i(D) := h^i(\mathcal{O}(D))$ for a divisor D.

$|D|$: the complete linear system associated with a divisor D.

$\kappa(D)$: the Iitaka dimension of a Cartier divisor D on X.

$\kappa(X)$: the Kodaira dimension of X.

$\mathbb{P}^n$: the projective space of dimension n.

$\mathbb{Q}^n$: a hyperquadric surface in $\mathbb{P}^{n+1}$.

$\sim$ (or $=$): linear equivalence.

$\equiv$: numerical equivalence.

For $r \in \mathbb{R}$, we define $[r] := \max\{\ t \in \mathbb{Z}\ :\ t \leq r\ \}$, $\lceil r \rceil := -[-r]$.

The pair (X, L) is called a *quasi-polarized* (resp. *polarized*) *manifold* if X is a smooth projective variety and L is a nef-big (resp. an ample) line bundle. Then (f, X, Y) is called a *fiber space* if X and Y are smooth projective varieties with $\dim X > \dim Y \geq 1$ and f is a surjective morphism $X \to Y$ with connected fibers. (f, X, Y, L) is called a *quasi-polarized* (resp. *polarized*) *fiber space* if (f, X, Y) is a fiber space and L is a nef and big (resp. an ample) line bundle.

We say that two quasi-polarized fiber spaces (f, X, Y, L) and (h, X, Y', L) are isomorphic if there is an isomorphism $\delta : Y \to Y'$ such that $h = \delta \circ f$. In this case we write $(f, X, Y, L) \cong (h, X, Y', L)$.

We say that (f, X, Y, L) is a *scroll* if Y is smooth, $f : X \to Y$ is $\mathbb{P}^t$-bundle, and $L|_F = \mathcal{O}(1)$, where F is a fiber of f and $t = \dim X - \dim Y$.

We say that (X, L) *has a structure of scroll over* Y if there exists a surjective morphism $f : X \to Y$ such that $(F, L|_F) \cong (\mathbb{P}^{n-m}, \mathcal{O}_{\mathbb{P}^{n-m}}(1))$ for any fiber F of f, where

$\dim X = n$ and $\dim Y = m$.

We say that a Cartier divisor D on a projective variety X is *pseudo-effective* if there is a big Cartier divisor H such that $\kappa(mD + H) \geq 0$ for all natural number m.

A *general fiber F of f* for a quasi-polarized fiber space (f, X, Y, L) means a fiber of a point of the set which is intersection of at most countable many Zariski open sets.

Let D be an effective divisor on X. We call D a *normal cossing divisor* if D has regular components which intersect transversally.

0 Preliminaries

In this section, we prove some lemmata which are used in the following sections.

First we prove Lemma 0.1 and Lemma 0.2. Theorem A in Appendix plays an important role there.

Lemma 0.1 *Let (f, X, Y, L) be a quasi-polarized fiber space. Assume that $\kappa(K_F + tL_F) \geq 0$ for some positive rational number t, where F is a general fiber of f. Then $(K_{X/Y} + tL)L^{n-1} \geq 0$, where $K_{X/Y} = K_X - f^*K_Y$.*

Proof. We note that for any natural number $p > 0$, $\kappa(K_F + tL_F + \frac{1}{p}A_F) \geq 0$, where A is an ample line bundle on X. By assumption, there exists a Zariski open set U of Y such that

(1) $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is smooth

(2) $h^0(m(K_{F_y} + tL_{F_y} + \frac{1}{p}A_{F_y}))$ is non zero constant for any fiber F_y over y in U, and some natural number m such that $mt, \frac{m}{p} \in \mathbb{N}$ and

$$\mathrm{Bs}\left| m(tL_{F_y} + \frac{1}{p}A_{F_y}) \right| = \phi.$$

We note that $f|_{f^{-1}(U)}$ is proper. By Grauert's theorem ([11]), we have

$$f_*(\mathcal{O}(m(K_{X/Y} + tL + (1/p)A))) \neq 0.$$

There is a natural map

$$f^*f_*\left(\mathcal{O}\left(m\left(K_{X/Y} + tL + \frac{1}{p}A\right)\right)\right) \to \mathcal{O}\left(m\left(K_{X/Y} + tL + \frac{1}{p}A\right)\right).$$

Then by the Hironaka theory [12] there is a birational morphism $\mu : X' \to X$ such that

$$\begin{aligned} & \mu^* f^* f_* \left(\mathcal{O} \left(m \left(K_{X/Y} + tL + \frac{1}{p} A \right) \right) \right) \\ (1) \qquad & \to \mu^* \left(\mathcal{O} \left(m \left(K_{X/Y} + tL + \frac{1}{p} A \right) - Z \right) \right) \otimes \mathcal{O}(-E) \end{aligned}$$

is surjective, where X' is a smooth projective variety, Z is an effective divisor on X, and E is a μ-exceptional effective divisor on X'.
By Theorem A in Appendix, $f_*(\mathcal{O}(m(K_{X/Y} + tL + (1/p)A)))$ is weakly positive. Hence $\mu^*\mathcal{O}(m(K_{X/Y} + tL + \frac{1}{p}A) - Z) \otimes \mathcal{O}(-E)$ is pseudo effective. Since Z and E is effective divisors, $\mu^*(\mathcal{O}(m(K_{X/Y} + tL + (1/p)A)))$ is pseudo-effective. Therefore $m(K_{X/Y} + tL + (1/p)A)L^{n-1} \geq 0$. Since p is any natural number, $(K_{X/Y} + tL)L^{n-1} \geq 0$. $\square$

Lemma 0.2 *Let (f, X, Y, L) be a quasi-polarized fiber space, where X is a normal projective variety with only $\mathbb{Q}$-factorial canonical singularities with $\dim X = n \geq 2$. Assume that $K_{X/Y} + tL$ is f-nef , where t is positive integer. Then $(K_{X/Y} + tL)L^{n-1} \geq 0$. Moreover if $\dim Y = 1$, then $K_{X/Y} + tL$ is nef.*

Proof. For any ample Cartier divisor A and any natural number p , $K_{X/Y} + tL + (1/p)A$ is f-nef by assumption. Let m be a natural number such that $m(K_{X/Y} + tL + (1/p)A)$ is a Cartier divisor. Since $m(K_{X/Y} + tL + (1/p)A) - K_X$ is f-ample, by the base point free theorem ([16, Theorem 3-1-1]),

$$f^* f_* \mathcal{O} \left(lm \left(K_{X/Y} + tL + \frac{1}{p} A \right) \right) \to \mathcal{O} \left(lm \left(K_{X/Y} + tL + \frac{1}{p} A \right) \right)$$

is surjective for any $l \gg 0$.

Let $\mu : X_1 \to X$ be a resolution of X. We put $h = f \circ \mu$. Since

$$\mu^* f^* f_* \mathcal{O} \left(lm \left(K_{X/Y} + tL + \frac{1}{p} A \right) \right) = h^* h_* \mathcal{O} \left(lm \left(K_{X_1/Y} + \mu^* \left(tL + \frac{1}{p} A \right) \right) \right),$$

we have

$$\begin{aligned} & h^* h_* \mathcal{O} \left(lm \left(K_{X_1/Y} + \mu^* \left(tL + \frac{1}{p} A \right) \right) \right) \\ (2) \qquad & \to \mu^* \mathcal{O} \left(lm \left(K_{X/Y} + tL + \frac{1}{p} A \right) \right) \end{aligned}$$

is surjective. We take l which satisfies the following condition.

$$\mathrm{Bs} \left| lm \left(tL + \frac{1}{p} A \right) \right| = \phi.$$

By Theorem A in Appendix and (2), we see that $\mu^*(lm(K_{X/Y}+tL+(1/p)A))$ is pseudo-effective. Since p is any natural number, $(K_{X/Y}+tL)L^{n-1}=\mu^*(K_{X/Y}+tL)(\mu^*L)^{n-1}\geq 0$.

If $\dim Y=1$, then $K_{X/Y}+tL+(1/p)A$ is nef. Since p is any natural number, $K_{X/Y}+tL$ is nef. □

Lemma 0.3 *Let (f,X,Y) be a fiber space with $\dim X>\dim Y\geq 1$. Then $q(X)\leq q(Y)+q(F)$, where F is a general fiber of f.*

Proof. See [9, Theorem B in Appendix]. □

Lemma 0.4 *Let (X,L) be a quasi-polarized manifold with $\dim X\leq 3$. Then $g(L)\geq 0$. Moreover if $g(L)\leq 1$, then $g(L)\geq q(X)$.*

Proof. See [9, Corollary 4.8 and Corollary 4.9]. □

1 The case of quasi-polarized fiber space over a smooth curve

In this section, we consider the case in which (f,X,Y,L) is a (quasi-)polarized fiber space with $\dim Y=1$.

Theorem 1.1 *Let (f,X,C,L) be a polarized fiber space, where X is a normal projecitve variety with only Cohen-Macaulay singularities and C is a smooth projective curve with $g(C)\geq 1$. Then $g(L)\geq g(C)$.*

Proof. Let $\mu: X'\to X$ be a resolution of X and $\dim X=n$. Then (g,X',C) is a fiber space, where $g=f\circ\mu$. Let F be a general fiber of f and let F' be a general fiber of g. Then F is normal, F' is smooth, and $\mu_{F'}: F'\to F$ is a resolution of F. Let $L'=\mu^*L$. Then $(L')^{n-1}F'=L^{n-1}F$. We note that $g(L)=g(L')$ and

$$g(L')=g(C)+\frac{1}{2}(K_{X'/Y}+(n-1)L')(L')^{n-1}+(g(C)-1)((L')^{n-1}F'-1).$$

If $\kappa(K_{F'}+(n-1)L'_{F'})\neq-\infty$, then by Lemma 0.1, we get $g(L)=g(L')\geq g(C)$ since $(L')^{n-1}F'\geq 1$ and $g(C)\geq 1$.

If $\kappa(K_{F'}+(n-1)L'_{F'})=-\infty$, then $(F,L_F)\cong(\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ by [9, Theorem (2.2)] and the ampleness of L. By [2, Proposition (1.4)], (f,X,C,L) is a scroll. But in this case, $g(L)=g(C)$. □

Next we consider the case in which (f,X,C,L) is a quasi-polarized fiber space with $\dim X=3$ and $\dim C=1$.

Definition 1.2 Let (f_1,X_1,Y,L_1) and (f_2,X_2,Y,L_2) be quasi-polarized fiber spaces, where X_i may have singularities for $i=1,2$. Then (f_1,X_1,Y,L_1) and (f_2,X_2,Y,L_2)

are said to be *birationally equivalent* if there is another variety G with birational morphisms $g_i : G \to X_i$ $(i = 1, 2)$ such that $g_1^* L_1 = g_2^* L_2$ and $f_1 \circ g_1 = f_2 \circ g_2$.

Theorem 1.3 *Let (f, X, C, L) be a quasi-polarized fiber space with* $\dim X = 3$ *and* $\dim C = 1$*. Then there exists a quasi-polarized fiber space (f', X', C, L') which is birationally equivalent to (f, X, C, L) such that (f', X', C, L') is one of the following types.*

(1) *$K_{X'} + 2L'$ is f'-nef.*

(2) *(f', X', C, L') is a scroll.*

Here X' is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities.

Proof. We prove this theorem by the similar method of the proof of [6, Theorem (4.2)].

If $K_X + 2L$ is f-nef, we put $(f', X', C, L') = (f, X, C, L)$. So we may assume that $K_X + 2L$ is not f-nef. Then there exists an extremal curve l_0 such that $(K_X + 2L)l_0 < 0$ and $f(l_0)$ is a point. Let $\phi_0 : X \to Z$ be the contraction morphism of l_0. Then there is a morphism $\pi_0 : Z \to C$ such that $f = \pi_0 \circ \phi_0$.
(a) The case where ϕ_0 is not birational.
If $L.l_0 = 0$, then $L \in \phi_0^* \mathrm{Pic}(Z)$. But this is a contradiction. (In fact, since L is nef and big, L_F is also nef and big, where F is a general fiber of ϕ_0. But since $L \in \phi_0^* \mathrm{Pic}(Z)$, $L_F^{n-1} = 0$. This is a contradiction.) Hence $L.l_0 > 0$ and L is relatively ϕ_0-ample. By [6, (3.7)], $\dim Z = 1$ and (ϕ_0, X, Z, L) is a scroll. But since f has connected fibers, we have $Z \cong C$.
Hence $(f, X, C, L) = (\phi_0, X, Z, L)$ is a scroll.
(b) The case where ϕ_0 is birational.
If $L.l_0 > 0$, then this cannot occur by the same argument as in [6, (3.6)]. So $L.l_0 = 0$ and $L = \phi_0^* L_1$ for some L_1 in $\mathrm{Pic}(Z)$.
If ϕ_0 is divisorial contraction, then we put $(f_0, X_0, C, L_0) := (\pi_0, Z, C, L_1)$. If ϕ_0 is flipping contraction, we take a flip $\phi^+ : X^+ \to Z$ and put $(f_0, X_0, C, L_0) := (f^+, X^+, C, (\phi^+)^*(L_1))$, where $f^+ = \pi_0 \circ \phi^+$.

In the above two cases, X_0 is a normal variety with only $\mathbb{Q}$-factorial terminal singularities and is smooth in codimension 2. Moreover (f_0, X_0, C, L_0) is birationally equivalent to (f, X, C, L).

Next we repeat the above process for (f_0, X_0, C, L_0). Then this process cannot continue infinitely by the minimal model conjecture. Therefore we get the assertion. □

Theorem 1.4 *Let (f, X, C, L) be a quasi-polarized fiber space with* $\dim X = 3$ *and* $\dim C = 1$*. Then $g(L) \geq g(C)$.*

Proof. We note that $g(L) = g(L')$, where (f', X', C, L') is birationally equivalent to (f, X, C, L) such that (f', X', C, L') satisfies (1) or (2) in Theorem 1.3.

(a) The case where (f', X', C, L') is (2) in Theorem 1.3
In this case, we have $g(L) = g(L') = g(C)$.
(b) The case where (f', X', C, L') is (1) in Theorem 1.3
Then by Lemma 0.2

$$(K_{X'/C} + 2L')(L')^2 \geq 0. \tag{1.4.1}$$

If $g(C) = 0$, then by [6, (4.8) Corollary], we have $g(L) \geq 0 = g(C)$. So we may assume that $g(C) \geq 1$. Since $(L')^2 F' \geq 1$ (where F' is a fiber of f'),

$$\begin{aligned} g(L) &= g(L') \\ &= g(C) + \frac{1}{2}(K_{X'/C} + 2L')(L')^2 + (g(C) - 1)((L'_F)^2 - 1) \\ &\geq g(C) \end{aligned}$$

by (1.4.1). □

Remark 1.5 If the Flip Conjectures I and II for the case where X is terminal (that is, the case where $\Delta = 0$ and X has only terminal singularities in [16, Conjecture 5-1-10 and Conjecture 5-1-13]) are true for $\dim X \geq 4$, then the following statement is true by the same argument as the proof of Theorem 1.3.
Let (f, X, C, L) be a quasi-polarized fiber space with $\dim X = n$ and $\dim C = 1$. Then there exists a quasi-polarized fiber space (f', X', C, L') which is birationally equivalent to (f, X, C, L) such that (f', X', C, L') is one of the following types.

(1) $K_{X'} + (n-1)L'$ is f'-nef.

(2) (f', X', C, L') is a scroll.

Here X' is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities. We note that if the Flip Conjectures are true, then $g(L) \geq 0$ for any quasi-polarized manifolds (X, L) (see [6, §4]). Therefore Theorem 1.4 is true for $\dim X \geq 4$ if the Flip Conjectures are true for $\dim X \geq 4$.

Remark 1.6 Recently it was proved that the Flip Conjecture I is true, that is, the flip contraction always exists (see [10]). Moreover it is known that the flips terminate for the case where $\Delta = 0$ and $\dim X \leq 4$ (see [16, Theorem 5-1-15]). Therefore Theorem 1.4 is true for $\dim X \leq 4$.

In general, for any $n = \dim X$, we can prove the following theorem.

Theorem 1.7 *Let (f, X, C, L) be a quasi-polarized fiber space with* $\dim C = 1$ *and* $g(C) \geq 1$. *Assume that there does not exist a birational morphism $\pi : F \to \mathbb{P}^{n-1}$ such that $L = \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for a general fiber F of f. Then $g(L) \geq g(C)$.*

Proof. If $\kappa(K_F + (n-1)L_F) \geq 0$ for a general fiber F of f, then by Lemma 0.1, $(K_{X/C} + (n-1)L)L^{n-1} \geq 0$. So we have $g(L) \geq g(C)$. Hence we may assume that $\kappa(K_F + (n-1)L_F) = -\infty$. Then $h^0(F, K_F + tL_F) = 0$ for $1 \leq t \leq n-1$. By the

Serre duality, we get $h^{n-1}(F, -tL_F) = 0$ for $1 \leq t \leq n-1$. By [6, (2.2) Theorem], there exists a birational morphism $\pi : F \to \mathbb{P}^{n-1}$ such that $\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) = L_F$. But this contradicts the assumption. $\Box$

2 The case where $\kappa(X) = -\infty$

First we prove the following lemma.

Lemma 2.1 *Let (f, X, Y, L) be a quasi-polarized fiber space. Then $(K_{X/Y} + (n - m + 1)L)L^{n-1} \geq 0$, where $n = \dim X$ and $m = \dim Y$.*

Proof. Since $\dim F = n - m$ for a general fiber F of f, we have $\kappa(K_F + (n - m + 1)L_F) \geq 0$ (See [4, (3.4) Lemma]). Hence by Lemma 0.1, we get $(K_{X/Y} + (n - m + 1)L)L^{n-1} \geq 0$. $\Box$

Proposition 2.2 *Let (X, L) be a quasi-polarized manifold with $\kappa(X) = -\infty$ and $q(X) \geq 1$. Then*

$$g(L) \geq 1 + \left\lceil \frac{m-2}{2} L^n \right\rceil,$$

where m is the dimension of the image of the Albanese map.

Proof. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map of X and let $f : X \to Y$ be its Stein factorization. Let $\mu_Y : Y_1 \to Y$ be a resolution of Y. Then there exists a smooth projective variety X_1, a birational morphism $\mu_X : X_1 \to X$, and a surjective morphism $f_1 : X_1 \to Y_1$ with connected fibers such that $\mu_Y \circ f_1 = f \circ \mu_X$. We note that $(f_1, X_1, Y_1, \mu_X^* L)$ is a quasi-polarized fiber space and $g(L) = g(\mu_X^* L)$.We put $L_1 := \mu_X^* L$. Then

$$\begin{aligned} g(L) &= g(L_1) \\ &= 1 + \frac{1}{2}(K_{X_1/Y_1} + (n - m + 1)L_1)L_1^{n-1} + \frac{m-2}{2} L_1^n + \frac{1}{2} f_1^* K_{Y_1} L_1^{n-1}. \end{aligned}$$

Since $\kappa(Y_1) \geq 0$ (see [21, Lemma 10.1]), we have $f_1^* K_{Y_1} L_1^{n-1} \geq 0$. By Lemma 2.1, we see that

$$g(L) \geq 1 + \left\lceil \frac{m-2}{2} L^n \right\rceil$$

because $g(L) \in \mathbb{Z}$. $\Box$

In Corollary 2.3 and Corollary 2.4, we use the same notation as in the proof of Proposition 2.2.

Corollary 2.3 *Let (X, L) be a quasi-polarized manifold with $\kappa(X) = -\infty$ and $q(X) \geq 1$. Suppose that (X, L) does not satisfy the following condition.*
$(*)$ $\dim Y_1 = 1$ and there is a birational morphism $\varphi : F_1 \to \mathbb{P}^{n-1}$ such that

$L_{1F_1} = \varphi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ *for a general fiber* F_1 *of* f_1.
Then $g(L) \geq 1$.

Proof. Let m be the dimension of the image of the Albanese map of X. If $m \geq 2$, then $g(L) \geq 1$ by Proposition 2.2. So we may assume $m = 1$. By assumption and [6, (2.2) Theorem] , $\kappa(K_{F_1} + (n-1)L_1|_{F_1}) \neq -\infty$. Therefore $g(L) \geq 1$ by Lemma 0.1. $\square$

Corollary 2.4 *Let* (X, L) *be a quasi-polarized manifold with* $\kappa(X) = -\infty$. *Suppose that* $L^n \leq 2q(X) + 1$ *and* $q(X) \geq 1$. *Then* $g(L) \geq 0$.

Proof. By Proposition 2.2, we may assume that $m = 1$. Then by the proof of Proposition 2.2, $g(L) = g(L_1) \geq g(Y_1) - \frac{1}{2}L^n = q(X) - \frac{1}{2}L^n$. By assumption and $g(L) \in \mathbb{Z}$, we have $g(L) \geq 0$. $\square$

In general, we can prove the following proposition by Fujita's results [6].

Proposition 2.5 *Let* (X, L) *be a quasi-polarized manifold of dimension* n *with* $L^n \leq 4$. *Then* $g(L) \geq 0$.

Proof. If $\kappa(K_X + (n-1)L) \neq -\infty$, then by the definition of the sectional genus we have $g(L) \geq 0$. Hence we may assume that $\kappa(K_X + (n-1)L) = -\infty$. Then $h^n(X, -tL) = 0$ for every integer t with $1 \leq t < n$. Hence

$$\begin{aligned} \chi(X, tL) &= \chi(t) \\ &= (t+1) \cdots (t+n-1)(dt+a)/n! \end{aligned}$$

(Here $d = L^n$ and $a \in \mathbb{Z}$.) Then we have $K_X L^{n-1} = d(1-n) - \frac{2a}{n}$. If $t = -n$, then

$$\chi(-n) = \frac{(-1)^n}{n}(dn - a). \tag{2.5.1}$$

On the other hand,

$$\chi(-n) = (-1)^n l, \tag{2.5.2}$$

where $l = h^n(X, -nL)$. Hence by (2.5.1) and (2.5.2), we have $a = n(d-l)$. Then

$$g(L) = 1 - \frac{a}{n} = 1 - d + l. \tag{2.5.3}$$

(1) The case of $l = 0$.
Then by [6, (2.2) Theorem], we have $g(L) \geq 0$.
(2) The case of $l \geq 1$.
If $d = 1$, then by (2.5.3) $g(L) \geq 0$. Hence we may assume that $2 \leq d \leq 4$.
(2-1) The case of $d = 4$.
In this case, $g(L) = l - 3$. If $l \geq 3$, then $g(L) \geq 0$. Hence $l = 1$ or 2.
(2-1-1) The case of $l = 1$.

In this case $g(L) \geq 0$ by [6, (2.3) Theorem].
(2-1-2) The case of $l = 2$.
In this case $L^{n-1}(K_X + nL) = 2l - d = 0$. Since $l = h^0(K_X + nL) \neq 0$, we have $l = 1$ by [6, (2.8) Corollary]. But this is a contradiction.
(2-2) The case of $d = 3$.
In this case $g(L) = l - 2$. If $l \geq 2$, then $g(L) \geq 0$. So we may assume that $l = 1$. But then $g(L) \geq 0$ by [6, (2.3) Theorem].
(2-3) The case of $d = 2$.
In this case, $g(L) = l - 1 \geq 0$. $\square$

Theorem 2.6 *Let X be a normal projective variety with only rational singularities, $\kappa(X) = -\infty$, and $\dim H^1(\mathcal{O}_X) \geq 1$, and let L be an ample Cartier divisor on X. Then $g(L) \geq 1$.*

Proof. Let $\alpha_X : X \to \mathrm{Alb}(X)$ be the Albanese map of X. Since X has only rational singularity, α_X is a morphism (see [20, (0.3.3) Lemma] [15, Lemma 8.1]). Let $\mu_0 : X_0 \to X$ be a resolution of X, and $X_0 \to Y_0 \to \mathrm{Alb}(X)$ be the Stein factraization of $\alpha_X \circ \mu_0$. We put $f_0 : X_0 \to Y_0$. Let $\mu_{Y,1} : Y_1 \to Y_0$ be a resolution of Y_0. Then there is a smooth projective variety X_1, a birational morphism $\mu_{X,1} : X_1 \to X_0$, and a surjective morphism $f_1 : X_1 \to Y_1$ with connected fibers such that $f_0 \circ \mu_{X,1} = \mu_{Y,1} \circ f_1$.

We consider a quasi-polarized manifold $(X_1, (\mu_0 \circ \mu_{X,1})^* L)$. We set $L_1 := (\mu_0 \circ \mu_{X,1})^* L$. We note that $\kappa(Y_1) \geq 0$.

If $m := \dim Y_1 \geq 2$, then by Proposition 2.2

$$\begin{aligned} g(L) &= g(L_1) \\ &= 1 + \frac{1}{2}(K_{X_1/Y_1} + (n-m+1)L_1)L_1^{n-1} + \frac{m-2}{2}L_1^n + \frac{1}{2}f_1^* K_{Y_1} L_1^{n-1} \\ &\geq 1. \end{aligned}$$

Hence we may assume that $m = 1$. We note that $Y_1 = Y_0$ and $X_1 = X_0$. Let F_1 be a general fiber of f_1.
(1) The case of $\kappa(K_{F_1} + (n-1)L_{1F_1}) \geq 0$.
Then by Lemma 0.1 $(K_{X_1/Y_1} + (n-1)L_1)L_1^{n-1} \geq 0$. Hence $g(L) = g(L_1) \geq 1$ since $g(Y_1) = h^1(\mathcal{O}_X) \geq 1$.
(2) The case of $\kappa(K_{F_1} + (n-1)L_{1F_1}) = -\infty$.
Let F be a general fiber of $g : X \to Y_0$ such that $\mu_0^{-1}(F) = F_1$. (We note that $f_0 = g \circ \mu_0$.)
Then we note that F is a normal projective variety with $\dim F = n-1$. In this case by [6, (2.2) Theorem], there is a birational morphism $\varphi : F \to \mathbb{P}^{n-1}$ such that $L_F = \varphi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Since L is ample, $(F, L_F) = (\mathbb{P}^{n-1}, \mathcal{O}(1))$. Hence (g, X, Y_0, L) is scroll by [2, Proposition 1.4]. Therefore $g(L) = g(Y_0) \geq 1$. $\square$

Next we propose the following conjecture.

Conjecture 2.7 *Let (X, L) be a quasi-polarized manifold with $\kappa(X) = -\infty$, let $\alpha_X : X \to Alb(X)$ be the Albanese map of X, and let*

$$m(X) = \begin{cases} \dim \alpha_X(X) & \textit{if } q(X) \geq 1, \\ 0 & \textit{if } q(X) = 0. \end{cases}$$

Then $g(L) \geq m(X)$.

We note that Conjecture 2.7 is true if Conjecture in Introduction is true.

Proposition 2.8 *Let (X, L) be an n-dimensional quasi-polarized manifold. Assume that $\kappa(X) = -\infty$, $n \geq 4$, $L^n \geq 3$, and $m(X) \geq 3$. Then $g(L) \geq m(X)$.*

Proof. We note that $q(X) \geq 1$ since $m(X) \geq 3$. By Proposition 2.2 we have

$$g(L) \geq 1 + \left\lceil \frac{m(X) - 2}{2} L^n \right\rceil .$$

By assumption,

$$\begin{aligned} 1 + \left\lceil \frac{m(X) - 2}{2} L^n \right\rceil &\geq 1 + \frac{3}{2}(m(X) - 2) \\ &= m(X) + \frac{1}{2} m(X) - 2. \end{aligned}$$

Since $g(L)$ is integer, we get $g(L) \geq m(X)$. $\square$

Proposition 2.9 *Let (X, L) be a quasi-polarized manifold with $\dim X \leq 3$ and $\kappa(X) = -\infty$. Then $g(L) \geq m(X)$.*

Proof. We note that $m(X) \leq q(X)$ and $m(X) < \dim X \leq 3$. If $g(L) \leq 1$, then $g(L) \geq q(X) \geq m(X)$ by Lemma 0.4. If $g(L) \geq 2$, then $g(L) \geq 2 \geq m(X)$. $\square$

Theorem 2.10 *Let (X, L) be a polarized manifold with $\kappa(X) = -\infty$. Assume that $L^n \geq 3$. Then $g(L) \geq m(X)$.*

Proof. We note that $g(L) \geq q(X)$ if $g(L) \leq 2$. (See [7, §12 and §15].) So we may assume that $g(L) \geq 3$. Then $g(L) \geq 3 \geq m(X)$ if $m(X) \leq 3$. Moreover if $\dim X \leq 4$, then $g(L) \geq m(X)$ because $m(X) \leq 3$ in this case. So we may assume that $\dim X \geq 5$ and $m(X) \geq 4$. Then by Proposition 2.8, we have $g(L) \geq m(X)$. $\square$

3 The case where $\dim X = 3$

In this section, we are going to investigate Conjecture in introduction for $\dim X = 3$.

Theorem 3.1 *Let (X, L) be a quasi-polarized manifold with $\dim X = 3$ and $\kappa(X) \leq 2$. Then $g(L) \geq q(X)$ holds if (X, L) is one of the following cases.*

(1.1) $\kappa(X)=-\infty$ *and* $m\leq 1$.

(1.2) $\kappa(X)=-\infty$, $m=2$, *and* $\kappa(Y)\leq 1$.

(2.1) $\kappa(X)=0$ *and* $L^3\geq 2$.

(2.2) $\kappa(X)=0$ *and* L *is ample.*

(3) $\kappa(X)=1$ *and* $L^3\geq 2$.

(4) $\kappa(X)=2$, $\kappa(Y)\leq 1$, *and* $L^3\geq 2$.

Here in (1.1) *and* (1.2), m *is the dimension of the image of the Albanese map* $\alpha_X : X\to \mathrm{Alb}(X)$, $X\to Y\to \alpha_X(X)$ *is the Stein factrization of* α_X, *and* Y *is the image of the Iitaka fibration of* X *in* (4).

Proof. (A) The case of $\kappa(X)=-\infty$.
(A.1) The case of $q(X)=0$.
In this case, $g(L)\geq 0=q(X)$ by Lemma 0.4.
(A.2) The case of $q(X)\geq 1$.
Let $\alpha_X : X\to \mathrm{Alb}(X)$ be the Albanese map of X and $m=\dim\alpha_X(X)$. Then $m=1$ or 2.
(A.2.1) The case of $m=1$.
In this case, $\alpha_X : X\to\alpha_X(X)$ is a fiber space, that is, α_X is surjective morphism with connected fibers and $\alpha_X(X)$ is a smooth curve of genus $q(X)$. Hence $g(L)\geq q(X)$ by Theorem 1.4.
(A.2.2) The case of $m=2$.
Let $X\to Y\to\alpha_X(X)$ be the Stein factraization of $\alpha_X(X)$. We put $f: X\to Y$. We note that Y is normal (not smooth in general). Let $\mu_2 : Y_1\to Y$ be a resolution of Y. Then there is a birational morphism $\mu_1 : X_1\to X$ and a surjective morphism $f_1 : X_1\to Y_1$ with connected fibers such that $\mu_2\circ f_1 = f\circ\mu_1$. We note that $g(\mu_1^*L)=g(L)$ and $\kappa(Y_1)\geq 0$. And also we note that $\kappa(F)=-\infty$ since $\kappa(X)=-\infty$, where F is a general fiber of f_1. Since $\dim F=1$, $F\cong\mathbb{P}^1$. Hence $q(X_1)=q(Y_1)$ by Lemma 0.3. So it is enough to show that $g(\mu_1^*L)\geq q(Y_1)$. We put $L_1:=\mu_1^*L$.
(A.2.2.1) The case of $\kappa(Y_1)=0$.
In this case $q(Y_1)\leq 2$ by the classification theory of smooth projective surfaces. By Proposition 2.2 we have $g(L_1)\geq 1+\lceil\frac{m-2}{2}L_1^n\rceil$. Since $m=2$, we have $g(L_1)\geq 1$.
If $q(Y_1)\leq 1$, then $g(L_1)\geq q(Y_1)$. So we may assume that $q(Y_1)=2$. Then Y_1 is birationally equivalent to an Abelian surface.
If $g(L_1)\geq 2$, then $g(L_1)\geq q(Y_1)$. So we may assume that $g(L_1)=1$. But then by Lemma 0.4, $g(L_1)\geq q(X_1)$. Hence this cannot occur.
(A.2.2.2) The case of $\kappa(Y_1)=1$.
In this case Y_1 has an elliptic fibration. Let $\pi : Y_1\to C$ be an elliptic fibration. Then $q(Y_1)=g(C)$ or $q(Y_1)=g(C)+1$ by Lemma 0.3. Hence $g: X_1\to Y_1\to C$ is a fiber space, where $g=\pi\circ f_1$. Hence $g(L_1)\geq g(C)$ by Theorem 1.4.
(A.2.2.2.a) The case of $q(Y_1)=g(C)$.

In this case $g(L_1) \geq g(C) = q(Y_1) = q(X)$ by Lemma 0.3.
(A.2.2.2.b) The case of $q(Y_1) = g(C) + 1$.
Let F_g be a general fiber of g. We note that

$$g(L_1) = g(C) + \frac{1}{2}(K_{X_1/C} + 2L_1)L_1^2 + (L_1^2 F_g - 1)(g(C) - 1).$$

If $g(L_1) \geq g(C) + 1$, then $g(L_1) \geq q(Y_1) = q(X)$. So we may assume that $g(L_1) = g(C)$ by Theorem 1.4. If $g(C) \leq 1$, then $g(L_1) \leq 1$ and $g(L_1) \geq q(X)$ by Lemma 0.4. So we may assume that $g(C) \geq 2$. By using Theorem 1.3, we may assume that X_1 is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities and $K_{X_1} + 2L_1$ is g-nef. (In fact if (X_1, C, g, L_1) is the type (2) in Theorem 1.3, then $q(F_g) = 0$ and $q(X_1) = g(C)$ by Lemma 0.3. But then $g(C) \geq q(Y_1)$ because $q(X_1) \geq q(Y_1)$. Hence this type cannot occur.) Hence $K_{X_1/C} + 2L_1$ is nef by Lemma 0.2. Since $g(C) \geq 2$ and $g(L_1) = g(C)$, we have $L_1^2 F_g = 1$ and $(K_{X_1/C} + 2L_1)L_1^2 = 0$.

Claim 3.2 *$K_{F_g} + L_{1F_g}$ is not nef.*

Proof. We assume that $K_{F_g} + L_{1F_g}$ is nef. Then $r(K_{F_g} + L_{1F_g} + (1/p)A_{F_g})$ is nef for any $p \in \mathbb{N}$ and any ample Cartier divisor A on X_1, where r is a natural number such that $(r/p) \in \mathbb{N}$. On the other hand $r(K_{F_g} + L_{1F_g} + \frac{1}{p}A_{F_g}) - K_{F_g}$ is ample. By the Nonvanishing theorem (See [16, Theorem 2-1-1]),

$$h^0\left(Nr\left(K_{F_g} + L_{1F_g} + \frac{1}{p}A_{F_g}\right)\right) \neq 0$$

for any sufficiently large natural number N. Here we choose N which satisfies the following condition.

$$\mathrm{Bs}\left|Nr\left(L_1 + \frac{1}{p}A\right)\right| = \phi.$$

By Grauert's theorem, $f_*(Nr(K_{X_1/C} + L_1 + (1/p)A)) \neq 0$. By the same argument as in the proof of Lemma 0.2, $(K_{X_1/C} + L_1)L_1^2 \geq 0$. Hence $(K_{X_1/C} + 2L_1)L_1^2 > 0$. But this contradicts assumption. Hence $K_{F_g} + L_{1F_g}$ is not nef. This completes the proof of Claim 3.2. □

Then there is an extremal rational curve l_g on F_g with $(K_{F_g} + L_{1F_g})l_g < 0$ such that $(F_g\ , l_g)$ is one of the following types.

(α) $F_g \cong \mathbb{P}^2$ and l_g is a line.

(β) $F_g \cong \mathbb{P}^1$-bundle and l_g is a fiber.

(γ) l_g is (-1)-curve.

Case (α) In this case, $q(F_g) = 0$. Hence $q(X_1) = g(C)$. But then $g(C) \geq q(Y_1)$ and this is a contradiction because $g(C) + 1 = q(Y_1)$. Hence this case cannot occur.

Case (β) Then $L_{1F_g}l_g = 1$. Hence (F_g, L_{1F_g}) is a scroll over a smooth curve. We put $F_g = \mathbb{P}_T(\mathcal{E})$ and $\pi_T : F_g \to T$, where T is a smooth curve and $\mathcal{E}$ is a normalized locally free sheaf of rank 2, that is, $h^0(\mathcal{E}) \neq 0$ and $h^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on T with $\deg\mathcal{L} < 0$. Then $K_{F_g} = -2\mathcal{H} + \pi_T^*(K_T + \det\mathcal{E})$ and $L_{1F_g} = \mathcal{H} + \pi_T^* D$, where $\mathcal{H}$ is the tautological line bundle of F_g and D is a Cartier divisor on T. We put $b := \deg D$ and $e := -\deg\mathcal{E}$. Since a general fiber of $X_1 \to Y$ is $\mathbb{P}^1$ and $\pi : Y_1 \to C$ is an elliptic fibration, we see that $q(F_g) = 1$. Hence $g(T) = 1$. Therefore $e \geq 0$ or -1 (see [11, Theorem 2.12 and Theorem 2.15, Section 2, Chapter V]).
Case (β.1) The case of $e \geq 0$.
Then $K_{F_g} + 2L_{1F_g} = \pi_T^*(\det\mathcal{E} + 2D)$. Since $K_{X_1/C} + 2L_1$ is nef, $K_{X_1/C} + 2L_1 \equiv 0$ by [6, (2.7) Lemma]. In particular, $K_{F_g} + 2L_{1F_g} \equiv 0$. Hence $2b - e = 0$. Since L_{1F_g} is nef and big, we have $b \geq e \geq 0$. Hence $b = e = 0$. But this contradicts the bigness of L_{1F_g}. Hence this case cannot occur.
Case (β.2) The case of $e = -1$.
First we note that $g(T) \geq 1$ because $e < 0$. Let $\mu_r : X_r \to X_1$ be a resolution of X_1 such that $X_r \backslash \mu_r^{-1}(\mathrm{Sing}(X_1)) \cong X_1 \backslash \mathrm{Sing}(X_1)$. Let $h := g \circ \mu_r$ and $L_r := \mu_r^* L_1$. Since X_1 has only $\mathbb{Q}$-factorial terminal singularities, we see $(F_h, L_{rF_h}) \cong (F_g, L_{1F_g})$, where F_g (resp F_h) is a general fiber of g (resp h). Then L_{rF_h} is ample if and only if L_{rF_h} is nef and big since $e < 0$. Hence L_{rF_h} is ample. Hence $b \geq 0$ by [11, Proposition 2.21, Section 2, Chapter V], and we get

$$\begin{aligned} h^0(K_{F_g} + 2L_{1F_g}) &= h^0(K_T + \det\mathcal{E} + 2D) \\ &\geq 1 - g(T) + \deg(K_T + \det\mathcal{E} + 2D) \\ &= g(T) - 1 + e + 2b \\ &> 0. \end{aligned}$$

Hence $h_*(K_{X_r/C} + 2L_r) \neq 0$ by Grauert's theorem. Since L_{rF_h} is ample, $h_*(K_{X_r/C} + 2L_r)$ is ample by ([3, Theorem 2.4 and Corollary 2.5]). By [9, Lemma 1.4.2], $(K_{X_1/C} + 2L_1)L_1^2 = (K_{X_r/C} + 2L_r)L_r^2 > 0$. Hence this case also cannot occur. Therefore case (β) cannot occur.
Case(γ) Then $L_{1F_g}l_g = 0$. Hence $(K_{F_g} + 2L_{1F_g})l_g < 0$ and $K_{F_g} + 2L_{1F_g}$ is not nef. Therefore this case cannot occur.
By the above argument, the case in which $g(L_1) = g(C)$ and $q(Y_1) = g(C) + 1$ cannot occur. Therefore we have $g(L_1) \geq g(C) + 1 = q(Y_1)$.
(B) The case of $\kappa(X) = 0$.
By [9, Theorem 1.3.5], $g(L) \geq q(X)$ holds if $L^3 \geq 2$. Next we prove that $g(L) \geq q(X)$ holds if L is ample. We note that $q(X) \leq 3$ holds by Kawamata's theorem ([14, Corollary 2]).
(B.1) The case of $q(X) = 3$.
Let $\alpha_X : X \to \mathrm{Alb}(X)$ be the Albanese map of X. Then α_X is a birational morphism.
If α_X is isomorphism, then $\frac{L^3}{3!} \in \mathbb{N}$ (see [19, Chapter III, Section 16]). Hence $L^3 \geq 6$. On the other hand $K_X \equiv 0$. Hence $g(L) \geq 7 > q(X)$.

Assume that α_X is not an isomorphism. By [17, Theorem 9.13], there exists a rational curve B such that $BK_X < 0$. If $K_X L^2 = 0$, then $K_X = \mathcal{O}_X$ because $h^0(K_X) > 0$ and L is ample. Hence K_X is nef. But this is impossible because $BK_X < 0$. Therefore $K_X L^2 > 0$. Hence $(K_X + 2L)L^2 \geq 3$ if L is ample. Therefore $g(L) \geq 3 = q(X)$.
(B.2) The case of $q(X) \leq 2$.
Then $g(L) = 1 + (1/2)(K_X + 2L)L^2 \geq 2 \geq q(X)$.
(C) The case of $\kappa(X) = 1$.
By [9, Theorem 1.3.4], $g(L) \geq q(X)$ holds if $L^3 \geq 2$.
(D) $\kappa(X) = 2$ case.
By using the Iitaka theory, there is a birational morphism $\mu_1 : X_1 \to X$ and a surjective morphism $f : X_1 \to Y$ with connected fibers such that $\kappa(F) = 0$, where Y is a smooth projective surface and F is a general fiber of f. In this case F is an elliptic curve. We put $L_1 := \mu_1^* L$. We note that $g(L) = g(L_1)$.
(D.1) The case of $\kappa(Y) = -\infty$.
(D.1.1) The case of $q(Y) = 0$.
Then $q(X) \leq 1$ by Lemma 0.3. Hence $g(L) \geq q(X)$ by Lemma 0.4.
(D.1.2) The case of $q(Y) \geq 1$.
Then there is a surjective morphism $g : Y \to C$ with connected fibers, where C is a smooth curve. We put $h := g \circ f : X_1 \to Y \to C$. Let F_f (resp. F_g, F_h) be a general fiber of f (resp. g, h).Then $q(F_h) \leq q(F_f) + q(F_g) = 1$. Since $\kappa(X) = 2$, we have $\kappa(F_h) \geq 0$. Therefore $K_{X_1/C} L_1^2 \geq 0$ by [9, Lemma 1.3.1].

If $g(C) = 0$, then $q(X) \leq g(C) + q(F_h) = 1$. Hence $g(L) = g(L_1) \geq q(X)$.

If $g(C) \geq 1$, then

$$\begin{aligned} g(L) &= g(L_1) \\ &= g(C) + \frac{1}{2}(K_{X_1/C} + 2L_1)L_1^2 + (g(C) - 1)(L_1^2 F_h - 1) \\ &\geq g(C) + 1 \\ &\geq g(C) + q(F_h) \\ &\geq q(X) \end{aligned}$$

because $L_1^2 F \geq 1$ and $K_{X_1/C} L_1^2 \geq 0$.
(D.2) The case of $\kappa(Y) = 0$ or 1.
By Lemma 0.3, $q(X_1) \leq q(F_f) + q(Y) = 1 + q(Y)$. By [9, Theorem 2.3] we have $g(L) \geq q(Y) + L^3 - 1$. So if $L^3 \geq 2$, then $g(L) \geq q(Y) + 1 \geq q(X_1)$. $\square$

Appendix

Here we give a proof of the following which was proved in [9, Theorem A']:

Theorem. *Let X and Y be smooth quasi-projective varieties over $\mathbb{C}$, $\mathcal{L}$ a semiample invertible sheaf over X, $f : X \to Y$ a projective surjective morphism. Then for any positive integer k and i, $f_*(\omega_{X/Y}^{\otimes k} \otimes \mathcal{L}^{\otimes i})$ is weakly positive.*

Proof. Let $\eta : X' \to X$ be a finite cyclic covering defined by the nonsingular divisor B such that $\mathcal{L}^{\otimes N} = \mathcal{O}(B)$. Then $\eta_* \omega_{X'/Y} = \bigoplus_{i=0}^{N-1} (\omega_{X/Y} \otimes \mathcal{L}^{\otimes i})$. Since X' is nonsingular and η is affine,

$$(\eta_* \omega_{X'/Y})^{\otimes k} = \eta_*(\omega_{X'/Y}^{\otimes k}).$$

Hence we have

$$(f \circ \eta)_*(\omega_{X'/Y}^{\otimes k}) = \bigoplus_{t=0}^{k(N-1)} f_*(\omega_{X/Y}^{\otimes k} \otimes \mathcal{L}^{\otimes t})^{\oplus \alpha(t)},$$

which is weakly positive by Viehweg [22], where $\left(\sum_{i=0}^{N-1} x^i\right)^k = \sum_{t=0}^{k(N-1)} \alpha(t) x^t$. Thus $f_*(\omega_{X/Y}^{\otimes k} \otimes \mathcal{L}^{\otimes t})$ is also weakly positive for $0 \le t \le k(N-1)$. Tend $N \to \infty$ and this completes the proof. $\Box$

As a corollary of the result, we get the following Theorem A:

Theorem A (Viehweg and Mori [18, P.319]) *Let X and Y be smooth quasi-projective varieties over the field of complex numbers $\mathbb{C}$, $\mathcal{L}$ a base point free Cartier divisor on X, and $f : X \to Y$ be a projective surjective morphism. Then $f_*(\omega_{X/Y}^{\otimes k} \otimes \mathcal{L})$ is weakly positive in the sense of Viehweg* [22] *for $\forall k > 0$. (Here $\omega_{X/Y} = \omega_X \otimes f^* \omega_Y^{-1}$.)*

References

[1] M. C. Beltrametti, A. Lanteri, and M. Palleschi, Algebraic surfaces containing an ample divisor of arithmetic genus two, Arkiv för Mat. 25 (1987), 189–210.

[2] M. C. Beltrametti and A.J.Sommese, *On the adjunction theoretic classification of polarized varieties,* J. reine angew. Math. 427 (1992), 157–192.

[3] H. Esnault and E. Viehweg, *Effective bounds for semipositive sheaves and for the height of points on curves over complex function fields,* Composit. Math. 76 (1990), 69–85.

[4] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive,* Advanced Studies in Pure Math. 10 (1985), 167–178.

[5] T. Fujita, *Classification of polarized manifolds of sectional genus two,* the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata, 1987, 73–98.

[6] T. Fujita, *Remarks on quasi-polarized varieties,* Nagoya Math. J. 115 (1989), 105–123.

[7] T. Fujita, *Classification Theories of Polarized Varieties,* London Math. Soc. Lecture Note Series 155, 1990.

[8] Y. Fukuma, *A lower bound for the sectional genus of quasi-polarized surfaces,* Geom. Dedicata 64 (1997), 229–251.

[9] Y. Fukuma, *A lower bound for sectional genus of quasi-polarized manifolds,* J. Math. Soc. Japan 49 (1997), 339–362.

[10] C. D. Hacon and J. McKernan, *Extension theorems and the existence of flips,* Flips for 3-folds and 4-folds, 76–110, Oxford Lecture Ser. Math. Appl., 35, Oxford Univ. Press, Oxford, 2007.

[11] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Math. 52 1977.

[12] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero I,II,* Ann. of Math. 79 (1964), 109–326.

[13] P. Ionescu, *Generalized adjunction and applications,* Math. Proc. Camb. Phil. Soc. 99 (1986), 457–472.

[14] Y. Kawamata, *Characterization of Abelian varieties,* Composit. Math. 43 (1981), 253–276.

[15] Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces,* J. reine angew Math. 363 (1985), 1–46.

[16] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem,* Advanced Studies in Pure Math. 10 (1985), 283–360.

[17] J. Kollár, *The structure of algebraic threefolds: an introduction to Mori's program,* Bull. Amer. Math. Soc. 17 (1987), 211–273.

[18] S. Mori, *Classification of higher-dimensional varieties,* Proc. Symp. Pure Math. 46 1987, 269–331.

[19] D. Mumford, *Abelian Varietes,* Oxford University Press, 1970.

[20] A. J. Sommese, *On the adjunction theoretic structure of projective varieties,* Lecture Notes in Math. 1194, 175–213.

[21] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex spaces,* Lecture Notes in Math. 439 (1970)

[22] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces,* Advanced Studies in Pure Math. 1 1983, 329–353.

Yoshiaki Fukuma
Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp