

# A LOWER BOUND FOR THE DIMENSION OF CERTAIN $G_\delta$ SETS IN COMPLETELY NORMAL SPACES<sup>1</sup>

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Nagami and Roberts have proved [3, Theorem 1] that if  $X$  is a normal space of dimension at least  $n$  satisfying certain conditions, then  $\dim(X - \bigcup_{i=1}^{\infty} A_i) \geq n - 1$  if the  $A_i$  are disjoint closed subsets of  $X$ . In this paper we allow the  $A_i$  to intersect provided that the dimension of the pairwise intersections is known. (The dimension of the remainder is reduced accordingly.)

By dimension (denoted  $\dim$ ) we mean the covering dimension. The symbols  $\text{bdy}$  and  $\text{int}$  are used to denote the topological boundary and interior. We will say that a  $T_1$  space  $X$  is  $n$ -solid ( $n$  a positive integer) if for every point  $x$  of  $X$  and every open neighborhood  $U$  of  $x$  there is a connected open neighborhood  $V$  of  $x$  such that  $\bar{V} \subseteq U$ ,  $\bar{V}$  is compact,  $\dim \bar{V} \geq n$ , and no  $(n - 2)$ -dimensional subset of  $\bar{V}$  separates it. Such a space is locally compact and locally connected. This property is hereditary on open subsets.

**THEOREM 1.** *Let  $m$  and  $n$  be integers such that  $m \geq -1$  and  $n \geq 1$ . Let  $X$  be a compact completely normal space with  $\dim X \geq n$ . Let  $A_1, A_2, \dots$  be a countable sequence of closed subsets of  $X$  such that*

- (1)  $\dim A_i \leq n - 1$  for every integer  $i$ ,
- (2)  $\dim(A_i \cap A_j) \leq m$  if  $i \neq j$ .

*Then  $\dim(X - \bigcup_{i=1}^{\infty} A_i) \geq n - m - 2$ .*

**PROOF.** The proof of the present theorem is similar to that of Theorem 1 in [3]. That proof has five steps:

- (1)  $X$  is assumed to be a closed set with a certain minimal property.
- (2) A function  $h$  from  $X$  into  $I^n$  is obtained.
- (3) An  $n$ -dimensional pyramid is constructed in  $I^n$ .
- (4) Assuming the theorem is false, the inverse images in  $X$  of the  $(n - 1)$  pairs of opposite faces of the pyramid are separated by  $n - 1$  closed sets whose intersection contains no continuum hitting the inverse images of both the base and the apex of the pyramid.
- (5) The pyramid is truncated and the inverse images of the  $n$  pairs of opposite faces of the lower portion are shown not to be a defining

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system [3, Definition 5], in  $X$ . This is shown to be false, and a contradiction results.

The only change required in order to prove our present theorem is the substitution of the following modified step four:

*Fourth step.* Suppose that  $\dim(X - \bigcup_{i=1}^{\infty} A_i) < n - m - 2$ . Then  $\dim(X - (\bigcup_{i=1}^{\infty} A_i \cup h^{-1}(p)))$  is less than  $n - m - 2$  since  $h^{-1}(p)$  is a  $G_\delta$ . By Lemma 1 of [3] there exist sets  $B_1, \dots, B_{n-m-2}$  which are closed in  $X - h^{-1}(p)$  and such that

- (1)  $B_i$  separates  $C_i$  from  $C'_i, 1 \leq i \leq n - m - 2$ ,
- (2)  $\bigcap_{i=1}^{n-m-2} B_i \subseteq \bigcup_{i=1}^{\infty} A_i$ .

Let  $A = \bigcup_{i \neq j} (A_i \cap A_j)$ ; then  $A$  is an  $F_\sigma$ , and  $\dim A \leq m$ . By [2, Theorem 3.4], there exist  $m + 1$  sets  $B_{n-m-1}, \dots, B_{n-1}$  which are closed in  $X - h^{-1}(p)$  and such that

- (1)  $B_i$  separates  $C_i$  from  $C'_i, n - m - 1 \leq i \leq n - 1$ ,
- (2)  $\bigcap_{i=n-m-1}^{n-1} B_i \subseteq X - A$ .

Let  $H = \bigcap_{i=1}^{n-1} B_i \subseteq \bigcup_{i=1}^{\infty} A_i - A$ .  $\bar{H} = H \cup h^{-1}(p)$  is compact, and  $H \cap h^{-1}(B)$  and  $h^{-1}(p)$  are disjoint closed subsets of  $\bar{H}$ . Recall from step 2 that  $h^{-1}(p) \subseteq X - \bigcup_{i=1}^{\infty} A_i$ . Then by the above  $\bar{H} = h^{-1}(p) \cup (H \cap A_1) \cup (H \cap A_2) \cup \dots$ , and these pieces are disjoint and closed ( $\bar{H} \cap A_i = H \cap A_i$ ). As in step 4 of [3, Theorem 1], there is no continuum  $K \subseteq \bar{H}$  which intersects both  $h^{-1}(p)$  and  $H \cap h^{-1}(B)$ .

Step 5 now proceeds unchanged, and the proof is complete.

Theorem 1 of [3] is obtained as a special case of this theorem when  $m = -1$ .

**THEOREM 2.** *If  $X$  is a connected  $n$ -solid space, then  $X$  is not the countable union of closed proper subsets  $A_1, A_2, \dots$  such that  $\dim(A_i \cap A_j) \leq n - 2$  when  $i \neq j$ .*

**PROOF.** Suppose the theorem is false, and  $X = \bigcup_{i=1}^{\infty} A_i$ , where the  $A_i$  satisfy the conditions of the theorem. We will construct a sequence  $\bar{V}_0, \bar{V}_1, \dots$  of compact nonempty subsets of  $X$  such that for every integer  $i, \bar{V}_{i+1} \subseteq \bar{V}_i$  and  $\bar{V}_i \cap A_i = \emptyset$ . This is a contradiction since the  $\bar{V}_i$  have the finite intersection property and an empty total intersection.

By Baire's theorem there is an  $i$  such that  $\text{int}(A_i)$  is not empty. Fix  $i$  at this value. Then  $\text{bdy}(\text{int}(A_i)) \neq \emptyset$  since  $X$  is connected. Let  $x_0 \in \text{bdy}(\text{int}(A_i))$ . Let  $x_0 \in V_0 \subseteq \bar{V}_0 \subseteq X$  be as in the definition of  $n$ -solid. Then  $\bar{V}_0$  is not a subset of  $A_i$  since  $x_0$  is not in  $\text{int}(A_i)$ . Suppose there is an integer  $j \neq i$  such that  $\bar{V}_0 \subseteq A_j$ . Then  $A_j \cap A_i \supseteq V_0 \cap \text{int}(A_i)$ , a nonempty open set. Hence  $A_j \cap A_i$  contains a closed set of dimension at least  $n$  by the definition of  $n$ -solid, so  $\dim(A_i \cap A_j)$

$\geq n$ , a contradiction. Then  $x_0$  and  $V_0$  satisfy (1)–(4) below if we let  $V_{-1} = X$  and  $A_0 = \emptyset$ .

*Induction step.* Suppose we are given  $x_p$  and  $V_p$  such that

- (1)  $x_p \in V_p$ , and  $\bar{V}_p$  is as in the definition of  $n$ -solid;
- (2) for every integer  $j$ ,  $\bar{V}_p \not\subseteq A_j$ ;
- (3)  $\bar{V}_p \subseteq \bar{V}_{p-1}$ ;
- (4)  $\bar{V}_p \cap A_p = \emptyset$ .

We will construct  $x_{p+1}$  and  $V_{p+1}$  satisfying (1)–(4).

Let  $W$  be the open set  $V_p - A_{p+1}$ . If  $W$  is connected, then it is not a subset of any  $A_j$ , since otherwise  $\bar{V}_p \subseteq A_j \cup A_{p+1}$ , so  $A_j \cap A_{p+1}$  separates  $\bar{V}_p$ , a contradiction.

If  $W$  is not connected, let  $K$  be a component;  $K$  is open. Let  $L = \text{int}(\bar{K})$ , so  $K \subseteq L \subseteq \bar{L} = \bar{K}$ . But some other component of  $W$  is not empty and  $V_p$  is connected. Therefore  $\text{bdy}(L) \cap V_p$  is nonempty, separates  $V_p$ , and is a subset of  $A_{p+1}$ .

There is no integer  $j$  such that  $\bar{K} \subseteq A_j$ , for suppose there is such a  $j$ . Then  $\bar{K} \subseteq A_j$ , so  $\text{bdy}(L) \cap V_p \subseteq A_j \cap A_{p+1}$ . Let  $x \in \text{bdy}(L) \cap V_p$  and let  $x \in V \subseteq \bar{V} \subseteq V_p$  be as in the definition of  $n$ -solid. Then  $V$  is not a subset of  $\bar{L}$  since  $x$  is not in  $L = \text{int} \bar{L}$ . Also  $V \cap L \neq \emptyset$  since  $x$  is a limit point of  $L$ . Therefore  $\text{bdy}(L) \cap \bar{V}$  separates  $\bar{V}$ , a contradiction.

In either case we get a connected open subset ( $K$  or  $W$ ) of  $X$  which is not a subset of any  $A_j$ . Since this set ( $K$  or  $W$ ) is  $n$ -solid, we can repeat the process used to get  $x_0$  and  $V_0$ . This gives us the  $x_{p+1}$  and  $V_{p+1}$  we want, completing the induction and the proof.

It is well known that no continuum is a countable union of disjoint closed subsets. In particular, no locally connected continuum is such a union. Since a locally connected continuum is 1-solid, Theorem 2 is a direct generalization of this special case.

Nagami and Roberts have given an example [3, Figure 2] which shows that the conclusion of Theorem 2 does not hold if we require only that  $X$  be a Cantor  $n$ -manifold. This example is a Cantor 2-manifold which is the countable union of disjoint closed sets  $A_{i,j}$  and a Cantor discontinuum.

*COROLLARY.* *If  $X$  is Hausdorff, connected, locally connected, and locally compact, then  $X$  is not a countable union of disjoint closed subsets.*

*PROOF.* Such a space is 1-solid.

Sierpinsky has given an example (see [1, p. 188]) which shows that local connectedness is not superfluous in the Corollary.

*THEOREM 3.* *Let  $m$  and  $n$  be integers such that  $m \geq -1$  and  $n \geq 1$ . If  $X$  is connected, and every point in  $X$  has a neighborhood homeomorphic*

to the  $n$ -cube  $I^n$ , and  $A_1, A_2, \dots$  are closed proper subsets of  $X$  such that  $\dim(A_i \cap A_j) \leq m$  whenever  $i \neq j$ , then  $\dim(X - \bigcup_{i=1}^{\infty} A_i) \geq n - m - 2$ .

PROOF.  $X$  is connected and  $n$ -solid. By Theorem 2 there is a point  $x \in X - \bigcup_{i=1}^{\infty} A_i$ . There is a neighborhood  $F$  of  $x$  which is homeomorphic to  $I^n$ . The dimension of  $X - \bigcup_{i=1}^{\infty} A_i$  is at least as great as the dimension of  $F - \bigcup_{i=1}^{\infty} A_i$ , so it is sufficient to prove the theorem for the case where  $X = I^n$ .

Let  $S^{n-1}$  denote the surface of  $I^n$ . Then  $(I^n - S^{n-1})$  is connected and  $n$ -solid, and no  $A_i$  contains all of it. By Theorem 2 there is a point  $q \in (I^n - S^{n-1}) - \bigcup_{i=1}^{\infty} A_i$ .

Recall step 3 of the proof of Theorem 1 in [3]. A pyramid with apex  $p$  and base  $B$  has been inscribed in  $I^n$ . Let  $h$  be a homeomorphism from  $I^n$  onto  $I^n$  which leaves  $S^{n-1}$  fixed and such that  $h(q) = p$ . Then  $h^{-1}(p) \cap (\bigcup_{i=1}^{\infty} A_i) = \emptyset$  as before. Let  $f = h|_{S^{n-1}}$ , the identity map. Now, using these mappings  $h$  and  $f$ , repeat step 3, the modified step 4, and step 5 to complete the proof.

Corollary 2 of [3] is the special case of this theorem obtained when  $m = -1$ .

#### REFERENCES

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