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A Lower Bound for the Length of the Shortest Carefully Synchronizing Words

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Abstract—We introduce the notion of careful synchronization for partial finite automata as a natural generalization of the synchronization notion for complete finite automata. We obtain a lower bound for the careful synchronization threshold for automata with a given number of states.

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1. INTRODUCTION

A partial finite automaton (PFA) is a triple (Q, Σ, δ), where Q is a finite set of states, Σ is a finite alphabet and δ is a partial function from $Q \times \Sigma$ to Q. Denote by 2^Q the set of all subsets of the set Q and by Σ^* the free Σ-generated monoid with the empty word λ. The function δ can be extended to $2^Q \times \Sigma^*$ in the following natural way. We put $\delta(q, \lambda) = q$ for every $q \in Q$. Let $q \in Q$, $w \in \Sigma^*$, $a \in \Sigma$. If the value of the function δ is defined on the state q and the word w, then we put $\delta(q, wa) = \delta(\delta(q, w), a)$. If $S \subseteq Q$, $w \in \Sigma^*$ and the function $\delta(q, w)$ is defined at all states $q \in S$, then we put $\delta(S, w) = \{\delta(q, w) : q \in S\}$.

A PFA $\mathscr{A} = (Q, \Sigma, \delta)$ is called *carefully synchronizable* if there is a word $w \in \Sigma^*$ such that the value $\delta(Q, w)$ is defined and $|\delta(Q, w)| = 1$. We say that such a word w *carefully synchronizes* the automaton $\mathscr{A}$. We denote by $\mathrm{Syn}(\mathscr{A})$ the set of all words carefully synchronizing the automaton $\mathscr{A}$. It is evident that for every carefully synchronizable automaton there is a carefully synchronizing word of minimum length.

Let us fix an integer n. The natural question arises: How long can be a carefully synchronizing word of minimum length for various automata with n states. Denote by $\omega(n)$ the maximal length of such word. The value $\omega(n)$ is the *careful synchronization threshold* for the set of all carefully synchronizable automata with n states in the following sense: For each automaton $\mathscr{A}$ in this set, the language $\Sigma^{\omega}(n)$ of all words of length $\omega(n)$ contains a word that carefully synchronizes $\mathscr{A}$ and $\omega(n)$ is the least number with this property.

The concept of careful synchronization of PFA is a natural generalization of the synchronization concept for deterministic finite automata (DFA) with everywhere defined transition functions. Let $\mathscr{A} = (Q, \Sigma, \delta)$ be a DFA. A word $w \in \Sigma^*$ is called *synchronizing* for the automaton $\mathscr{A}$ if $|\delta(Q, w)| = 1$. A conjecture proposed by Černý (see [1]) states that every synchronizable automaton with n states can be synchronized by a word of length at most $(n-1)^2$. There have been many attempts to prove it but they all have failed so far. The conjecture has been proved only for some special cases of automata [2–4]. In the general case there is only a cubic upper bound [5]. The corresponding lower bound has been proved by Černý [1].

The problem of estimating the value $\omega(n)$ has been considered by Ito and Shikishima-Tsuji in [6] and [7] (in these sources the value $\omega(n)$ has been denoted by $d_3(n)$). These authors have considered

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