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A Lower Bound for Weak ε-Nets in High Dimension*

Jiří Matoušek

Department of Applied Mathematics, Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz
and
Institut für Theoretische Informatik, ETH Zentrum,
Zürich, Switzerland

Abstract. A finite set $N \subset \mathbf{R}^d$ is a *weak* ε*-net* for an n-point set $X \subset \mathbf{R}^d$ (with respect to convex sets) if it intersects each convex set K with $|K \cap X| \geq \varepsilon n$. It is shown that there are point sets $X \subset \mathbf{R}^d$ for which every weak $\frac{1}{50}$-net has at least $const \cdot e^{\sqrt{d/2}}$ points. This distinguishes the behavior of weak ε-nets with respect to convex sets from ε-nets with respect to classes of shapes like balls or ellipsoids in $\mathbf{R}^d$, where the size can be bounded from above by a polynomial function of d and ε.

Weak ε-nets with respect to convex sets, as defined in the abstract, were introduced by Haussler and Welzl [7] and later applied in results in discrete geometry, most notably in the spectacular proof of the Hadwiger–Debrunner (p, q)-conjecture by Alon and Kleitman [2].

For a finite $X \subset \mathbf{R}^d$, let $f(X, \varepsilon)$ denote the smallest size of a weak ε-net for X, $0 < \varepsilon < 1$, and let

$$f(d, \varepsilon) = \sup\{f(X, \varepsilon) : X \subset \mathbf{R}^d \text{ finite}\}.$$

Alon et al. [1] proved that $f(d, \varepsilon)$ is finite for every $d \geq 1$ and every $\varepsilon > 0$. They established the bounds $f(2, \varepsilon) = O(\varepsilon^{-2})$ and $f(d, \varepsilon) \leq C_d \varepsilon^{-(d+1-\delta(d))}$, where C_d depends only on d and $\delta(d)$ is a positive number tending to zero (exponentially fast) as $d \to \infty$. With a simpler proof, they obtained the slightly worse bound $C'_d \varepsilon^{-(d+1)}$, and here their proof yields $C'_d = d^{O(d)}$. Chazelle et al. [6] improved the bound for all fixed dimensions $d \geq 3$, to $O(\varepsilon^{-d}(\log(1/\varepsilon))^{b(d)})$ with a suitable constant $b(d)$.

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It seems that no lower bound better than the obvious $f(d, \varepsilon) = \Omega(1/\varepsilon)$ is known. Proving a lower bound superlinear in $1/\varepsilon$ for some fixed dimension remains a challenging open problem. In the present note it is shown that, for ε fixed and sufficiently small and $d \to \infty$, $f(d, \varepsilon)$ is at least $e^{\Omega(\sqrt{d})}$. The lower bound is still meaningful up to $\varepsilon \approx e^{-\sqrt{d}}$, but for d fixed it is entirely useless. For simplicity, we do the calculations for a particular value of ε.

Theorem 1. *For all* $d \geq 4$, *we have*

$$f(d, \tfrac{1}{50}) \geq \tfrac{1}{\sqrt{200}} \cdot e^{\sqrt{d/2}}.$$

One can also consider weak ε-nets for set systems $\mathcal{F}$ other than convex sets in $\mathbf{R}^d$, such as the family of all balls, or all ellipsoids, and so on (then N should intersect all $F \in \mathcal{F}$ such that $|F \cap X| \geq \varepsilon n$). For these two examples, balls and ellipsoids, there exist weak ε-nets of size bounded by a polynomial in d (for fixed $\varepsilon > 0$). This follows from a general result of Haussler and Welzl [7]; roughly speaking, their bound applies whenever the sets of $\mathcal{F}$ can be defined by a formula of length polynomial in d. Theorem 1 shows that the weak ε-nets for convex sets behave quite differently.

Proof of Theorem 1. Instead of a finite point set X, we consider the uniform probability measure μ on the $(d-1)$-dimensional unit sphere S^{d-1}. We prove a lower bound for the cardinality of a finite $N \subset \mathbf{R}^d$ intersecting every convex set K with $\mu(K \cap S^{d-1}) \geq \frac{1}{50}$. As will be apparent later, the sets K actually used in the proof have a simple structure, and as far as they are concerned, the measure μ can be approximated with arbitrary precision by the uniform probability measure concentrated on a finite $X \subset S^{d-1}$. As easy limit argument shows that if weak $\frac{1}{50}$-nets of size s exist for all finite X, then a weak $\frac{1}{50}$-net of size s exists for the measure μ as well.

First we need to recall bounds for $\mu(C_t)$, where C_t denotes the spherical cap $S^{d-1} \cap h$, with h being a halfspace at distance t from the origin. For all $t \in [0, 1]$, we have the well-known upper bound

$$\mu(C_t) \leq 2e^{-t^2 d/2}; \tag{1}$$

see, e.g., [8] or [4]. It is also known that this is nearly the right order of magnitude. A convenient explicit lower bound was calculated by Brieden et al. [5]: for $\sqrt{2/d} \leq t < 1$, we have

$$\mu(C_t) \geq \frac{1}{6t\sqrt{d}} (1 - t^2)^{(d-1)/2}. \tag{2}$$

Let $N \subset \mathbf{R}^d$ be an arbitrary set; we may suppose that it is contained in the unit ball bounded by S^{d-1}. Supposing that $n = |N|$ is small, we construct a collection H of halfspaces such that $N \subseteq \bigcup H$ and $\mu(S^{d-1} \setminus \bigcup H) \geq \frac{1}{50}$. Setting $K = \mathbf{R}^d \setminus \bigcup H$, we see that N is not a weak $\frac{1}{50}$-net with respect to convex sets.

We choose a suitable parameter $r \in (0, 1)$. For each point $p \in N$ lying outside the (closed) ball $B(0, r)$, we let h_p be the halfspace containing p and cutting off the smallest cap of S^{d-1} (so the boundary of h_p passes through p and is perpendicular to $0p$). By (1), $\mu(h_p \cap S^{d-1}) < 2e^{-r^2 d/2}$. We choose $r = \sqrt{(2\ln(100n))/d}$ so that the total measure of these caps for all $p \in N$ is smaller than $\frac{1}{50}$.

It remains to deal with the points of N inside $B(0, r)$. Here we want to enclose all points of $N_1 = N \cap B(0, r)$ in a single halfspace h_0 (containing the origin) whose boundary is at distance ρ from the origin, where $\rho \in (0, r)$ is another suitable parameter. First, we can project the points of N_1 centrally from the origin on the boundary of $B(0, r)$; if h_0 encloses the projected set, then it encloses the original set too. We consider a random halfspace h_0 with boundary at distance ρ from the origin (this approach was taken, e.g., by Bárány and Füredi [3]). The expected number of points of N_1 missed by h_0 is $|N_1|$ times the relative measure of the cap cut off by the complement of h_0 on the boundary of $B(0, r)$. The latter equals $\mu(C_{\rho/r}) \leq 2e^{-\rho^2 d/2r^2}$ by (1), and so if $2ne^{-\rho^2 d/2r^2} < 1$, then there is an h_0 enclosing all of N_1. Calculation shows that ρ can be set to $\ln(200n^2)/d$ (one can use $\ln(200n^2) \geq 2\sqrt{\ln(100n)\ln(2n)}$, which follows from the inequality between the arithmetic and geometric means).

If the cap $S^{d-1} \backslash h_0$ has measure at least $\frac{1}{25}$, then N is not a weak $\frac{1}{50}$-net. Therefore $\mu(C_\rho) \leq \frac{1}{25}$. Now for $d \geq 4$ and $t = \sqrt{2/d}$, it follows from (2) that $\mu(C_t) > \frac{1}{25}$, and hence $\rho > \sqrt{2/d}$. Substituting for ρ, we have $\ln(200n^2)/d > \sqrt{2/d}$, and calculation yields the lower bound for n claimed by the theorem. □

Remark. The best available upper bound for the size of a weak ε-net with respect to convex sets in $\mathbf{R}^d$, with $\varepsilon > 0$ fixed, is $d^{O(d)}$. Even for weak ε-nets for the uniform measure μ on the sphere S^{d-1} (used in the proof), I am aware of no substantially better bound (although there are several alternative ways of proving an upper bound in this particular case). Determining the right order of magnitude in this situation might perhaps be less challenging than in the general case.

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