

A LUNA ÉTALE SLICE THEOREM FOR ALGEBRAIC STACKS

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ABSTRACT. We prove that every algebraic stack, locally of finite type over an algebraically closed field with affine stabilizers, is étale-locally a quotient stack in a neighborhood of a point with a linearly reductive stabilizer group. The proof uses an equivariant version of Artin’s algebraization theorem proved in the appendix. We provide numerous applications of the main theorems.

1. INTRODUCTION

Quotient stacks form a distinguished class of algebraic stacks which provide intuition for the geometry of general algebraic stacks. Indeed, equivariant algebraic geometry has a long history with a wealth of tools at its disposal. Thus, it has long been desired—and more recently believed [Alp10, AK14]—that certain algebraic stacks are locally quotient stacks. This is fulfilled by the main result of this paper: at a point with linearly reductive stabilizer, an algebraic stack is étale-locally a quotient stack of an affine scheme by the stabilizer. In the case of smooth algebraic stacks, we can provide a more refined description which resolves the algebro-geometric counterpart to the Weinstein conjectures [Wei00]—now known as Zung’s Theorem [Zun06, CF11, CS13, PPT14]—on the linearization of proper Lie groupoids in differential geometry.

What the main theorems (Theorem 1.1 and 1.2) of this paper really justify is a philosophy that quotient stacks of the form $[\mathrm{Spec} A/G]$, where G is a linearly reductive group, are the building blocks of algebraic stacks near points with linearly reductive stabilizers. These theorems yield a number of applications to old and new problems, including:

- a generalization of Luna’s étale slice theorem to non-affine schemes (§2.2);
- a generalization of Sumihiro’s theorem on torus actions to Deligne–Mumford stacks (§2.3), confirming an expectation of Oprea [Opr06, §2];
- Białynicki-Birula decompositions for smooth Deligne–Mumford stacks (§2.4), generalizing Skowera [Sko13];
- a criterion for the existence of a good moduli space (§2.9), generalizing [KM97, AFSv14];
- a criterion for étale-local equivalence of algebraic stacks (§2.8), extending Artin’s corresponding results for schemes [Art69a, Cor. 2.6];
- the existence of equivariant miniversal deformation spaces for curves (§2.5), generalizing [AK14];
- a characterization of toric Artin stacks in terms of stacky fans [GS15, Thm. 6.1] (§2.12);
- the étale-local quotient structure of a good moduli space (§2.6);

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- formal GAGA for good moduli space morphisms (§2.6), resolving a conjecture of Geraschenko–Zureick-Brown [GZ12, Conj. 32];
- a short proof of Drinfeld’s results [Dri13] on algebraic spaces with a \mathbb{G}_m -action (§2.11); and
- compact generation of derived categories of algebraic stacks (§2.13).

Our first theorem gives a precise description of the étale-local structure of an algebraic stack at a non-singular point with linearly reductive stabilizer. This is the algebro-geometric analogue of Zung’s resolution of the Weinstein conjectures.

Before we state the theorem, we introduce the following notation: if \mathcal{X} is an algebraic stack over a field k and $x \in \mathcal{X}(k)$ is a closed point with stabilizer group scheme G_x , then we let N_x denote the normal space to x viewed as a G_x -representation. If $\mathcal{J} \subseteq \mathcal{O}_{\mathcal{X}}$ denotes the sheaf of ideals defining x , then $N_x = (\mathcal{J}/\mathcal{J}^2)^\vee$. If G_x is smooth, then N_x is identified with the tangent space of \mathcal{X} at x ; see Section 4.1.

Theorem 1.1. *Let \mathcal{X} be a quasi-separated algebraic stack, locally of finite type over an algebraically closed field k , with affine stabilizers. Let $x \in |\mathcal{X}|$ be a smooth and closed point with linearly reductive stabilizer group G_x . Then there exists an affine and étale morphism $(U, u) \rightarrow (N_x//G_x, 0)$, where $N_x//G_x$ denotes the GIT quotient, and a cartesian diagram*

$$\begin{array}{ccc} ([N_x/G_x], 0) & \longleftarrow & ([W/G_x], w) \xrightarrow{f} (\mathcal{X}, x) \\ \downarrow & \square & \downarrow \\ (N_x//G_x, 0) & \longleftarrow & (U, u) \end{array}$$

such that W is affine and f is étale and induces an isomorphism of stabilizer groups at w . In addition, if \mathcal{X} has affine diagonal, the morphism f can be arranged to be affine.

In particular, this theorem implies that \mathcal{X} and $[N_x/G_x]$ have a common étale neighborhood of the form $[\mathrm{Spec} A/G_x]$. Our second theorem gives a local description of an algebraic stack at a (potentially singular) point with respect to a linearly reductive subgroup of the stabilizer.

Theorem 1.2. *Let \mathcal{X} be a quasi-separated algebraic stack, locally of finite type over an algebraically closed field k , with affine stabilizers. Let $x \in \mathcal{X}(k)$ be a point and $H \subseteq G_x$ be a subgroup scheme of the stabilizer such that H is linearly reductive and G_x/H is smooth (resp. étale). Then there exists an affine scheme $\mathrm{Spec} A$ with an action of H , a k -point $w \in \mathrm{Spec} A$ fixed by H , and a smooth (resp. étale) morphism*

$$f: ([\mathrm{Spec} A/H], w) \rightarrow (\mathcal{X}, x)$$

such that $BH \cong f^{-1}(BG_x)$; in particular, f induces the given inclusion $H \rightarrow G_x$ on stabilizer group schemes at w . In addition, if \mathcal{X} has affine diagonal, then the morphism f can be arranged to be affine.

The main techniques employed in the proof of Theorem 1.1 are

- (1) deformation theory,
- (2) coherent completeness,
- (3) Tannaka duality, and
- (4) Artin approximation.

Deformation theory produces an isomorphism between the n th infinitesimal neighborhood $\mathcal{N}_x^{[n]}$ of 0 in $\mathcal{N}_x = [N_x/G_x]$ and $\mathcal{X}^{[n]}$, the n th infinitesimal neighborhood of x in \mathcal{X} . It is not at all obvious, however, that the system of morphisms $\{f^{[n]}: \mathcal{N}_x^{[n]} \rightarrow \mathcal{X}\}$ algebraizes. We establish algebraization in two steps.

The first step is effectivization. To accomplish this, we introduce *coherent completeness*, a key concept of the article. Recall that if (A, \mathfrak{m}) is a complete local ring, then $\mathrm{Coh}(A) = \varprojlim_n \mathrm{Coh}(A/\mathfrak{m}^{n+1})$. Coherent completeness is a generalization of this, which is more refined than the formal GAGA results of [EGA, III.5.1.4] and [GZ12] (see §2.6). What we prove in §3.1 is the following.

Theorem 1.3. *Let G be a linearly reductive affine group scheme over an algebraically closed field k . Let $\mathrm{Spec} A$ be a noetherian affine scheme with an action of G , and let $x \in \mathrm{Spec} A$ be a k -point fixed by G . Suppose that A^G is a complete local ring. Let $\mathcal{X} = [\mathrm{Spec} A/G]$ and let $\mathcal{X}^{[n]}$ be the n th infinitesimal neighborhood of x . Then the natural functor*

$$(1.1) \quad \mathrm{Coh}(\mathcal{X}) \rightarrow \varprojlim_n \mathrm{Coh}(\mathcal{X}^{[n]})$$

is an equivalence of categories.

Tannaka duality for algebraic stacks with affine stabilizers was recently established by the second two authors [HR14b, Thm. 1.1] (also see Theorem 3.5). This proves that morphisms between algebraic stacks $\mathcal{Y} \rightarrow \mathcal{X}$ are equivalent to symmetric monoidal functors $\mathrm{Coh}(\mathcal{X}) \rightarrow \mathrm{Coh}(\mathcal{Y})$. Therefore, to prove Theorem 1.1, we can combine Theorem 1.3 with Tannaka duality (Corollary 3.6) and the above deformation-theoretic observations to show that the morphisms $\{f^{[n]}: \mathcal{N}_x^{[n]} \rightarrow \mathcal{X}\}$ effectivize to $\hat{f}: \hat{\mathcal{N}}_x \rightarrow \mathcal{X}$, where $\hat{\mathcal{N}}_x = \mathcal{N}_x \times_{\mathcal{N}_x//G_x} \mathrm{Spec} \hat{\mathcal{O}}_{\mathcal{N}_x//G_x, 0}$. The morphism \hat{f} is then algebraized using Artin approximation [Art69a].

The techniques employed in the proof of Theorem 1.2 are similar, but the methods are more involved. Since we no longer assume that $x \in \mathcal{X}(k)$ is a non-singular point, we cannot expect an étale or smooth morphism $\mathcal{N}_x \rightarrow \mathcal{X}$. Using Theorem 1.3 and Tannaka duality, however, we can produce a closed substack $\hat{\mathcal{H}}$ of $\hat{\mathcal{N}}_x$ and a formally versal morphism $\hat{f}: \hat{\mathcal{H}} \rightarrow \mathcal{X}$. To algebraize \hat{f} , we apply an equivariant version of Artin algebraization (Corollary A.14), which we believe is of independent interest.

For tame stacks with finite inertia, Theorems 1.1 and 1.2 are one of the main results of [AOV08]. The structure of algebraic stacks with infinite stabilizers has been poorly understood until the present article. For algebraic stacks with infinite stabilizers that are not—or are not known to be—quotient stacks, Theorems 1.1 and 1.2 were only known when $\mathcal{X} = \mathfrak{M}_{g,n}^{\mathrm{ss}}$ is the moduli stack of semistable curves. This is the central result of [AK14], where it is also shown that f can be arranged to be representable. For certain quotient stacks, Theorems 1.1 and 1.2 can be obtained using traditional methods in equivariant algebraic geometry, see §2.2 for details.

1.1. Some remarks on the hypotheses. We mention here two counterexamples to Theorems 1.1 and 1.2 if some of the hypotheses are weakened.

Example 1.4. Some reductivity assumption of the stabilizer G_x is necessary in Theorem 1.2. For instance, consider the group scheme $G = \mathrm{Spec} k[x, y]_{xy+1} \rightarrow \mathbb{A}^1 = \mathrm{Spec} k[x]$ (with multiplication defined by $y \mapsto xyy' + y + y'$), where the generic fiber is \mathbb{G}_m but the fiber over the origin is \mathbb{G}_a . Let $\mathcal{X} = BG$ and $x \in |\mathcal{X}|$ be the point corresponding to the origin. There does not exist an étale morphism $([W/\mathbb{G}_a], w) \rightarrow (\mathcal{X}, x)$, where W is an algebraic space over k with an action of \mathbb{G}_a .

Example 1.5. It is essential to require that the stabilizer groups are affine in a neighborhood of $x \in |\mathcal{X}|$. For instance, let X be a smooth curve and $\mathcal{E} \rightarrow X$ be a group scheme whose generic fiber is a smooth elliptic curve but the fiber over a point $x \in X$ is isomorphic to \mathbb{G}_m . Let $\mathcal{X} = B\mathcal{E}$. There is no étale morphism $([W/\mathbb{G}_m], w) \rightarrow (\mathcal{X}, x)$, where W is an affine k -scheme with an action of \mathbb{G}_m .

1.2. Generalizations. Using a similar argument, one can in fact establish a generalization of Theorem 1.2 to the relative and mixed characteristic setting. This requires developing some background material on deformations of linearly reductive group schemes, a more general version of Theorem 1.3 and a generalization of the formal functions theorem for good moduli spaces. To make this paper more accessible, we have decided to postpone the relative statement until a future paper.

If G_x is not reductive, it is possible that one could find an étale neighborhood $([\mathrm{Spec} A/GL_n], w) \rightarrow (\mathcal{X}, x)$. However, this is not known even if $\mathcal{X} = B_{k[\epsilon]}G_\epsilon$ where G_ϵ is a deformation of a non-reductive algebraic group [Con10].

In characteristic p , the linearly reductive hypothesis in Theorem 1.2 is quite restrictive. Indeed, an algebraic group G over an algebraically closed field k of characteristic p is linearly reductive if and only if G^0 is a torus and $|G/G^0|$ is coprime to p [Nag62]. We ask however:

Question 1.6. *Does a variant of Theorems 1.1 and 1.2 remain true if “linearly reductive” is replaced with “reductive”?*

We remark that if \mathcal{X} is a Deligne–Mumford stack, then the conclusion of Theorem 1.2 holds. We also ask:

Question 1.7. *If \mathcal{X} has separated (resp. quasi-affine) diagonal, then can the morphism f in Theorems 1.1 and 1.2 be chosen to be representable (resp. quasi-affine)?*

If \mathcal{X} does not have separated diagonal, then the morphism f cannot necessarily be chosen to be representable. For instance, consider the non-separated affine line as a group scheme $G \rightarrow \mathbb{A}^1$ whose generic fiber is trivial but the fiber over the origin is \mathbb{Z}_2 . Then BG admits an étale neighborhood $f: [\mathbb{A}^1/\mathbb{Z}_2] \rightarrow BG$ which induces an isomorphism of stabilizer groups at 0, but f is not representable in a neighborhood.

1.3. Notation. An algebraic stack \mathcal{X} is quasi-separated if the diagonal and the diagonal of the diagonal are quasi-compact. An algebraic stack \mathcal{X} has *affine stabilizers* if for every field k and point $x: \mathrm{Spec} k \rightarrow \mathcal{X}$, the stabilizer group G_x is affine. If \mathcal{X} is an algebraic stack and $\mathcal{Z} \subseteq \mathcal{X}$ is a closed substack, we will denote by $\mathcal{X}_{\mathcal{Z}}^{[n]}$ the n th nilpotent thickening of $\mathcal{Z} \subseteq \mathcal{X}$ (i.e., if $\mathcal{J} \subseteq \mathcal{O}_{\mathcal{X}}$ is the ideal sheaf defining \mathcal{Z} , then $\mathcal{X}_{\mathcal{Z}}^{[n]} \rightarrow \mathcal{X}$ is defined by \mathcal{J}^{n+1}). If $x \in |\mathcal{X}|$ is a closed point, the *n th infinitesimal neighborhood of x* is the n th nilpotent thickening of the inclusion of the residual gerbe $\mathcal{G}_x \rightarrow \mathcal{X}$.

Recall from [Alp13] that a quasi-separated and quasi-compact morphism $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if the push-forward functor ϕ_* on the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -modules is exact. If \mathcal{Y} has quasi-affine diagonal and ϕ has affine diagonal, then ϕ is cohomologically affine if and only if $R\phi_*: D_{\mathrm{QCoh}}^+(\mathcal{X}) \rightarrow D_{\mathrm{QCoh}}^+(\mathcal{Y})$ is t -exact, cf. [Alp13, Prop. 3.10 (vii)] and [HNR14, Prop. 2.1]; this equivalence is false if \mathcal{Y} does not have affine stabilizers [HR14a, Rem. 1.6]. If $G \rightarrow \mathrm{Spec} k$ is an affine group scheme of finite type, we say that G is *linearly reductive* if $BG \rightarrow \mathrm{Spec} k$ is cohomologically affine. We say that G is an *algebraic group over k* if G is a smooth, affine group scheme over k . A quasi-separated and quasi-compact morphism $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic stacks is a *good moduli space* if \mathcal{Y} is an algebraic space, ϕ is cohomologically affine and $\mathcal{O}_{\mathcal{Y}} \rightarrow \phi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

If G is an affine group scheme of finite type over k acting on an algebraic space X , we say that a G -invariant morphism $\pi: X \rightarrow Y$ of algebraic spaces is a *good GIT quotient* if the induced map $[X/G] \rightarrow Y$ is a good moduli space; we often write $Y = X//G$. In the case that G is linearly reductive, a G -equivariant morphism $\pi: X \rightarrow Y$ is a good GIT quotient if and only if π is affine and $\mathcal{O}_Y \rightarrow (\pi_*\mathcal{O}_X)^G$ is an isomorphism.

If \mathcal{X} is a noetherian algebraic stack, we denote by $\text{Coh}(\mathcal{X})$ the category of coherent $\mathcal{O}_{\mathcal{X}}$ -modules.

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2. APPLICATIONS

Theorems 1.1 and 1.2 have many striking applications. We first record some immediate consequences of Theorems 1.1 and 1.2. Unless stated otherwise, for this section k will denote an algebraically closed field.

2.1. Immediate consequences. If \mathcal{X} is a quasi-separated algebraic stack, locally of finite type over k with affine stabilizers, and $x \in \mathcal{X}(k)$ has linearly reductive stabilizer G_x , then

- (1) there is an étale neighborhood of x with a closed embedding into a smooth algebraic stack;
- (2) there is an étale-local description of the cotangent complex $L_{\mathcal{X}/k}$ of \mathcal{X} in terms of the cotangent complex $L_{\mathcal{W}/k}$ of $\mathcal{W} = [\text{Spec } A/G_x]$. If $x \in |\mathcal{X}|$ is a smooth point (so that \mathcal{W} can be taken to be smooth), then $L_{\mathcal{W}/k}$ admits an explicit description. In general, the $[0, 1]$ -truncation of $L_{\mathcal{W}/k}$ can be described explicitly by appealing to (1);
- (3) for any representation V of G_x , there exists a vector bundle over an étale neighborhood of x extending V ; and
- (4) the G_x -invariants of a formal miniversal deformation space of x is isomorphic to the completion of a finite type k -algebra.

We now state some further applications of Theorems 1.1 and 1.2. We will defer their proofs until §5.

2.2. Generalization of Luna's étale slice theorem. We now provide a refinement of Theorem 1.2 in the case that $\mathcal{X} = [X/G]$ is a quotient stack. The following theorem provides a generalization of Luna's étale slice theorem.

Theorem 2.1. *Let X be a quasi-separated algebraic space, locally of finite type over k , with an action of an algebraic group G . Let $x \in X(k)$ be a point with a linearly reductive stabilizer G_x . Then there exists an affine scheme W with an action of G_x which fixes a point w , and an unramified G_x -equivariant morphism $(W, w) \rightarrow (X, x)$ such that $\tilde{f}: W \times^{G_x} G \rightarrow X$ is étale.¹*

If X admits a good GIT quotient $X \rightarrow X//G$, then it is possible to arrange that the induced morphism $W//G_x \rightarrow X//G$ is étale and $W \times^{G_x} G \cong W//G_x \times_{X//G} X$.

Let $N_x = T_{X,x}/T_{G \cdot x,x}$ be the normal space to the orbit at x ; this inherits a natural linear action of G_x . If $x \in X$ is smooth, then it can be arranged that there is an étale G_x -equivariant $W \rightarrow N_x$ such that $W//G_x \rightarrow N_x//G_x$ is étale and

$$\begin{array}{ccc} N_x \times^{G_x} G & \longleftarrow & W \times^{G_x} G \xrightarrow{\tilde{f}} X \\ \downarrow & \square & \downarrow \\ N_x//G_x & \longleftarrow & W//G_x \end{array}$$

is cartesian.

¹Here, $W \times^{G_x} G$ denotes the quotient $(W \times G)/G_x$. Note that there is an identification of GIT quotients $(W \times^{G_x} G)//G \cong W//G_x$.

Remark 2.2. The theorem above follows from Luna’s étale slice theorem [Lun73] if X is affine. In this case, Luna’s étale slice theorem is stronger than Theorem 2.1 as it asserts additionally that $W \rightarrow X$ can be arranged to be a locally closed immersion (which is obtained by choosing a G_x -equivariant section of $T_{X,x} \rightarrow N_x$ and then restricting to an open subscheme of the inverse image of N_x under a G_x -equivariant étale morphism $X \rightarrow T_{X,x}$). Note that while [Lun73] assumes that $\text{char}(k) = 0$ and G is reductive, the argument goes through unchanged in arbitrary characteristic if G is smooth, and G_x is smooth and linearly reductive. Moreover, with minor modifications, the argument in [Lun73] is also valid if G_x is not necessarily smooth.

Remark 2.3. More generally, if X is a normal scheme, it is shown in [AK14, §2.1] that $W \rightarrow X$ can be arranged to be a locally closed immersion. However, when X is not normal or is not a scheme, one cannot always arrange $W \rightarrow X$ to be a locally closed immersion and therefore we must allow unramified “slices” in the theorem above.

2.3. Generalization of Sumihiro’s theorem on torus actions. In [Opr06, §2], Oprea speculates that every quasi-compact Deligne–Mumford stack \mathcal{X} with a torus action has an equivariant étale atlas $\text{Spec } A \rightarrow \mathcal{X}$. He proves this when $\mathcal{X} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is the moduli space of stable maps and the action is induced by any action of \mathbb{G}_m on \mathbb{P}^r and obtains some nice applications. We show that Oprea’s speculation holds in general.

Let T be a torus acting on an algebraic stack \mathcal{X} , locally of finite type over k , via $\sigma: T \times \mathcal{X} \rightarrow \mathcal{X}$. Let $\mathcal{Y} = [\mathcal{X}/T]$. Let $x \in \mathcal{X}(k)$ be a point with image $y \in \mathcal{Y}(k)$. There is an exact sequence

$$(2.1) \quad 1 \rightarrow G_x \rightarrow G_y \rightarrow T_x \rightarrow 1$$

where the stabilizer $T_x \subseteq T$ is defined by the fiber product

$$(2.2) \quad \begin{array}{ccc} T_x \times BG_x & \xrightarrow{\sigma_x} & BG_x \\ \downarrow & \square & \downarrow \\ T \times BG_x & \xrightarrow{\sigma|_x} & \mathcal{X} \end{array}$$

and $\sigma|_x: T \times BG_x \xrightarrow{(\text{id}, \iota_x)} T \times \mathcal{X} \xrightarrow{\sigma} \mathcal{X}$. Observe that $G_y = \text{Spec } k \times_{BG_x} T_x$. The exact sequence (2.1) is trivially split if and only if the induced action σ_x of T_x on BG_x is trivial. The sequence is split if and only if the action σ_x comes from a group homomorphism $T \rightarrow \text{Aut}(G_x)$.

Theorem 2.4. *Let \mathcal{X} be a quasi-separated algebraic (resp. Deligne–Mumford) stack, locally of finite type over k , with affine stabilizers. Let T be a torus with an action on \mathcal{X} . Let $x \in \mathcal{X}(k)$ be a point such that G_x is smooth and the exact sequence (2.1) is split (e.g., \mathcal{X} is an algebraic space). There exists a T -equivariant smooth (resp. étale) neighborhood $(\text{Spec } A, u) \rightarrow (\mathcal{X}, x)$ that induces an isomorphism of stabilizers at u .*

The theorem above fails when (2.1) does not split. For a simple example, consider the action of $T = \mathbb{G}_m$ on $\mathcal{X} = B\mu_n$ defined by: for $t \in \mathbb{G}_m(S) = \Gamma(S, \mathcal{O}_S)^*$ and $(\mathcal{L}, \alpha) \in B\mu_n(S)$ (where \mathcal{L} is a line bundle on S and $\alpha: \mathcal{L}^{\otimes n} \rightarrow \mathcal{O}_S$ is an isomorphism), then $t \cdot (\mathcal{L}, \alpha) = (\mathcal{L}, t \circ \alpha)$. The exact sequence of (2.1) is $1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \rightarrow 1$ which does not split. In this case though, there is an étale presentation $\text{Spec } k \rightarrow B\mu_n$ which is equivariant under $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$. More generally, we have:

Theorem 2.5. *Let \mathcal{X} be a quasi-separated Deligne–Mumford stack, locally of finite type over k . Let T be a torus with an action on \mathcal{X} . If $x \in \mathcal{X}(k)$, then there exist a*

reparameterization $\alpha: T \rightarrow T$ and an étale neighborhood $(\text{Spec } A, u) \rightarrow (\mathcal{X}, x)$ that is equivariant with respect to α .

In the case that \mathcal{X} is a normal scheme, Theorem 2.4 was proved by Sumihiro [Sum74, Cor. 2], [Sum75, Cor. 3.11]; then $\text{Spec } A \rightarrow \mathcal{X}$ can be taken to be an open neighborhood. The nodal cubic with a \mathbb{G}_m -action provides an example where there does not exist a \mathbb{G}_m -invariant affine open cover. Theorem 2.4 was also known if \mathcal{X} is a quasi-projective scheme [Bri13, Thm. 1.1(iii)] or if \mathcal{X} is a smooth, proper, tame and irreducible Deligne–Mumford stack, whose generic stabilizer is trivial and whose coarse moduli space is a scheme [Sko13, Prop. 3.2]. We can also prove:

Theorem 2.6. *Let X be a quasi-separated algebraic space, locally of finite type over k , with an action of an affine group scheme G of finite type over k . Let $x \in X(k)$ be a point with linearly reductive stabilizer G_x . Then there exists an affine scheme W with an action of G and a G -equivariant étale neighborhood $W \rightarrow X$ of x .*

This is a partial generalization of another result of Sumihiro [Sum74, Lemma. 8], [Sum75, Thm. 3.8]. He proves the existence of an open G -equivariant covering by quasi-projective subschemes when X is a normal scheme and G is connected.

2.4. Białyński-Birula decompositions. In [Opr06, Prop. 5], Oprea proved the existence of a Białyński-Birula decomposition [BB73] for a smooth Deligne–Mumford stack \mathcal{X} with a \mathbb{G}_m -action provided that there exists a \mathbb{G}_m -equivariant, separated, étale atlas $\text{Spec } A \rightarrow \mathcal{X}$. Therefore, Theorem 2.5 implies:

Theorem 2.7. *Let \mathcal{X} be a smooth, proper Deligne–Mumford stack over k with separated diagonal, equipped with a \mathbb{G}_m -action. Let $\mathcal{F} = \sqcup_i \mathcal{F}_i$ be the decomposition of the fixed substack into connected components. Then \mathcal{X} decomposes into disjoint, locally closed, \mathbb{G}_m -equivariant substacks \mathcal{X}_i which are \mathbb{G}_m -equivariant affine fibrations over \mathcal{F}_i .*

This is proven in [Sko13, Thm. 3.5] under the additional assumptions that \mathcal{X} is tame, and the coarse moduli space of \mathcal{X} is a scheme. See also Section 2.11 for a similar statement due to Drinfeld for algebraic spaces with a \mathbb{G}_m -action.

2.5. Existence of equivariant versal deformations of curves. By a *curve*, we mean a proper scheme over k of pure dimension one.

Theorem 2.8. *Let C be an n -pointed curve. Suppose that every connected component of C is either reduced of arithmetic genus $g \neq 1$ or contains a marked point. Suppose that a linearly reductive group scheme H acts on C . If $\text{Aut}(C)$ is smooth, then there exist an affine scheme W of finite type over k with an action of H fixing a point $w \in W$ and a miniversal deformation*

$$\begin{array}{ccc} \mathcal{C} & \longleftarrow & C \\ \downarrow & \square & \downarrow \\ W & \xleftarrow{w} & \text{Spec } k \end{array}$$

of $C \cong \mathcal{C}_w$ such that there exists an action of H on the total family \mathcal{C} compatible with the action of H on W and \mathcal{C}_w .

The theorem above was proven for Deligne–Mumford semistable curves in [AK14].

2.6. Good moduli spaces. In the following result, we determine the étale-local structure of good moduli space morphisms.

Theorem 2.9. *Let \mathcal{X} be a locally noetherian algebraic stack over k . Suppose there exists a good moduli space X such that the moduli map $\pi: \mathcal{X} \rightarrow X$ is of finite type with affine diagonal. If $x \in \mathcal{X}(k)$ is a closed point, then there exists an affine scheme $\text{Spec } A$ with an action of G_x and a cartesian diagram*

$$\begin{array}{ccc} [\text{Spec } A/G_x] & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \pi \\ \text{Spec } A//G_x & \longrightarrow & X \end{array}$$

such that $\text{Spec } A//G_x \rightarrow X$ is an étale neighborhood of $\pi(x)$.

The following corollary answers negatively a question of Geraschenko–Zureick-Brown [GZ12, Qstn. 32]: does there exist an algebraic stack, with affine diagonal and good moduli space a field, that is not a quotient stack? In the equicharacteristic setting, this result also settles a conjecture of theirs: formal GAGA holds for good moduli spaces with affine diagonal [GZ12, Conj. 28]. The general case will be treated in forthcoming work.

Corollary 2.10. *Let \mathcal{X} be a noetherian algebraic stack over a field k (not assumed to be algebraically closed) with affine diagonal. Suppose there exists a good moduli space $\pi: \mathcal{X} \rightarrow \text{Spec } R$ of finite type, where (R, \mathfrak{m}) is a complete local ring.*

- (1) *Then $\mathcal{X} \cong [\text{Spec } B/\text{GL}_n]$; in particular, \mathcal{X} has the resolution property; and*
- (2) *the natural functor*

$$\text{Coh}(\mathcal{X}) \rightarrow \varprojlim_n \text{Coh}(\mathcal{X} \times_{\text{Spec } R} \text{Spec } R/\mathfrak{m}^{n+1})$$

is an equivalence of categories.

Remark 2.11. If k is algebraically closed, then in (1) above, \mathcal{X} is in fact isomorphic to a quotient stack $[\text{Spec } A/G_x]$ where G_x is the stabilizer of the unique closed point.

2.7. Existence of coherent completions. Let \mathcal{X} be a noetherian algebraic stack with affine stabilizers and $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack. Denote by $\mathcal{X}_{\mathcal{Z}}^{[n]}$ the n th nilpotent thickening of $\mathcal{Z} \subseteq \mathcal{X}$. We say that \mathcal{X} is *coherently complete along \mathcal{Z}* if the natural functor

$$\text{Coh}(\mathcal{X}) \rightarrow \varprojlim_n \text{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})$$

is an equivalence of categories. When $|\mathcal{Z}|$ consists of a single point x , we say that (\mathcal{X}, x) is a *complete local stack* if \mathcal{X} is coherently complete along the residual gerbe \mathcal{G}_x . See Section 3.1 for more details on coherent completion. The next result asserts that the coherent completion always exists under very mild hypotheses.

Theorem 2.12. *Let \mathcal{X} be a quasi-separated algebraic stack, locally of finite type over k , with affine stabilizers. For any point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer G_x , there exists a complete local stack $(\widehat{\mathcal{X}}_x, \widehat{x})$ and a morphism $\eta: (\widehat{\mathcal{X}}_x, \widehat{x}) \rightarrow (\mathcal{X}, x)$ inducing isomorphisms of n th infinitesimal neighborhoods of \widehat{x} and x . The pair $(\widehat{\mathcal{X}}_x, \eta)$ is unique up to unique 2-isomorphism.*

We call $\widehat{\mathcal{X}}_x$ the *coherent completion of \mathcal{X} at x* . If $\mathcal{W} = [\text{Spec } A/G_x] \rightarrow \mathcal{X}$ is an étale morphism as in Theorem 1.2 and $\pi: \mathcal{W} \rightarrow W = \text{Spec } A^{G_x}$ is the good moduli space of \mathcal{W} , then

$$\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \widehat{W}_{\pi(x)} = \mathcal{W} \times_W \widehat{\text{Spec } A^{G_x}}.$$

That is, $\widehat{\mathcal{X}}_x = [\mathrm{Spec} B/G_x]$ where $B = A \otimes_{A^{G_x}} \widehat{A^{G_x}}$ and $\widehat{A^{G_x}}$ denotes the completion at $\pi(x)$ (Theorem 1.3). In particular, $B^G \rightarrow B$ is of finite type and B^G is the completion of an algebra of finite type over k .

The *henselization of \mathcal{X} at x* is the stack $\mathcal{X}_x^h = \mathcal{W} \times_W \mathrm{Spec}(A^{G_x})^h$. This stack also satisfies a universal property (initial among pro-étale neighborhoods of the residual gerbe at x) and will be treated in forthcoming work.

2.8. Étale-local equivalences. Before we state the next result, let us recall that if \mathcal{X} is an algebraic stack, locally of finite type over k , and $x \in \mathcal{X}(k)$ is a point, then a formal miniversal deformation space of x is a formal affine scheme $\widehat{\mathrm{Def}}(x)$ together with a formally smooth morphism $\widehat{\mathrm{Def}}(x) \rightarrow \mathcal{X}$ which is an isomorphism on tangent spaces. If the stabilizer group scheme G_x is smooth and linearly reductive, $\widehat{\mathrm{Def}}(x)$ inherits an action of G_x .

Theorem 2.13. *Let \mathcal{X} and \mathcal{Y} be quasi-separated algebraic stacks, locally of finite type over k , with affine stabilizers. Suppose $x \in \mathcal{X}(k)$ and $y \in \mathcal{Y}(k)$ are points with smooth linearly reductive stabilizer group schemes G_x and G_y , respectively. Then the following are equivalent:*

- (1) *There exist an isomorphism $G_x \rightarrow G_y$ of group schemes and an isomorphism $\widehat{\mathrm{Def}}(x) \rightarrow \widehat{\mathrm{Def}}(y)$ of formal miniversal deformation spaces which is equivariant with respect to $G_x \rightarrow G_y$.*
- (2) *There exists an isomorphism $\widehat{\mathcal{X}}_x \rightarrow \widehat{\mathcal{Y}}_y$.*
- (3) *There exist an affine scheme $\mathrm{Spec} A$ with an action of G_x , a point $w \in \mathrm{Spec} A$ fixed by G_x , and a diagram of étale morphisms*

$$\begin{array}{ccc} & [\mathrm{Spec} A/G_x] & \\ f \swarrow & & \searrow g \\ \mathcal{X} & & \mathcal{Y} \end{array}$$

such that $f(w) = x$ and $g(w) = y$, and both f and g induce isomorphisms of stabilizer groups at w .

If additionally $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$ are smooth, then the conditions above are equivalent to the existence of an isomorphism $G_x \rightarrow G_y$ of group schemes and an isomorphism $T_{\mathcal{X},x} \rightarrow T_{\mathcal{Y},y}$ of tangent spaces which is equivariant under $G_x \rightarrow G_y$.

Remark 2.14. If the stabilizers G_x and G_y are not smooth, then the theorem above remains true (with the same argument) if the formal miniversal deformation spaces are replaced with flat adic presentations (Definition A.2) and the tangent spaces are replaced with normal spaces.

2.9. Characterization of when \mathcal{X} admits a good moduli space. Using the existence of completions, we can give an intrinsic characterization of those algebraic stacks that admit a good moduli space.

We will need one preliminary definition. We say that a geometric point $y: \mathrm{Spec} l \rightarrow \mathcal{X}$ is *geometrically closed* if the image of $(y, \mathrm{id}): \mathrm{Spec} l \rightarrow \mathcal{X} \times_k l$ is a closed point of $|\mathcal{X} \times_k l|$.

Theorem 2.15. *Let \mathcal{X} be an algebraic stack, locally of finite type over k , with affine diagonal. Then \mathcal{X} admits a good moduli space if and only if*

- (1) *For every point $y \in \mathcal{X}(k)$, there exists a unique closed point in the closure $\overline{\{y\}}$.*
- (2) *For every closed point $x \in \mathcal{X}(k)$, the stabilizer group scheme G_x is linearly reductive and the morphism $\widehat{\mathcal{X}}_x \rightarrow \mathcal{X}$ from the coherent completion of \mathcal{X} at x satisfies:*

- (a) *The morphism $\widehat{\mathcal{X}}_x \rightarrow \mathcal{X}$ is stabilizer preserving at every point; that is, $\widehat{\mathcal{X}}_x \rightarrow \mathcal{X}$ induces an isomorphism of stabilizer groups for every point $\xi \in |\widehat{\mathcal{X}}_x|$.*
- (b) *The morphism $\widehat{\mathcal{X}}_x \rightarrow \mathcal{X}$ maps geometrically closed points to geometrically closed points.*
- (c) *The map $\widehat{\mathcal{X}}_x(k) \rightarrow \mathcal{X}(k)$ is injective.*

Remark 2.16. The quotient $[\mathbb{P}^1/\mathbb{G}_m]$ (where \mathbb{G}_m acts on \mathbb{P}^1 via multiplication) does not satisfy (1). If $\mathcal{X} = [X/\mathbb{Z}_2]$ is the quotient of the non-separated affine line X by the \mathbb{Z}_2 -action which swaps the origins (and acts trivially elsewhere), then the map $\mathrm{Spec} k[[x]] = \widehat{\mathcal{X}}_0 \rightarrow \mathcal{X}$ from the completion at the origin does not satisfy (2a). If $\mathcal{X} = [(\mathbb{A}^2 \setminus 0)/\mathbb{G}_m]$ where \mathbb{G}_m -acts via $t \cdot (x, y) = (x, ty)$ and $p = (0, 1) \in |\mathcal{X}|$, then the map $\mathrm{Spec} k[[x]] = \widehat{\mathcal{X}}_p \rightarrow \mathcal{X}$ does not satisfy (2b). If $\mathcal{X} = [C/\mathbb{G}_m]$ where C is the nodal cubic curve with a \mathbb{G}_m -action and $p \in |\mathcal{X}|$ denotes the image of the node, then $[\mathrm{Spec}(k[x, y]/xy)/\mathbb{G}_m] = \widehat{\mathcal{X}}_p \rightarrow \mathcal{X}$ does not satisfy (2c). (Here \mathbb{G}_m acts on coordinate axes via $t \cdot (x, y) = (tx, t^{-1}y)$.) These pathological examples in fact appear in many natural moduli stacks; see [AFSv14, Appendix A].

Remark 2.17. Consider the non-separated affine line as a group scheme $G \rightarrow \mathbb{A}^1$ whose generic fiber is trivial but the fiber over the origin is \mathbb{Z}_2 . In this case (2a) is not satisfied. Nevertheless, the stack quotient $\mathcal{X} = [\mathbb{A}^1/G]$ does have a good moduli space $X = \mathbb{A}^1$ but $\mathcal{X} \rightarrow X$ has non-separated diagonal.

Remark 2.18. When \mathcal{X} has finite stabilizers, then conditions (1), (2b) and (2c) are always satisfied. Condition (2a) is satisfied if and only if the inertia stack is finite over \mathcal{X} . In this case, the good moduli space of \mathcal{X} coincides with the coarse space of \mathcal{X} , which exists by [KM97].

2.10. Algebraicity results. In this subsection, we fix a field k (not necessarily algebraically closed), an algebraic space S locally of finite type over k , and an algebraic stack \mathcal{W} of finite type over S with affine diagonal over S such that $\mathcal{W} \rightarrow S$ is a good moduli space. We prove the following algebraicity results.

Theorem 2.19 (Stacks of coherent sheaves). *The S -stack $\underline{\mathrm{Coh}}_{\mathcal{W}/S}$, whose objects over $S' \rightarrow S$ are finitely presented quasi-coherent sheaves on $\mathcal{W} \times_S S'$ flat over S' , is an algebraic stack, locally of finite type over S , with affine diagonal over S .*

Corollary 2.20 (Quot schemes). *If \mathcal{F} is a quasi-coherent $\mathcal{O}_{\mathcal{W}}$ -module, then the S -sheaf $\underline{\mathrm{Quot}}_{\mathcal{W}/S}(\mathcal{F})$, whose objects over $S' \rightarrow S$ are quotients $p_1^* \mathcal{F} \rightarrow \mathcal{G}$ (where $p_1: \mathcal{W} \times_S S' \rightarrow \mathcal{W}$) such that \mathcal{G} is a finitely presented quasi-coherent $\mathcal{O}_{\mathcal{W} \times_S S'}$ -module flat over S' , is a separated algebraic space over S . If \mathcal{F} is of finite presentation, then $\underline{\mathrm{Quot}}_{\mathcal{W}/S}(\mathcal{F})$ is locally of finite presentation.*

Corollary 2.21 (Hilbert schemes). *The S -sheaf $\underline{\mathrm{Hilb}}_{\mathcal{W}/S}$, whose objects over $S' \rightarrow S$ are closed substacks $\mathcal{Z} \subseteq \mathcal{W} \times_S S'$ such that \mathcal{Z} is flat and locally of finite presentation over S , is a separated algebraic space locally of finite type over S .*

Theorem 2.22 (Hom stacks). *Let \mathcal{X} be a quasi-separated algebraic stack, locally of finite type over S with affine stabilizers. If $\mathcal{W} \rightarrow S$ is flat, then the S -stack $\underline{\mathrm{Hom}}_S(\mathcal{W}, \mathcal{X})$, whose objects are pairs consisting of a morphism $S' \rightarrow S$ of algebraic spaces and a morphism $\mathcal{W} \times_S S' \rightarrow \mathcal{X}$ of algebraic stacks over S , is an algebraic stack, locally of finite type over S , with quasi-separated diagonal. If $\mathcal{X} \rightarrow S$ has affine (resp. quasi-affine, resp. separated) diagonal, then the same is true for $\underline{\mathrm{Hom}}_S(\mathcal{W}, \mathcal{X}) \rightarrow S$.*

Variants of the above results were considered in [HLP14, Thm. 1.6]. We also prove the following, which we have not seen in the literature before.

Corollary 2.23 (*G*-equivariant Hom sheaves). *Let W , X and S be quasi-separated algebraic spaces, locally of finite type over k . Let G be a linearly reductive affine group scheme acting on W and X . Let $W \rightarrow S$ and $X \rightarrow S$ be G -invariant morphisms. Suppose that $W \rightarrow S$ is flat and a good GIT quotient. Then the S -sheaf $\underline{\mathrm{Hom}}_S^G(W, X)$, whose objects over $S' \rightarrow S$ are G -equivariant S -morphisms $W \times_S S' \rightarrow X$, is a quasi-separated algebraic space, locally of finite type over S .*

2.11. Drinfeld's results on algebraic spaces with \mathbb{G}_m -actions. Let Z be a quasi-separated algebraic space, locally of finite type over a field k (not assumed to be algebraically closed), with an action of \mathbb{G}_m . Define the following sheaves on Sch/k :

$$\begin{aligned} Z^0 &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}(\mathrm{Spec} k, Z) && \text{(the fixed locus)} \\ Z^+ &:= \underline{\mathrm{Hom}}^{\mathbb{G}_m}(\mathbb{A}^1, Z) && \text{(the attractor)} \end{aligned}$$

where \mathbb{G}_m acts on \mathbb{A}^1 by multiplication, and define the sheaf \tilde{Z} on $\mathrm{Sch}/\mathbb{A}^1$ by

$$\tilde{Z} := \underline{\mathrm{Hom}}_{\mathbb{A}^1}^{\mathbb{G}_m}(\mathbb{A}^2, Z \times \mathbb{A}^1)$$

where \mathbb{G}_m acts on \mathbb{A}^2 via $t \cdot (x, y) = (tx, t^{-1}y)$ and acts on \mathbb{A}^1 trivially, and the morphism $\mathbb{A}^2 \rightarrow \mathbb{A}^1$ is defined by $(x, y) \mapsto xy$.

Theorem 2.24. [Dri13, Prop. 1.2.2, Thm. 1.4.2 and Thm. 2.2.2] *With the hypotheses above, Z^0 , Z^+ and \tilde{Z} are quasi-separated algebraic spaces locally of finite type over k . Moreover, the natural morphism $Z^0 \rightarrow Z$ is a closed immersion, and the natural morphism $Z^+ \rightarrow Z^0$ obtained by restricting to the origin is affine.*

The algebraicity follows directly from Corollary 2.23. The final statements follow from Theorem 2.4 above and Lemma 5.8 proved in Section 5.12.

2.12. The resolution property holds étale-locally.

Theorem 2.25. *Let \mathcal{X} be a quasi-separated algebraic stack, of finite type over a perfect (resp. arbitrary) field k , with affine stabilizers. Assume that for every closed point $x \in |\mathcal{X}|$, the unit component G_x^0 of the stabilizer group scheme G_x is linearly reductive. Then there exists*

- (1) a finite field extension k'/k ;
- (2) a linearly reductive group scheme G over k' ;
- (3) a k' -algebra A with an action of G ; and
- (4) an étale (resp. quasi-finite flat) surjection $p: [\mathrm{Spec} A/G] \rightarrow \mathcal{X}$.

Moreover,

- (a) If \mathcal{X} has affine diagonal, then p can be arranged to be affine.
- (b) We can replace G with GL_n (which is linearly reductive in characteristic zero).

A stack of the form $[\mathrm{Spec} A/GL_n]$ has the *resolution property*, that is, every coherent sheaf is a quotient of a vector bundle [Tot04]. Although we do not know if \mathcal{X} has the resolution property, we conclude that \mathcal{X} has the resolution property étale-locally. In [Ryd15, Def. 2.1], an algebraic stack \mathcal{X} having the resolution property locally for a representable (resp. representable and separated) étale covering $p: W \rightarrow \mathcal{X}$ is said to be of *global type* (resp. *s-global type*). Thus, if \mathcal{X} has linearly reductive stabilizers at closed points and affine diagonal, then \mathcal{X} is of s-global type.

Geraschenko and Satriano define toric Artin stacks in terms of stacky fans. They show that a stack \mathcal{X} is toric if and only if it is normal, has affine diagonal, has an open dense torus T acting on the stack, has linearly reductive stabilizers, and $[\mathcal{X}/T]$ is of global type [GS15, Thm. 6.1]. If \mathcal{X} has linearly reductive stabilizers at closed points, then so has $[\mathcal{X}/T]$. Theorem 2.25 thus shows that the last condition is superfluous.

2.13. Compact generation of derived categories. For results involving derived categories of quasi-coherent sheaves, perfect (or compact) generation of the unbounded derived category $D_{\text{QCoh}}(\mathcal{X})$ continues to be an indispensable tool at one's disposal [Nee96, BZFN10]. We prove:

Theorem 2.26. *Let \mathcal{X} be an algebraic stack of finite type over a field k (not assumed to be algebraically closed) with affine diagonal. If the stabilizer group G_x has linearly reductive identity component G_x^0 for every closed point of \mathcal{X} , then*

- (1) $D_{\text{QCoh}}(\mathcal{X})$ is compactly generated by a countable set of perfect complexes; and
- (2) for every open immersion $\mathcal{U} \subseteq \mathcal{X}$, there exists a compact and perfect complex $P \in D_{\text{QCoh}}(\mathcal{X})$ with support precisely $\mathcal{X} \setminus \mathcal{U}$.

Theorem 2.26 was previously only known for stacks with finite stabilizers [HR14c, Thm. A] or quotients of quasi-projective schemes by a linear action of an algebraic group in characteristic 0 [BZFN10, Cor. 3.22].

In positive characteristic, the theorem is almost sharp: if the reduced identity component $(G_x)_{\text{red}}^0$ is not linearly reductive, i.e., not a torus, at some point x , then $D_{\text{QCoh}}(\mathcal{X})$ is not compactly generated [HNR14, Thm. 1.1].

If \mathcal{X} is an algebraic stack of finite type over k with affine stabilizers such that either

- (1) the characteristic of k is 0; or
- (2) every stabilizer is linearly reductive;

then \mathcal{X} is concentrated, that is, a complex of $\mathcal{O}_{\mathcal{X}}$ -modules with quasi-coherent cohomology is perfect if and only if it is a compact object of $D_{\text{QCoh}}(\mathcal{X})$ [HR14a, Thm. C]. If \mathcal{X} admits a good moduli space $\pi: \mathcal{X} \rightarrow X$ with affine diagonal, then one of the two conditions hold by Theorem 2.9. If \mathcal{X} does not admit a good moduli space and is of positive characteristic, then it is not sufficient that closed points have linearly reductive stabilizers as the following example shows.

Example 2.27. Let $\mathcal{X} = [X/\mathbb{G}_m \times \mathbb{Z}_2]$ be the quotient of the non-separated affine line X by the natural \mathbb{G}_m -action and the \mathbb{Z}_2 -action that swaps the origins. Then \mathcal{X} has two points, one closed with stabilizer group \mathbb{G}_m and one open point with stabilizer group \mathbb{Z}_2 . Thus if k has characteristic two, then not every stabilizer group is linearly reductive and there are non-compact perfect complexes [HR14a, Thm. C].

3. COHERENTLY COMPLETE STACKS AND THE TANNAKIAN FORMALISM

3.1. Coherently complete algebraic stacks. We now prove Theorem 1.3.

Proof of Theorem 1.3. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x . A coherent $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} corresponds to a finitely generated A -module M with an action of G . Note that since G is linearly reductive, M^G is a finitely generated A^G -module. We claim that the following two sequences of A^G -submodules $\{(\mathfrak{m}^n M)^G\}$ and $\{(\mathfrak{m}^G)^n M^G\}$ of M^G define the same topology, or in other words that

$$(3.1) \quad M^G \rightarrow \varprojlim M^G / (\mathfrak{m}^n M)^G$$

is an isomorphism of A^G -modules.

To this end, we first establish that

$$(3.2) \quad \bigcap_{n \geq 0} (\mathfrak{m}^n M)^G = 0,$$

which immediately informs us that (3.1) is injective. Let $N = \bigcap_{n \geq 0} \mathfrak{m}^n M$. Krull's intersection theorem implies that $N \otimes_A A/\mathfrak{m} = 0$. Since A^G is a local ring, $\text{Spec } A$

has a unique closed orbit $\{x\}$. Since the support of N is a closed G -invariant subscheme of $\text{Spec } A$ which does not contain x , it follows that $N = 0$.

We next establish that (3.1) is an isomorphism if A^G is artinian. In this case, $\{(\mathfrak{m}^n M)^G\}$ automatically satisfies the Mittag-Leffler condition (it is a sequence of artinian A^G -modules). Therefore, taking the inverse limit of the exact sequences $0 \rightarrow (\mathfrak{m}^n M)^G \rightarrow M^G \rightarrow M^G/(\mathfrak{m}^n M)^G \rightarrow 0$ and applying (3.2), yields an exact sequence

$$0 \rightarrow 0 \rightarrow M^G \rightarrow \varprojlim M^G/(\mathfrak{m}^n M)^G \rightarrow 0.$$

Thus, we have established (3.1) when A^G is artinian.

To establish (3.1) in the general case, let $J = (\mathfrak{m}^G)A \subseteq A$ and observe that

$$(3.3) \quad M^G = \varprojlim M^G/(\mathfrak{m}^G)^n M^G = \varprojlim (M/J^n M)^G.$$

For each n , we know that

$$(3.4) \quad (M/J^n M)^G = \varprojlim_l M^G/((J^n + \mathfrak{m}^l)M)^G$$

using the artinian case proved above. Finally, combining (3.3) and (3.4) together with the observation that $J^n \subseteq \mathfrak{m}^l$ for $n \geq l$, we conclude that

$$\begin{aligned} M^G &= \varprojlim_n (M/J^n M)^G \\ &= \varprojlim_n \varprojlim_l M^G/((J^n + \mathfrak{m}^l)M)^G \\ &= \varprojlim_l M^G/(\mathfrak{m}^l M)^G. \end{aligned}$$

We now show that (1.1) is fully faithful. Suppose that \mathcal{G} and \mathcal{F} are coherent $\mathcal{O}_{\mathcal{X}}$ -modules, and let \mathcal{G}_n and \mathcal{F}_n denote the restrictions to $\mathcal{X}^{[n]}$, respectively. We need to show that

$$\text{Hom}(\mathcal{G}, \mathcal{F}) \rightarrow \varprojlim \text{Hom}(\mathcal{G}_n, \mathcal{F}_n)$$

is bijective. Since \mathcal{X} satisfies the resolution property, we can find locally free $\mathcal{O}_{\mathcal{X}}$ -modules \mathcal{E}' and \mathcal{E} and an exact sequence

$$\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0.$$

This induces a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\mathcal{G}, \mathcal{F}) & \longrightarrow & \text{Hom}(\mathcal{E}, \mathcal{F}) & \longrightarrow & \text{Hom}(\mathcal{E}', \mathcal{F}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varprojlim \text{Hom}(\mathcal{G}_n, \mathcal{F}_n) & \longrightarrow & \varprojlim \text{Hom}(\mathcal{E}_n, \mathcal{F}_n) & \longrightarrow & \varprojlim \text{Hom}(\mathcal{E}'_n, \mathcal{F}_n) \end{array}$$

with exact rows. Therefore, it suffices to assume that \mathcal{G} is locally free. In this case,

$$\text{Hom}(\mathcal{G}, \mathcal{F}) = \text{Hom}(\mathcal{O}_{\mathcal{X}}, \mathcal{G}^\vee \otimes \mathcal{F}) \quad \text{and} \quad \text{Hom}(\mathcal{G}_n, \mathcal{F}_n) = \text{Hom}(\mathcal{O}_{\mathcal{X}^{[n]}}, (\mathcal{G}_n^\vee \otimes \mathcal{F}_n)).$$

Therefore, we can also assume that $\mathcal{G} = \mathcal{O}_{\mathcal{X}}$ and we need to verify that the map

$$\Gamma(\mathcal{X}, \mathcal{F}) \rightarrow \varprojlim \Gamma(\mathcal{X}^{[n]}, \mathcal{F}_n)$$

is an isomorphism, but this is precisely the isomorphism from (3.1), and the full faithfulness of (1.1) follows.

We now prove that the functor (1.1) is essentially surjective. Since \mathcal{X} has the resolution property, there is a vector bundle \mathcal{E} on \mathcal{X} together with a surjection $\phi_0: \mathcal{E} \rightarrow \mathcal{F}_0$. We claim that ϕ_0 lifts to a compatible system of morphisms $\phi_n: \mathcal{E} \rightarrow \mathcal{F}_n$ for every $n > 0$. It suffices to show that for $n > 0$, the natural map $\text{Hom}(\mathcal{E}, \mathcal{F}_{n+1}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}_n)$ is surjective but this is clear as $\text{Ext}_{\mathcal{X}}^1(\mathcal{E}, \mathfrak{m}^{n+1}\mathcal{F}_{n+1}) = 0$ since \mathcal{E} is locally free and G is linearly reductive. By Nakayama's Lemma,

each ϕ_n is surjective. It follows that we obtain an induced morphism of systems $\{\phi_n\}: \{\mathcal{E}_n\} \rightarrow \{\mathcal{F}_n\}$. Applying this procedure to $\{\ker(\phi_n)\}$ (which is not necessarily an adic system), there is another vector bundle \mathcal{H} and a morphism of systems $\{\psi_n\}: \{\mathcal{H}_n\} \rightarrow \{\mathcal{E}_n\}$ such that $\text{coker}(\psi_n) \cong \mathcal{F}_n$. By the full faithfulness of (1.1), the morphism $\{\psi_n\}$ arises from a unique morphism $\psi: \mathcal{H} \rightarrow \mathcal{E}$. Letting $\tilde{\mathcal{F}} = \text{coker } \psi$, the universal property of cokernels proves that there is an isomorphism $\tilde{\mathcal{F}}_n \cong \mathcal{F}_n$; the result follows. \square

Remark 3.1. In this remark, we show that with the hypotheses of Theorem 1.3 the coherent \mathcal{O}_X -module \mathcal{F} extending a given system $\{\mathcal{F}_n\} \in \varprojlim \text{Coh}(\mathcal{X}^{[n]})$ can in fact be constructed explicitly. Let Γ denote the set of irreducible representations of G with $0 \in \Gamma$ denoting the trivial representation. For $\rho \in \Gamma$, we let V_ρ be the corresponding irreducible representation. For any G -representation V , we set

$$V^{(\rho)} = (V \otimes V_\rho^\vee)^G \otimes V_\rho.$$

Note that $V = \bigoplus_{\rho \in \Gamma} V^{(\rho)}$ and that $V^{(0)} = V^G$ is the subspace of invariants. In particular, there is a decomposition $A = \bigoplus_{\rho \in \Gamma} A^{(\rho)}$. The data of a coherent \mathcal{O}_X -module \mathcal{F} is equivalent to a finitely generated A -module M together with a G -action, i.e., an A -module M with a decomposition $M = \bigoplus_{\rho \in \Gamma} M^{(\rho)}$, where each $M^{(\rho)}$ is a direct sum of copies of the irreducible representation V_ρ , such that the A -module structure on M is compatible with the decompositions of A and M . Given a coherent \mathcal{O}_X -module $\mathcal{F} = \widetilde{M}$ and a representation $\rho \in \Gamma$, then $M^{(\rho)}$ is a finitely generated A^G -module and

$$M^{(\rho)} \rightarrow \varprojlim (M/\mathfrak{m}^k M)^{(\rho)}$$

is an isomorphism (which follows from (3.1)).

Conversely, given a system of $\{\mathcal{F}_n = \widetilde{M}_n\} \in \varprojlim \text{Coh}(\mathcal{X}^{[n]})$ where each M_n is a finitely generated A/\mathfrak{m}^{n+1} -module with a G -action, then the extension $\mathcal{F} = \widetilde{M}$ can be constructed explicitly by defining:

$$M^{(\rho)} := \varprojlim M_n^{(\rho)} \quad \text{and} \quad M := \bigoplus_{\rho \in \Gamma} M^{(\rho)}.$$

One can show directly that each $M^{(\rho)}$ is a finitely generated A^G -module, M is a finitely generated A -module with a G -action, and $M/\mathfrak{m}^{n+1}M = M_n$.

Remark 3.2. Theorem 1.3 also implies that every vector bundle on \mathcal{X} is the pullback of a G -representation under the projection $\mathcal{X} \rightarrow BG$. In particular, suppose that G is a diagonalizable group scheme. Then using the notation of Remark 3.1, every irreducible G -representation $\rho \in \Gamma$ is one-dimensional so that a G -action on A corresponds to a Γ -grading $A = \bigoplus_{\rho \in \Gamma} A^{(\rho)}$, and an A -module with a G -action corresponds to a Γ -graded A -module.

Therefore, if $A = \bigoplus_{\rho \in \Gamma} A^{(\rho)}$ is a Γ -graded noetherian k -algebra with $A^{(0)}$ a complete local k -algebra, then every finitely generated projective Γ -graded A -module is free. When $G = \mathbb{G}_m$ and $A^G = k$, this is the well known statement (e.g., [Eis95, Thm. 19.2]) that every finitely generated projective graded module over a Noetherian graded k -algebra $A = \bigoplus_{d \geq 0} A_d$ with $A_0 = k$ is free.

The theorem above motivates the following definition:

Definition 3.3. Let \mathcal{X} be a noetherian algebraic stack with affine stabilizers and let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack. We say that \mathcal{X} is *coherently complete along \mathcal{Z}* if the natural functor

$$\text{Coh}(\mathcal{X}) \rightarrow \varprojlim_n \text{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})$$

is an equivalence of categories.

Remark 3.4. If (A, \mathfrak{m}) is complete local noetherian ring, then $\mathrm{Spec} A$ is coherently complete along $\mathrm{Spec} A/\mathfrak{m}$, and more generally if an algebraic stack \mathcal{X} is proper over $\mathrm{Spec} A$, then \mathcal{X} is coherently complete along $\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} A/\mathfrak{m}$. See [EGA, III.5.1.4] for the case of schemes and [Ols05, Thm. 1.4], [Con05, Thm. 4.1] for algebraic stacks. Theorem 1.3 concludes that \mathcal{X} is coherently complete along BG .

3.2. Tannakian formalism. The following Tannaka duality theorem proved by the second and third author is crucial in our argument.

Theorem 3.5. [HR14b, Thm. 1.1] *Let \mathcal{X} be an excellent stack and \mathcal{Y} be a noetherian algebraic stack with affine stabilizers. Then the natural functor*

$$\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathrm{Hom}_{r\otimes, \simeq}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$$

is an equivalence of categories, where $\mathrm{Hom}_{r\otimes, \simeq}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$ denotes the category whose objects are right exact monoidal functors $\mathrm{Coh}(\mathcal{Y}) \rightarrow \mathrm{Coh}(\mathcal{X})$ and morphisms are natural isomorphisms of functors.

We will apply the following consequence of Tannakian duality:

Corollary 3.6. *Let \mathcal{X} be an excellent algebraic stack with affine stabilizers and $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack. Suppose that \mathcal{X} is coherently complete along \mathcal{Z} . If \mathcal{Y} is a noetherian algebraic stack with affine stabilizers, then the natural functor*

$$\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) \rightarrow \varprojlim_n \mathrm{Hom}(\mathcal{X}_{\mathcal{Z}}^{[n]}, \mathcal{Y})$$

is an equivalence of categories.

Proof. There are natural equivalences

$$\begin{aligned} \mathrm{Hom}(\mathcal{X}, \mathcal{Y}) &\simeq \mathrm{Hom}_{r\otimes, \simeq}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})) && \text{(Tannakian formalism)} \\ &\simeq \mathrm{Hom}_{r\otimes, \simeq}(\mathrm{Coh}(\mathcal{Y}), \varprojlim \mathrm{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})) && \text{(coherent completeness)} \\ &\simeq \varprojlim \mathrm{Hom}_{r\otimes, \simeq}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}_{\mathcal{Z}}^{[n]})) \\ &\simeq \varprojlim \mathrm{Hom}(\mathcal{X}_{\mathcal{Z}}^{[n]}, \mathcal{Y}) && \text{(Tannakian formalism). } \square \end{aligned}$$

4. PROOFS OF THEOREMS 1.1 AND 1.2

4.1. The normal and tangent space of an algebraic stack. Let \mathcal{X} be a quasi-separated algebraic stack, locally of finite type over a field k , with affine stabilizers. Let $x \in \mathcal{X}(k)$ be a closed point. Denote by $i: BG_x \rightarrow \mathcal{X}$ the closed immersion of the residual gerbe of x , and by \mathcal{J} the corresponding ideal sheaf. The *normal space to x* is $N_x := (\mathcal{J}/\mathcal{J}^2)^\vee = (i^*\mathcal{J})^\vee$ viewed as a G_x -representation. The *tangent space to \mathcal{X} at x* is the k -vector space of equivalence classes of pairs (τ, α) consisting of morphisms $\tau: \mathrm{Spec} k[\epsilon]/\epsilon^2 \rightarrow \mathcal{X}$ and 2-isomorphisms $\alpha: x \rightarrow \tau|_{\mathrm{Spec} k}$. The stabilizer G_x acts linearly on the tangent space $T_{\mathcal{X}, x}$ by precomposition on the 2-isomorphism. If G_x is smooth, there is an identification $T_{\mathcal{X}, x} \cong N_x$ of G_x -representations. Moreover, if $\mathcal{X} = [X/G]$ is a quotient stack where G is an algebraic group and $x \in X(k)$ (with G_x not necessarily smooth), then N_x is identified with the normal space $T_{X, x}/T_{G \cdot x, x}$ to the orbit $G \cdot x$ at x .

4.2. The smooth case. We now prove Theorem 1.1 even though it follows directly from Theorem 1.2 coupled with Luna's fundamental lemma [Lun73, p. 94]. We feel that since the proof of Theorem 1.1 is more transparent and less technical than Theorem 1.2, digesting the proof first in this case will make the proof of Theorem 1.2 more accessible.

Proof of Theorem 1.1. Define the quotient stack $\mathcal{N} = [N_x/G_x]$, where N_x is viewed as an affine scheme via $\text{Spec}(\text{Sym } N_x^\vee)$. Since G_x is linearly reductive, we claim that there are compatible isomorphisms $\mathcal{X}^{[n]} \cong \mathcal{N}^{[n]}$.

To see this, first note that we can lift $\mathcal{X}^{[0]} = BG_x$ to a unique morphism $t_n: \mathcal{X}^{[n]} \rightarrow BG_x$ for all n : the obstruction to a lift from $t_n: \mathcal{X}^{[n]} \rightarrow BG_x$ to $t_{n+1}: \mathcal{X}^{[n+1]} \rightarrow BG_x$ is an element of the group $\text{Ext}_{\mathcal{O}_{BG_x}}^1(L_{BG_x/k}, \mathcal{J}^n/\mathcal{J}^{n+1})$ [Ols06] which is zero since BG_x is cohomologically affine and $L_{BG_x/k}$ is a perfect complex supported in degrees $[0, 1]$ since $BG_x \rightarrow \text{Spec } k$ is smooth. In particular, $BG_x = \mathcal{X}^{[0]} \hookrightarrow \mathcal{X}^{[1]}$ has a retraction. This implies that $\mathcal{X}^{[1]} \cong \mathcal{N}^{[1]}$ since both are trivial deformations by the same module. Since $\mathcal{N} \rightarrow BG_k$ is smooth, the obstruction to lifting the morphism $\mathcal{X}^{[1]} \cong \mathcal{N}^{[1]} \hookrightarrow \mathcal{N}$ to $\mathcal{X}^{[n]} \rightarrow \mathcal{N}$ vanishes as $H^1(BG_x, \Omega_{N/BG_k}^\vee \otimes \mathcal{J}^n/\mathcal{J}^{n+1}) = 0$. We have induced isomorphisms $\mathcal{X}^{[n]} \cong \mathcal{N}^{[n]}$ by Proposition A.5 (2).

Let $N \rightarrow N = N_x//G_x$ be the good moduli space and denote by $0 \in N$ the image of the origin. Set $\widehat{N} := \text{Spec } \widehat{\mathcal{O}}_{N,0} \times_N N$. Since \widehat{N} is coherently complete (Theorem 1.3), we may apply the Tannakian formalism (Corollary 3.6) to find a morphism $\widehat{N} \rightarrow \mathcal{X}$ filling in the diagram

$$\begin{array}{ccccc} & & & & \mathcal{X} \\ & & \curvearrowright & & \\ \mathcal{X}^{[n]} \cong \mathcal{N}^{[n]} & \longrightarrow & \widehat{N} & \xrightarrow{\quad} & \mathcal{N} \\ & & \downarrow & \square & \downarrow \\ & & \text{Spec } \widehat{\mathcal{O}}_{N,0} & \longrightarrow & N. \end{array}$$

Let us now consider the functor $F: \text{Sch}/N \rightarrow \text{Sets}$ which assigns to a morphism $S \rightarrow N$ the set of morphisms $S \times_N N \rightarrow \mathcal{X}$ modulo 2-isomorphisms. This functor is locally of finite presentation and we have an element of F over $\text{Spec } \widehat{\mathcal{O}}_{N,0}$. By Artin approximation [Art69a, Cor. 2.2], there exist an étale morphism $(U, u) \rightarrow (N, 0)$ where U is an affine scheme and a morphism $(U \times_N N, (u, 0)) \rightarrow (\mathcal{X}, x)$ agreeing with $(\widehat{N}, 0) \rightarrow (\mathcal{X}, x)$ to first order. Since \mathcal{X} is smooth at x , it follows that $U \times_N N \rightarrow \mathcal{X}$ is étale at $(u, 0)$ by Proposition A.5 (2). This establishes the theorem after shrinking U suitably; the final statement follows from Proposition 4.2. \square

4.3. The general case. We now prove Theorem 1.2 by a similar method to the proof in the smooth case but using Corollary A.14 in place of Artin approximation.

Proof of Theorem 1.2. We may assume that $x \in |\mathcal{X}|$ is a closed point. Let $\mathcal{N} := [N_x/H]$, $\mathcal{N} \rightarrow N := N_x//H$, and $\widehat{N} := \text{Spec } \widehat{\mathcal{O}}_{N,0} \times_N \mathcal{N}$ where 0 denotes the image of the origin. Let $\eta_0: BH \rightarrow BG_x = \mathcal{X}^{[0]}$ be the morphism induced from the inclusion $H \subseteq G_x$; this is a smooth (resp. étale) morphism. We first prove by induction that there are compatible 2-cartesian diagrams

$$\begin{array}{ccc} \mathcal{H}_n & \hookrightarrow & \mathcal{H}_{n+1} \\ \eta_n \downarrow & \square & \downarrow \eta_{n+1} \\ \mathcal{X}^{[n]} & \hookrightarrow & \mathcal{X}^{[n+1]}, \end{array}$$

where $\mathcal{H}_0 = BH$ and the vertical maps are smooth (resp. étale). Indeed, given $\eta_n: \mathcal{H}_n \rightarrow \mathcal{X}^{[n]}$, by [Ols06], the obstruction to the existence of η_{n+1} is an element of $\text{Ext}_{\mathcal{O}_{BH}}^2(\Omega_{BH/BG_x}, \eta_0^*(\mathcal{J}^n/\mathcal{J}^{n+1}))$, but this group vanishes as H is linearly reductive and Ω_{BH/BG_x} is a vector bundle.

Let $\tau_0: \mathcal{H}_0 = BH \rightarrow N$. Since H is linearly reductive, the deformation $\mathcal{H}_0 \hookrightarrow \mathcal{H}_1$ is a trivial extension by N and hence we have an isomorphism $\tau_1: \mathcal{H}_1 \cong \mathcal{N}^{[1]}$ (see proof of smooth case). Using linearly reductivity of H once again and deformation

theory, we obtain compatible morphisms $\tau_n: \mathcal{H}_n \rightarrow \mathcal{N}$ extending τ_0 and τ_1 . These are closed immersions by Proposition A.5 (1).

The closed embeddings $\mathcal{H}_n \hookrightarrow \mathcal{N}$ factor through $\widehat{\mathcal{N}}$ so that we have a compatible family of diagrams

$$\begin{array}{ccc} \mathcal{H}_n & \hookrightarrow & \widehat{\mathcal{N}} \\ \eta_n \downarrow & & \\ \mathcal{X}^{[n]} & \longrightarrow & \mathcal{X}. \end{array}$$

Since $\widehat{\mathcal{N}}$ is coherently complete, there exists a closed immersion $\widehat{\mathcal{H}} \hookrightarrow \widehat{\mathcal{N}}$ extending $\mathcal{H}_n \hookrightarrow \widehat{\mathcal{N}}$. Since $\widehat{\mathcal{H}}$ is also coherently complete, the Tannakian formalism yields a morphism $\eta: \widehat{\mathcal{H}} \rightarrow \mathcal{X}$ extending $\eta_n: \mathcal{H}_n \rightarrow \mathcal{X}^{[n]}$. By Proposition A.9, $\widehat{\mathcal{H}} \rightarrow \mathcal{X}$ is formally versal (resp. universal). Also note that $\widehat{\mathcal{H}} \rightarrow \text{Spec } \widehat{\mathcal{O}}_{N,0}$ is of finite type. We may therefore apply Corollary A.14 to obtain a stack $\mathcal{H} = [\text{Spec } A/H]$ together with a morphism $f: (\mathcal{H}, w) \rightarrow (\mathcal{X}, x)$ of finite type and a flat morphism $\varphi: \widehat{\mathcal{H}} \rightarrow \mathcal{H}$, identifying $\widehat{\mathcal{H}}$ with the completion of \mathcal{H} at w , such that $f \circ \varphi = \eta$. In particular, f is smooth (resp. étale) at w . Moreover, $(f \circ \varphi)^{-1}(\mathcal{X}^{[0]}) = \mathcal{H}^{[0]}$ so we have a flat morphism $BH = \mathcal{H}^{[0]} \rightarrow f^{-1}(BG_x)$ which equals the inclusion of the residual gerbe at w . It follows that w is an isolated point in the fiber $f^{-1}(BG_x)$. We can thus find an open neighborhood $\mathcal{W} \subseteq \mathcal{H}$ of w such that $\mathcal{W} \rightarrow \mathcal{X}$ is smooth (resp. étale) and $\mathcal{W} \cap f^{-1}(BG_x) = BH$. Since w is a closed point of \mathcal{H} , we may further shrink \mathcal{W} so that it becomes cohomologically affine (Lemma 4.1).

The final statement follows from Proposition 4.2. \square

4.4. The refinement. The following trivial lemma will be frequently applied to a good moduli space morphism $\pi: \mathcal{X} \rightarrow X$. Note that any closed subset $\mathcal{Z} \subseteq \mathcal{X}$ satisfies the assumption in the lemma in this case.

Lemma 4.1. *Let $\pi: \mathcal{X} \rightarrow X$ be a closed morphism of topological spaces and let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subset. Assume that every open neighborhood of \mathcal{Z} contains $\pi^{-1}(\pi(\mathcal{Z}))$. If $\mathcal{Z} \subseteq \mathcal{U}$ is an open neighborhood of \mathcal{Z} , then there exists an open neighborhood $U' \subseteq X$ of $\pi(\mathcal{Z})$ such that $\pi^{-1}(U') \subseteq \mathcal{U}$.*

Proof. Take $U' = X \setminus \pi(\mathcal{X} \setminus \mathcal{U})$. \square

Proposition 4.2. *Let $f: \mathcal{W} \rightarrow \mathcal{X}$ be a morphism of noetherian algebraic stacks such that \mathcal{W} is cohomologically affine with affine diagonal. Suppose $w \in |\mathcal{W}|$ is a closed point such that f induces an injection of stabilizer groups at w .*

- (1) *If there exists an affine and faithfully flat morphism of finite type $\mathcal{X}' \rightarrow \mathcal{X}$ such that \mathcal{X}' has quasi-finite and separated diagonal, then there exists a cohomologically affine open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of w such that $f|_{\mathcal{U}}$ is quasi-compact, representable and separated.*
- (2) *If \mathcal{X} has affine diagonal, then there exists a cohomologically affine open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of w such that $f|_{\mathcal{U}}$ is affine.*

Proof. We first establish (1). By shrinking \mathcal{W} , we may assume that $\Delta_{\mathcal{W}/\mathcal{X}}$ is quasi-finite and after further shrinking, we may arrange so that \mathcal{W} remains cohomologically affine (Lemma 4.1). Let $\mathcal{W}' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{W}$ and let $f': \mathcal{W}' \rightarrow \mathcal{X}'$ be the induced morphism. Then \mathcal{W}' is cohomologically affine with quasi-finite and affine diagonal. By applying [Alp13, Prop. 6.4] to $\Delta_{\mathcal{W}'}$, we obtain that $\Delta_{\mathcal{W}'}$ is finite. Since \mathcal{X}' has separated diagonal, it follows that $f': \mathcal{W}' \rightarrow \mathcal{X}'$ is separated. By descent, f is separated. In particular, the relative inertia of f , $i: I_{\mathcal{W}/\mathcal{X}} \rightarrow \mathcal{W}$, is finite. By Nakayama's lemma, there is an open substack \mathcal{U} of \mathcal{W} , containing w , with trivial inertia relative to \mathcal{X} . Thus $\mathcal{U} \rightarrow \mathcal{X}$ is quasi-finite, representable and separated.

Shrinking \mathcal{U} appropriately, \mathcal{U} also becomes cohomologically affine and the claim follows.

For (2), by (1) we may assume that f is representable and separated. Since \mathcal{W} is cohomologically affine and Δ_X is affine, it follows that f is cohomologically affine. But cohomologically affine, quasi-compact, representable and separated implies that f is affine (Serre's Criterion [Alp13, Prop. 3.3]). \square

Note that the condition in (1) implies that Δ_X is quasi-affine. It is possible that the condition in (1) can be replaced by this weaker condition.

5. PROOFS OF APPLICATIONS

We now prove the results stated in §2.

5.1. Generalization of Sumihiro's theorem. We first establish Theorems 2.4–2.6 since they will be used to establish Theorem 2.1.

Proof of Theorem 2.4. Let $\mathcal{Y} = [\mathcal{X}/T]$ and $y \in \mathcal{Y}(k)$ be the image of x . As the sequence (2.1) splits, we can consider T_x as a subgroup of G_y . By applying Theorem 1.2 to \mathcal{Y} at y with respect to the subgroup $T_x \subseteq G_y$, we obtain a smooth (resp. étale) morphism $f: [W/T_x] \rightarrow \mathcal{Y}$, where W is an affine scheme with an action of T_x , which induces the given inclusion $T_x \subseteq G_y$ at stabilizer groups at a preimage $w \in [W/T_x]$ of y . Consider the cartesian diagram

$$\begin{array}{ccccc} [W/T_x] \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} & \longrightarrow & \mathrm{Spec} k \\ \downarrow & & \downarrow & & \downarrow \\ [W/T_x] & \longrightarrow & \mathcal{Y} & \longrightarrow & BT \end{array}$$

The map $[W/T_x] \rightarrow \mathcal{Y} \rightarrow BT$ induces the injection $T_x \hookrightarrow T$ on stabilizer groups at w . Thus, by Proposition 4.2 (2), there is an open neighborhood $\mathcal{U} \subseteq [W/T_x]$ of w such that \mathcal{U} is cohomologically affine and $\mathcal{U} \rightarrow BT$ is affine. The fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U}$ is thus an affine scheme $\mathrm{Spec} A$ and the induced map $\mathrm{Spec} A \rightarrow \mathcal{X}$ is T -equivariant. If $u \in \mathrm{Spec} A$ is a closed point above w and x , then the map $\mathrm{Spec} A \rightarrow \mathcal{X}$ induces an isomorphism $T_x \rightarrow T_x$ of stabilizer groups at u . \square

Proof of Theorem 2.5. In the exact sequence (2.1), G_x is étale and T_x is diagonalizable. This implies that $(G_y)^0$ is diagonalizable. Indeed, first note that we have exact sequences:

$$\begin{aligned} 1 &\rightarrow G_x \cap (G_y)^0 \rightarrow (G_y)^0 \rightarrow (T_x)^0 \rightarrow 1 \\ 1 &\rightarrow G_x \cap (G_y)^0 \rightarrow (G_y)_{\mathrm{red}}^0 \rightarrow (T_x)_{\mathrm{red}}^0 \rightarrow 1 \end{aligned}$$

The second sequence shows that $(G_y)_{\mathrm{red}}^0$ is a torus (as it is connected, reduced and surjects onto a torus with finite kernel) and, consequently, that $G_x \cap (G_y)^0$ is diagonalizable. It then follows that $(G_y)^0$ is diagonalizable from the first sequence [SGA3_{II}, Exp. 17, Prop. 7.1.1 b)].

Theorem 1.2 produces an étale neighborhood $f: ([\mathrm{Spec} A/(G_y)^0], w) \rightarrow (\mathcal{Y}, y)$ such that the induced morphism on stabilizer groups is $(G_y)^0 \rightarrow G_y$. Replacing $\mathcal{X} \rightarrow \mathcal{Y}$ with the pull-back along f , we may thus assume that G_y is connected and diagonalizable.

If we let $G_y = D(N)$, $T_x = D(M)$ and $T = D(\mathbb{Z}^r)$, then we have a surjective map $q: \mathbb{Z}^r \rightarrow M$ and an injective map $\varphi: M \rightarrow N$. The quotient N/M is torsion but without p -torsion, where p is the characteristic of k . Since all torsion of M and

N is p -torsion, we have that φ induces an isomorphism of torsion subgroups. We can thus find splittings of φ and q as in the diagram

$$\begin{array}{ccc} \mathbb{Z}^r = \mathbb{Z}^s \oplus M/M_{\text{tor}} & \xrightarrow{\alpha = \text{id} \oplus \varphi_2} & \mathbb{Z}^s \oplus N/N_{\text{tor}} = \mathbb{Z}^r \\ \downarrow q = q_1 \oplus \text{id} & & \downarrow q' = \varphi_1 q_1 \oplus \text{id} \\ M = M_{\text{tor}} \oplus M/M_{\text{tor}} & \xrightarrow{\varphi = \varphi_1 \oplus \varphi_2} & N_{\text{tor}} \oplus N/N_{\text{tor}} = N. \end{array}$$

The map q' corresponds to an embedding $G_y \hookrightarrow T$ and the map α to a reparameterization $T \rightarrow T$. After reparameterizing the action of T on \mathcal{X} via α , the surjection $G_y \twoheadrightarrow T_x$ becomes split. The result now follows from Theorem 2.4. \square

Proof of Theorem 2.6. By Theorem 1.2, there exists an étale neighborhood $f: (\mathcal{W}, w) \rightarrow ([X/G], x)$ such that \mathcal{W} is cohomologically affine, f induces an isomorphism of stabilizers at w , and w is a closed point. By Proposition 4.2 (2), we can assume after shrinking \mathcal{W} that the composition $\mathcal{W} \rightarrow [X/G] \rightarrow BG$ is affine. It follows that $W = \mathcal{W} \times_{[X/G]} X$ is affine and that $W \rightarrow X$ is an G -equivariant étale neighborhood of x . \square

5.2. Generalization of Luna's étale slice theorem.

Proof of Theorem 2.1. By applying Theorem 2.6, we can find an affine scheme X' with an action of G and a G -equivariant, étale morphism $X' \rightarrow X$. This reduces the theorem to the case when X is affine which was established in [Lun73, p. 97], cf. Remark 2.2. \square

5.3. Białyński-Birula decompositions. Theorem 2.7 follows immediately from Theorem 2.5 and [Opr06, Prop. 5].

5.4. Existence of equivariant versal deformations for curves. Theorem 2.8 follows directly from Theorem 1.2 and the following lemma (because the image of H in $\text{Aut}(C)$ is linearly reductive and $BH \rightarrow B\text{Aut}(C)$ is smooth):

Lemma 5.1. *If $(C, \{p_j\}_{j=1}^n)$ is an n -pointed proper scheme of pure dimension 1 over an algebraically closed field k and no connected component of C_{red} is a smooth unpointed irreducible curve of genus 1, then $\text{Aut}(C, \{p_j\})$ is an affine group scheme over k .*

Proof. We first handle the case when C is reduced. Let $(\tilde{C}, \{p_j\}_{j=1}^n, \{q_j\}_{j=1}^k)$ be the pointed normalization of $(C, \{p_j\})$. The subgroup $K \subseteq \text{Aut}(C, \{p_j\})$ of automorphisms fixing the singular locus of C has finite index, and there is an injective homomorphism $h: K \rightarrow \text{Aut}(\tilde{C}, \{p_j\}, \{q_j\})$. As the automorphism group of any n -pointed smooth genus g curve with $(g, n) \neq (1, 0)$ is affine, the hypotheses imply $\text{Aut}(\tilde{C}, \{p_j\}, \{q_j\})$ is affine. It follows that K is affine and thus $\text{Aut}(C, \{p_j\})$.

In our argument for the general case, the marked points will not play a role and will be dropped from the notation. As $C_{\text{red}} \rightarrow C$ can be factored by square-zero closed immersions, by induction, it suffices to verify the following claim: if $C \rightarrow D$ is a closed immersion of proper curves defined by an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_D$ such that $\mathcal{J}^2 = 0$ and such that $\text{Aut}(C)$ is affine, then $\text{Aut}(D)$ is affine. Let $K_1 = \ker(\text{Aut}(D) \rightarrow \text{Aut}(C))$. Since $\text{Aut}(C)$ is affine, it suffices to prove that K_1 is affine. Any element of K_1 induces naturally an automorphism of the coherent \mathcal{O}_C -module $\mathcal{J} = \mathcal{J}/\mathcal{J}^2$. Since $\text{Aut}_{\mathcal{O}_C}(\mathcal{J})$ is affine, it suffices to show that $K_2 = \ker(K_1 \rightarrow \text{Aut}_{\mathcal{O}_C}(\mathcal{J}))$ is affine. Each element $\alpha \in K_2$ defines naturally an \mathcal{O}_D -derivation

$$\mathcal{O}_C \rightarrow \mathcal{J}, \quad s \mapsto \alpha(s) - s$$

since α is acting trivial on J and induces the identity on \mathcal{O}_D . The vector space $\mathrm{Der}_{\mathcal{O}_D}(\mathcal{O}_C, J)$ is finite dimensional and there is an injective group homomorphism $K_2 \rightarrow \mathrm{Der}_{\mathcal{O}_D}(\mathcal{O}_C, J)$, and we conclude that K_2 is affine. \square

Remark 5.2. From the lemma above, we see that the conclusion of Theorem 2.8 holds for pointed curves C such that C and every deformation of C has no connected component whose reduction is a smooth unpointed irreducible curve of genus 1.

Remark 5.3. If $\mathcal{C} \rightarrow S$ is a family of curves such that every fiber satisfies the hypothesis of Lemma 5.1, then the automorphism group scheme $\mathrm{Aut}(\mathcal{C}/S) \rightarrow S$ of \mathcal{C} over S need not be affine (or even quasi-affine). This even fails for families of Deligne–Mumford semistable curves; see [AK14, §4.1].

5.5. Good moduli spaces.

Proof of Theorem 2.9. We may assume that $X = \mathrm{Spec} R$, where R is a noetherian k -algebra. By noetherian approximation along $k \rightarrow R$, there is a finite type k -algebra R_0 and an algebraic stack \mathcal{X}_0 of finite type over $\mathrm{Spec} R_0$ with affine diagonal. We may also arrange that the image x_0 of x in \mathcal{X}_0 is closed with linearly reductive stabilizer G_x . We now apply Theorem 1.2 to find a pointed affine étale k -morphism $f_0: ([\mathrm{Spec} A_0/G_x], w_0) \rightarrow (\mathcal{X}_0, x_0)$ that induces an isomorphism of stabilizers at w_0 . Pulling this back along $\mathrm{Spec} R \rightarrow \mathrm{Spec} R_0$, we obtain an affine étale morphism $f: [\mathrm{Spec} A/G_x] \rightarrow \mathcal{X}$ inducing an isomorphism of stabilizers at all points lying over the preimage of w_0 . The result now follows from a generalization of Luna’s fundamental lemma [Alp10, Thm. 6.10]. \square

Proof of Corollary 2.10. By [GZ12, Thm. 1], we have (1) \Rightarrow (2); thus, it suffices to prove (1). If $R/\mathfrak{m} = k$ and k is algebraically closed, then the result follows from Theorem 2.9 since GL_n/G_x is affine for any embedding $G_x \hookrightarrow GL_n$. In this case (1) holds even if R is not complete but merely henselian.

If $R/\mathfrak{m} = k$ and k is not algebraically closed, then we proceed as follows. Let \bar{k} be an algebraic closure of k . By [EGA, 0III.10.3.1.3], $\bar{R} = R \otimes_k \bar{k} = \varinjlim_{k \subseteq k' \subseteq \bar{k}} R \otimes_k k'$ is local, noetherian, $\bar{\mathfrak{m}} = \mathfrak{m}\bar{R}$ is the maximal ideal, $\bar{R}/\bar{\mathfrak{m}} \cong \bar{k}$, and the induced map $R/\mathfrak{m} \rightarrow \bar{R}/\bar{\mathfrak{m}}$ coincides with $k \rightarrow \bar{k}$. Since each $R \otimes_k k'$ is henselian, \bar{R} is henselian. Let $\bar{\mathcal{X}} = \mathcal{X} \otimes_R \bar{R}$. By the case considered above, $\bar{\mathcal{X}}$ has the resolution property. Having the resolution property descends to some $\mathcal{X}_{k'} = \mathcal{X} \otimes_k k'$, where $k \subseteq k' \subseteq \bar{k}$ is a finite extension. Since $\mathcal{X}_{k'} \rightarrow \mathcal{X}$ is finite and faithfully flat, \mathcal{X} has the resolution property [Gro13, Prop. 4.3(vii)].

In general, let $K = R/\mathfrak{m}$. Since R is a complete k -algebra, it admits a coefficient field; thus, it is also a K -algebra. We are now free to replace k with K and the result follows. \square

5.6. Existence of coherent completions. Theorem 1.2 gives an étale morphism $(\mathcal{W} = [\mathrm{Spec} A/G_x], w) \rightarrow (\mathcal{X}, x)$. If we let $\pi: \mathcal{W} \rightarrow W = \mathrm{Spec} A^{G_x}$, then Theorem 2.12 follows by taking $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \mathrm{Spec} \widehat{\mathcal{O}}_{W, \pi(w)}$. Indeed, this stack is coherently complete by Theorem 1.3 and the uniqueness follows by the Tannakian formalism (Corollary 3.6).

5.7. Étale-local equivalences.

Proof of Theorem 2.13. The implications (3) \Rightarrow (2) \Rightarrow (1) are immediate. We also have (1) \Rightarrow (2) as $\mathcal{X}^{[n]} = [\widehat{\mathrm{Def}}(x)^{[n]}/G_x]$ and $\mathcal{Y}^{[n]} = [\widehat{\mathrm{Def}}(y)^{[n]}/G_y]$. We now show that (2) \Rightarrow (3). We are given an isomorphism $\alpha: \widehat{\mathcal{X}}_x \xrightarrow{\sim} \widehat{\mathcal{Y}}_y$. Let $(\mathcal{W} = [\mathrm{Spec} A/G_x], w) \rightarrow (\mathcal{X}, x)$ be an étale neighborhood as in Theorem 1.2. Let $W = \mathrm{Spec} A^{G_x}$ denote the good moduli space of \mathcal{W} and let w_0 be the image of w . Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \mathrm{Spec} \widehat{\mathcal{O}}_{W, w_0}$. The functor $F: (T \rightarrow W) \mapsto \mathrm{Hom}(\mathcal{W} \times_W T, \mathcal{Y})$ is locally of

finite presentation. Artin approximation applied to F and $\alpha \in F(\mathrm{Spec} \widehat{\mathcal{O}}_{W, w_0})$ thus gives an étale morphism $(W', w') \rightarrow (W, w)$ and a morphism $\varphi: \mathcal{W}' := \mathcal{W} \times_W W' \rightarrow \mathcal{Y}$ such that $\varphi|_{\mathcal{W}'^{[1]}}: \mathcal{W}'^{[1]} \rightarrow \mathcal{Y}^{[1]}$ is an isomorphism. Since $\widehat{\mathcal{W}}'_{w'} \cong \widehat{\mathcal{X}}_x \cong \widehat{\mathcal{Y}}_y$, it follows that φ induces an isomorphism $\widehat{\mathcal{W}}' \rightarrow \widehat{\mathcal{Y}}$ by Proposition A.7. After replacing W' with an open neighborhood we thus obtain an étale morphism $(W', w') \rightarrow (\mathcal{Y}, y)$. The final statement is clear from Theorem 1.1. \square

5.8. Characterization of when \mathcal{X} admits a good moduli space.

Proof of Theorem 2.15. The necessity of the conditions follow from Theorem 2.9. For the sufficiency, by [AFSv14, Theorem 4.1]², it is enough to verify:

- (I) For every closed point $x \in |\mathcal{X}|$, there exists an affine étale morphism

$$f: ([\mathrm{Spec} A/G_x], w) \rightarrow (\mathcal{X}, x)$$

such that for each closed point $w' \in [\mathrm{Spec} A/G_x]$,

- (a) f is stabilizer preserving at w' (i.e., f induces an isomorphism of stabilizer groups at w'); and
 (b) $f(w')$ is closed.

- (II) For any point $y \in \mathcal{X}(k)$, the closed substack $\overline{\{y\}}$ admits a good moduli space.

We first verify condition (I). Let $x \in \mathcal{X}(k)$ be a closed point. By Theorem 1.2, there exist a quotient stack $\mathcal{W} = [\mathrm{Spec} A/G_x]$ with a closed point $w \in |\mathcal{W}|$ and an affine étale morphism $f: (\mathcal{W}, w) \rightarrow (\mathcal{X}, x)$ such that f is stabilizer preserving at w . As the coherent completion of \mathcal{W} at w is identified with $\widehat{\mathcal{X}}_x$, we have a 2-commutative diagram

$$(5.1) \quad \begin{array}{ccc} \widehat{\mathcal{X}}_x & & \\ \downarrow & \searrow & \\ \mathcal{W} & \xrightarrow{f} & \mathcal{X}. \end{array}$$

The subset $Q_a \subseteq |\mathcal{W}|$ consisting of points $\xi \in |\mathcal{W}|$ such that f is stabilizer preserving at ξ is constructible. Since Q_a contains every point in the image of $\widehat{\mathcal{X}}_x \rightarrow \mathcal{W}$ by hypothesis (2a), it follows that Q_a contains a neighborhood of w . Thus after replacing \mathcal{W} with an open saturated neighborhood containing w (Lemma 4.1), we may assume that $f: \mathcal{W} \rightarrow \mathcal{X}$ satisfies condition (Ia).

Let $\pi: \mathcal{W} \rightarrow W$ be the good moduli space of \mathcal{W} and consider the morphism $g = (f, \pi): \mathcal{W} \rightarrow \mathcal{X} \times W$. For a point $\xi \in |\mathcal{W}|$, let $\xi^0 \in |\mathcal{W}|$ denote the unique point that is closed in the fiber \mathcal{W}_ξ . Let $Q_b \subseteq |\mathcal{W}|$ be the locus of points $\xi \in |\mathcal{W}|$ such that $g(\xi^0)$ is closed in $|(\mathcal{X} \times W)_\xi| = |\mathcal{X}_{\kappa(\xi)}|$. This locus is constructible. Indeed, the subset $\mathcal{W}^0 = \{\xi^0 : \xi \in |\mathcal{W}|\} \subseteq |\mathcal{W}|$ is easily seen to be constructible; hence so is $g(\mathcal{W}^0)$ by Chevalley's theorem. The locus Q_b equals the set of points $\xi \in |\mathcal{W}|$ such that $g(\mathcal{W}^0)_\xi$ is closed which is constructible by [EGA, IV.9.5.4]. The locus Q_b contains $\mathrm{Spec} \mathcal{O}_{W, \pi(w)}$ by hypothesis (2b) (recall that $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \mathrm{Spec} \widehat{\mathcal{O}}_{W, \pi(w)}$). Therefore, after replacing \mathcal{W} with an open saturated neighborhood of w , we may assume that $f: \mathcal{W} \rightarrow \mathcal{X}$ satisfies condition (Ib).

For condition (II), we may replace \mathcal{X} by $\overline{\{y\}}$. By (1), there is a unique closed point $x \in \overline{\{y\}}$ and we can find a commutative diagram as in (5.1) for x . By (2b) we can, since f is étale, also assume that \mathcal{W} has a unique closed point. This implies that $\Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}}) = k$ and $\widehat{\mathcal{X}}_x = \mathcal{W}$. By hypothesis (2c), $f: \mathcal{W} \rightarrow \mathcal{X}$ is an

²The underlying hypotheses in [AFSv14] is that the base field k has characteristic 0 but this hypothesis is not necessary.

étale monomorphism which is also surjective by hypothesis (1). We conclude that $f: \mathcal{W} \rightarrow \mathcal{X}$ is an isomorphism establishing condition (II). \square

5.9. The resolution property holds étale-locally.

Proof of Theorem 2.25. First assume that k is algebraically closed. Since \mathcal{X} is quasi-compact, Theorem 1.2 gives an étale surjective morphism $q: [U_1/G_1] \amalg \cdots \amalg [U_n/G_n] \rightarrow \mathcal{X}$ where G_i is a linearly reductive group scheme over k acting on affine scheme U_i . If we let $G = G_1 \times G_2 \times \cdots \times G_n$ and let U be the disjoint union of the $U_i \times G/G_i$, we obtain an étale surjective morphism $p: [U/G] \rightarrow \mathcal{X}$. If \mathcal{X} has affine diagonal, then we can assume that q , and hence p , are affine.

For general k , write the algebraic closure \bar{k} as a union of its finite subextensions k'/k . A standard limit argument gives a solution over some k' .

Since G is reductive, the quotient GL_n/G is affine for any embedding $G \hookrightarrow GL_n$. Note that $[U/G] = [U \times^G GL_n/GL_n]$ and $U \times^G GL_n$ is affine since $U \times^G GL_n \rightarrow U$ is a GL_n/G -fibration. \square

5.10. Compact generation of derived categories. Theorem 2.26 follows immediately from Theorem 2.25 together with [HR14c, Thm. B] (characteristic 0) or [HR14a, Thm. D] (positive characteristic).

5.11. Algebraicity results. These will be established using Artin's criterion, as formulated in [Hal14b, Thm. A]. Consequently, we will need a preparatory result on coherence (in the sense of Auslander [Aus66]) of the relevant deformation and obstruction functors, which will also help with the separation conditions. Throughout this subsection, we assume the following:

- k is a field (not necessarily algebraically closed);
- S is an algebraic space, locally of finite type over k ; and
- \mathcal{W} is an algebraic stack of finite type over k with affine diagonal that admits a good moduli space $\mathcal{W} \rightarrow S$.

The following proposition is a variant of [Hal14a, Thm. C] and [HR14c, Thm. D].

Proposition 5.4. *Assume S is an affine scheme. If $\mathcal{F}^\bullet \in \mathrm{D}_{\mathrm{QCoh}}(\mathcal{W})$ and $\mathcal{G}^\bullet \in \mathrm{D}_{\mathrm{Coh}}^b(\mathcal{W})$, then the functor*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{W}}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes_{\mathcal{O}_{\mathcal{W}}}^{\mathrm{L}} \mathrm{L}\pi_{\mathrm{QCoh}}^*(-)): \mathrm{QCoh}(S) \rightarrow \mathrm{QCoh}(S)$$

is coherent.

Proof. By Theorem 2.26, $\mathrm{D}_{\mathrm{QCoh}}(\mathcal{W})$ is compactly generated. Also, the restriction of $\mathrm{R}f_{*, \mathrm{QCoh}}: \mathrm{D}_{\mathrm{QCoh}}(\mathcal{W}) \rightarrow \mathrm{D}_{\mathrm{QCoh}}(S)$ to $\mathrm{D}_{\mathrm{Coh}}^+(\mathcal{W})$ factors through $\mathrm{D}_{\mathrm{Coh}}^+(S)$ [Alp13, Thm. 4.16(x)]. By [HR14c, Cor. 4.19], the result follows. \square

The following corollary is a variant of [Hal14a, Thm. D], whose proof is identical.

Corollary 5.5. *Let \mathcal{F} be a quasi-coherent $\mathcal{O}_{\mathcal{W}}$ -module and let \mathcal{G} be a coherent $\mathcal{O}_{\mathcal{W}}$ -module. If \mathcal{G} is flat over S , then the S -presheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{W}}/S}(\mathcal{F}, \mathcal{G})$ whose objects over $S' \xrightarrow{\tau} S$ are homomorphisms $\tau_{\mathcal{W}}^* \mathcal{F} \rightarrow \tau_{\mathcal{W}}^* \mathcal{G}$ of $\mathcal{O}_{\mathcal{W} \times_S S'}$ -modules (where $\tau_{\mathcal{W}}: \mathcal{W} \times_S S' \rightarrow \mathcal{W}$ is the projection) is representable by an affine S -scheme.*

Proof. Argue exactly as in the proof of [Hal14a, Thm. D], but using Proposition 5.4 in place of [Hal14a, Thm. C] to deduce that automorphisms, deformations and obstructions are coherent. \square

Proof of Theorem 2.19. This only requires small modifications to the proof of [Hal14b, Thm. 8.1]: the formal GAGA statement of Corollary 2.10 implies that formally versal deformations are effective and Proposition 5.4 implies that the automorphism, deformation and obstruction functors are coherent. Therefore, Artin's criterion (as

formulated in [Hal14b, Thm. A]) is satisfied and the result follows. Corollary 5.5 implies the diagonal is affine. \square

Corollaries 2.20 and 2.21 follow immediately from Theorem 2.19. Indeed, the natural functor $\underline{\text{Quot}}_{W/S}(\mathcal{F}) \rightarrow \underline{\text{Coh}}_{W/S}$ is quasi-affine by Corollary 5.5 and Nakayama's Lemma (see [Lie06, Lem. 2.6] for details).

Proof of Theorem 2.22. This only requires small modifications to the proof of [HR14b, Thm. 1.2], which uses Artin's criterion as formulated in [Hal14b, Thm. A]. Indeed, using Corollary 2.10 in place of [Ols05, Thm. 1.4] (effectivity), Proposition 5.4 in place of [Hal14a, Thm. C] (coherence) and Corollary 5.5 in place of [Hal14a, Thm. D] (conditions on the diagonal), the result follows. \square

Proof of Corollary 2.23. By Theorem 2.22, it suffices to prove that the natural map

$$(5.2) \quad \underline{\text{Hom}}_S^G(W, X) \rightarrow \underline{\text{Hom}}_S([W/G], [X/G])$$

is representable and quasi-separated. If $T \rightarrow \text{Hom}([W/G], [X/G])$ is a morphism, the corresponding morphism $T \times [W/G] \rightarrow [X/G]$ is induced from a G -equivariant morphism $T \times W \rightarrow X$ if and only if the two G -bundles over $T \times [W/G]$ corresponding to $r: T \times [W/G] \rightarrow [W/G] \rightarrow BG$ and $s: T \times [W/G] \rightarrow [X/G] \rightarrow BG$ are isomorphic. Therefore, the map in (5.2) is a pull-back of the diagonal of $\text{Hom}([W/G], BG)$ which is affine. \square

5.12. Drinfeld's results on algebraic spaces with \mathbb{G}_m -actions. We begin this subsection with the following coherent completeness lemma.

Lemma 5.6. *If S is a noetherian affine scheme, then $[\mathbb{A}_S^1/\mathbb{G}_m]$ is coherently complete along $[S/\mathbb{G}_m]$.*

Proof. Let $A = \Gamma(S, \mathcal{O}_S)$; then $\mathbb{A}_S^1 = \text{Spec } A[t]$ and $V(t) = [S/\mathbb{G}_m]$. If $\mathcal{F} \in \text{Coh}([\mathbb{A}_S^1/\mathbb{G}_m])$, then we claim that there exists an integer $n \gg 0$ such that the natural surjection $\Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{F}/t^n\mathcal{F})$ is bijective. Now every coherent sheaf on $[\mathbb{A}_S^1/\mathbb{G}_m]$ is a quotient of a finite direct sum of coherent sheaves of the form $p^*\mathcal{E}_l$, where \mathcal{E}_l is the weight l representation of \mathbb{G}_m and $p: [\mathbb{A}_S^1/\mathbb{G}_m] \rightarrow [S/\mathbb{G}_m]$ is the natural map. It is enough to prove that $\Gamma(p^*\mathcal{E}_l) \rightarrow \Gamma(p^*\mathcal{E}_l/t^n p^*\mathcal{E}_l)$ is bijective, or equivalently, that $\Gamma((t^n) \otimes p^*\mathcal{E}_l) = 0$. But $(t^n) = p^*\mathcal{E}_n$ and $\Gamma(p^*\mathcal{E}_{n+l}) = 0$ if $n+l > 0$, hence for all $n \gg 0$. We conclude that $\Gamma(\mathcal{F}) \rightarrow \varprojlim_n \Gamma(\mathcal{F}/t^n\mathcal{F})$ is bijective. What remains can be proven analogously to Theorem 1.3. \square

Proposition 5.7. *Let W be an excellent algebraic space over a field k and G be an algebraic group acting on W . Let $Z \subseteq W$ be a G -invariant closed subspace. Suppose that $[W/G]$ is coherently complete along $[Z/G]$. Let X be a noetherian algebraic space over k with an action of G . Then the natural map*

$$\text{Hom}^G(W, X) \rightarrow \varprojlim_n \text{Hom}^G(W_Z^{[n]}, X)$$

is bijective.

Proof. We have a cartesian diagram

$$\begin{array}{ccc} \text{Hom}^G(W, X) & \longrightarrow & \text{Hom}([W/G], BG) \\ \downarrow & & \downarrow \\ \text{Hom}([W/G], [X/G]) & \longrightarrow & \text{Hom}([W/G], BG) \times \text{Hom}([W/G], BG) \end{array}$$

and a similar cartesian diagram for W replaced with $W_Z^{[n]}$ for any n which gives the cartesian diagram

$$\begin{array}{ccc} \varprojlim_n \mathrm{Hom}^G(W_Z^{[n]}, X) & \longrightarrow & \varprojlim_n \mathrm{Hom}([W_Z^{[n]}/G], BG) \\ \downarrow & & \downarrow \\ \varprojlim_n \mathrm{Hom}([W_Z^{[n]}/G], [X/G]) & \longrightarrow & \varprojlim_n \mathrm{Hom}([W_Z^{[n]}/G], BG) \times \mathrm{Hom}([W_Z^{[n]}/G], BG) \end{array}$$

Since $[W/G]$ is coherently complete along $[Z/G]$, it follows by Tannaka duality that the natural maps from the first square to the second square are isomorphisms. \square

Lemma 5.8. *Let $f: U \rightarrow Z$ be a \mathbb{G}_m -equivariant étale morphism of quasi-separated algebraic spaces of finite type over a field k . Then $U^0 = Z^0 \times_Z U$ and $U^+ = Z^+ \times_{Z^0} U^0$.*

Proof. The inclusion of stabilizer group schemes $\mathrm{Stab}(U) \rightarrow \mathrm{Stab}(Z) \times_Z U$ is an open immersion since f is étale. The first statement follows since any open subgroup of \mathbb{G}_m is \mathbb{G}_m . For the second statement, we need to show that there exists a unique \mathbb{G}_m -equivariant morphism filling in the \mathbb{G}_m -equivariant diagram

$$(5.3) \quad \begin{array}{ccc} \mathrm{Spec} k \times S & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow f \\ \mathbb{A}^1 \times S & \longrightarrow & Z \end{array}$$

where S is an affine scheme of finite type over k , and the vertical left arrow is the inclusion of the origin. For each $n \geq 1$, the formal lifting property of étaleness yields a unique \mathbb{G}_m -equivariant map $\mathrm{Spec} k[x]/x^n \times S \rightarrow U$ such that

$$\begin{array}{ccc} \mathrm{Spec} k \times S & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(k[x]/x^n) \times S & \longrightarrow & Z \end{array}$$

commutes. By Lemma 5.6 and Proposition 5.7, there exists a unique \mathbb{G}_m -equivariant morphism $\mathbb{A}^1 \times S \rightarrow U$ such that (5.3) commutes. \square

Proof of Theorem 2.24. The algebraicity of Z^0 , Z^+ and \tilde{Z} follows directly from Corollary 2.23. The final statements may be verified after passing to an algebraic closure of k . Our generalization of Sumihiro's theorem (Theorem 2.4) and Lemma 5.8 now further reduce these to the case when Z is an affine scheme, which can be established directly; see [Dri13, §1.3.4]. \square

APPENDIX A. EQUIVARIANT ARTIN ALGEBRAIZATION

In this appendix, we give an equivariant generalization of Artin's algebraization theorem [Art69b, Thm. 1.6]. We follow the approach of [CJ02] relying on Popescu's theorem that establishes general Néron desingularization: any regular homomorphism of noetherian rings is a filtered colimit of smooth homomorphisms.

The main results of this appendix (Theorems A.12 and A.13) are formulated in greater generality than necessary to prove Theorem 1.2. We feel that these results are of independent interest and will have further applications. In particular, in a subsequent article we will apply the results of this appendix to prove a relative version of Theorem 1.2.

A.1. Artinian stacks.

Definition A.1. We say that an algebraic stack \mathcal{X} is *artinian* if it is noetherian and $|\mathcal{X}|$ is discrete. We say that a quasi-compact and quasi-separated algebraic stack \mathcal{X} is *local* if there exists a unique closed point $x \in |\mathcal{X}|$.

Let \mathcal{X} be a noetherian algebraic stack and let $x \in |\mathcal{X}|$ be a closed point with maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_x$. The *n*th infinitesimal neighborhood of x is the closed algebraic stack $\mathcal{X}^{[n]} \hookrightarrow \mathcal{X}$ defined by \mathfrak{m}_x^{n+1} . Note that $\mathcal{X}^{[n]}$ is artinian and that $\mathcal{X}^{[0]} = \mathcal{G}_x$ is the residual gerbe. A local artinian stack \mathcal{X} is a local artinian scheme if and only if $\mathcal{X}^{[0]}$ is the spectrum of a field.

Definition A.2. Let \mathcal{X} and \mathcal{Y} be algebraic stacks, and let $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$ be closed points. We say that a morphism $f: (\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$ is *adic* if $f^{-1}(y^{[0]}) = \mathcal{X}^{[0]}$.

Note that f is adic precisely when $f^*\mathfrak{m}_y \rightarrow \mathfrak{m}_x$ is surjective. When f is adic, we thus have that $f^{-1}(\mathcal{Y}^{[n]}) = \mathcal{X}^{[n]}$ for all $n \geq 0$. Every closed immersion is adic.

Proposition A.3. *Let \mathcal{X} be a quasi-separated algebraic stack and let $x \in |\mathcal{X}|$ be a closed point. Then there exists an adic flat presentation, that is, there exists an adic flat morphism of finite presentation $p: (\text{Spec } A, u) \rightarrow (\mathcal{X}, x)$. If the stabilizer group of x is smooth, then there exists an adic smooth presentation.*

Proof. The question is local on \mathcal{X} so we can assume that \mathcal{X} is quasi-compact. Start with any smooth presentation $q: V = \text{Spec } A \rightarrow \mathcal{X}$. The fiber $V_x = q^{-1}(\mathcal{G}_x) = \text{Spec } A/I$ is smooth over the residue field $\kappa(x)$ of the residual gerbe. Pick a closed point $v \in V_x$ such that $\kappa(v)/\kappa(x)$ is separable. After replacing V with an open neighborhood of v , we may pick a regular sequence $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n \in A/I$ that generates \mathfrak{m}_v . Lift this to a sequence $f_1, f_2, \dots, f_n \in A$ and let $Z \hookrightarrow V$ be the closed subscheme defined by this sequence. The sequence is transversely regular over \mathcal{X} in a neighborhood $W \subseteq V$ of v . In particular, $U = W \cap Z \rightarrow V \rightarrow \mathcal{X}$ is flat. By construction $U_x = Z_x = \text{Spec } \kappa(v)$ so $(U, v) \rightarrow (\mathcal{X}, x)$ is an adic flat presentation. Moreover, $\text{Spec } \kappa(v) \rightarrow \mathcal{G}_x \rightarrow \kappa(x)$ is étale so if the stabilizer group is smooth, then $U_x \rightarrow \mathcal{G}_x$ is smooth and $U \rightarrow \mathcal{X}$ is smooth at u . \square

Corollary A.4. *Let \mathcal{X} be a noetherian algebraic stack. The following statements are equivalent.*

- (1) *There exists an artinian ring A and a flat presentation $p: \text{Spec } A \rightarrow \mathcal{X}$ which is adic at every point of \mathcal{X} .*
- (2) *There exists an artinian ring A and a flat presentation $p: \text{Spec } A \rightarrow \mathcal{X}$.*
- (3) *\mathcal{X} is artinian.*

Proof. The implications (1) \implies (2) \implies (3) are trivial. The implication (3) \implies (1) follows from the proposition. \square

If \mathcal{X} is an algebraic stack, $\mathcal{J} \subseteq \mathcal{O}_{\mathcal{X}}$ is a sheaf of ideals and \mathcal{F} is a quasi-coherent sheaf, we set $\text{Gr}_{\mathcal{J}}(\mathcal{F}) := \bigoplus_{n \geq 0} \mathcal{J}^n \mathcal{F} / \mathcal{J}^{n+1} \mathcal{F}$, which is a quasi-coherent sheaf of graded modules on the closed substack defined by \mathcal{J} .

Proposition A.5. *Let $f: (\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$ be a morphism of local artinian stacks.*

- (1) *f is a closed immersion if and only if $f|_{\mathcal{X}^{[1]}}: \mathcal{X}^{[1]} \rightarrow \mathcal{Y}$ is a closed immersion.*
- (2) *f is an isomorphism if and only if f is a closed immersion and there exists an isomorphism $\varphi: \text{Gr}_{\mathfrak{m}_x}(\mathcal{O}_X) \rightarrow (f^{[0]})^* \text{Gr}_{\mathfrak{m}_y}(\mathcal{O}_Y)$ of graded $\mathcal{O}_{\mathcal{X}^{[0]}}$ -modules.*

Proof. The conditions are clearly necessary. For the sufficiency, pick an adic flat presentation $p: \text{Spec } A \rightarrow \mathcal{Y}$. After pulling back f along p , we may assume that $\mathcal{Y} = \text{Spec } A$ is a local artinian scheme. If $f|_{\mathcal{X}^{[0]}}$ is a closed immersion, then $\mathcal{X} = \text{Spec } B$

is also a local artinian scheme. If in addition $f|_{\mathcal{X}^{[1]}}$ is a closed immersion, then $\mathfrak{m}_A \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective; hence so is $\mathfrak{m}_A B \rightarrow \mathfrak{m}_B$ by Nakayama's Lemma. We conclude that f is adic and that $A \rightarrow B$ is surjective (Nakayama's lemma again).

Assume that in addition we have an isomorphism $\varphi: \mathrm{Gr}_{\mathfrak{m}_A} A \cong \mathrm{Gr}_{\mathfrak{m}_B} B$ of graded k -vector spaces where $k = A/\mathfrak{m}_A = B/\mathfrak{m}_B$. Then $\dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} = \dim_k \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$. It follows that the surjections $\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} \rightarrow \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$ induced by f are isomorphisms. It follows that f is injective. \square

Recall from Section 2.7 that a *complete local stack* is a noetherian local stack (\mathcal{X}, x) such that \mathcal{X} is coherently complete along the residual gerbe \mathcal{G}_x .

Lemma A.6. *Let $f: (\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$ be a morphism of complete local stacks that induces an isomorphism $f^{[0]}: \mathcal{X}^{[0]} \rightarrow \mathcal{Y}^{[0]}$.*

- (1) *f is a closed immersion if and only if $f|_{\mathcal{X}^{[n]}}: \mathcal{X}^{[n]} \rightarrow \mathcal{Y}$ is a closed immersion for all $n \geq 0$.*
- (2) *f is an isomorphism if and only if $f|_{\mathcal{Y}^{[n]}}: \mathcal{X}^{[n]} \rightarrow \mathcal{Y}^{[n]}$ is an isomorphism for all $n \geq 0$.*

Proof. In the first statement, we have a compatible system of closed immersions $\mathcal{X}^{[n]} \hookrightarrow \mathcal{Y}^{[n]}$. Since \mathcal{Y} is coherently complete, we obtain a unique closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ such that $\mathcal{X}^{[n]} = \mathcal{Z}^{[n]}$ for all n . Since \mathcal{X} and \mathcal{Y} are coherently complete, it follows by Tannaka duality (Theorem 3.5) that we have an isomorphism $\mathcal{X} \rightarrow \mathcal{Z}$. The second statement follows directly from Tannaka duality since \mathcal{X} and \mathcal{Y} are coherently complete. \square

Combining the lemma and Proposition A.5 we obtain.

Proposition A.7. *Let $f: (\mathcal{X}, x) \rightarrow (\mathcal{Y}, y)$ be a morphism of complete local stacks.*

- (1) *f is a closed immersion if and only if $f|_{\mathcal{X}^{[1]}}: \mathcal{X}^{[1]} \rightarrow \mathcal{Y}$ is a closed immersion.*
- (2) *f is an isomorphism if and only if f is a closed immersion and there exists an isomorphism $\varphi: \mathrm{Gr}_{\mathfrak{m}_x}(\mathcal{O}_X) \rightarrow (f^{[0]})^* \mathrm{Gr}_{\mathfrak{m}_y}(\mathcal{O}_Y)$ of graded $\mathcal{O}_{\mathcal{X}^{[0]}}$ -modules.*

A.2. Formal versality.

Definition A.8. Let \mathcal{W} be a noetherian algebraic stack, let $w \in |\mathcal{W}|$ be a closed point and let $\mathcal{W}^{[n]}$ denote the n th infinitesimal neighborhood of w . Let \mathcal{X} be a category fibered in groupoids and let $\eta: \mathcal{W} \rightarrow \mathcal{X}$ be a morphism. We say that η is *formally versal* (resp. *formally universal*) at w if the following lifting condition holds. Given a 2-commutative diagram of solid arrows

$$\begin{array}{ccc} \mathcal{W}^{[0]} \hookrightarrow \mathcal{Z} & \xrightarrow{f} & \mathcal{W} \\ & \searrow \scriptstyle f' & \downarrow \scriptstyle \eta \\ & \mathcal{Z}' & \longrightarrow \mathcal{X} \end{array}$$

(Note: In the original image, there is a curved arrow labeled g from \mathcal{Z} to \mathcal{Z}' and a dashed arrow from \mathcal{Z}' to \mathcal{W} .)

where \mathcal{Z} and \mathcal{Z}' are local artinian stacks and ι and g are closed immersions, there exists a morphism (resp. a unique morphism) f' and 2-isomorphisms such that the whole diagram is 2-commutative.

Proposition A.9. *Let $\eta: (\mathcal{W}, w) \rightarrow (\mathcal{X}, x)$ be a morphism of noetherian algebraic stacks. Assume that w and x are closed points. Suppose that $\eta^{[0]}: \mathcal{W}^{[0]} \rightarrow \mathcal{X}^{[0]}$ is representable.*

- (1) *If $\mathcal{W}^{[n]} \rightarrow \mathcal{X}^{[n]}$ is étale for every n , then η is formally universal at w .*
- (2) *If $\mathcal{W}^{[n]} \rightarrow \mathcal{X}^{[n]}$ is smooth for every n and the stabilizer G_w is linearly reductive, then η is formally versal at w .*

Proof. The first part is clear from descent. For the second part, we begin with the following observation: if (\mathcal{Z}, z) is a local artinian stack and $h: (\mathcal{Z}, z) \rightarrow (\mathcal{Q}, q)$ is a morphism of algebraic stacks, where q is a closed point, then there exists an n such that h factors through $\mathcal{Q}^{[n]}$. Now, if we are given a lifting problem, then the previous observation shows that we may assume that \mathcal{Z}' and \mathcal{Z} factor through some $\mathcal{W}^{[n]} \rightarrow \mathcal{X}^{[n]}$, which is representable. The obstruction to the existence of a lift belongs to the group $\mathrm{Ext}_{\mathcal{Z}}^1(f^*\Omega_{\mathcal{W}^{[n]}/\mathcal{X}^{[n]}}/I, I)$, where I is the square zero ideal defining the closed immersion g . But \mathcal{Z} is cohomologically affine and $\Omega_{\mathcal{W}^{[n]}/\mathcal{X}^{[n]}}$ is a vector bundle, so the Ext-group vanishes. The result follows. \square

A.3. Refined Artin–Rees for algebraic stacks. The results in this section is a generalization of [CJ02, §3] (also see [Stacks, Tag 07VD]) from rings to algebraic stacks.

Definition A.10. Let \mathcal{X} be a noetherian algebraic stack and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack defined by the ideal $\mathcal{J} \subseteq \mathcal{O}_{\mathcal{X}}$. Let $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ be a homomorphism of coherent sheaves on \mathcal{X} . Let $c \geq 0$ be an integer. We say that $(\mathrm{AR})_c$ holds for φ along \mathcal{Z} if

$$\varphi(\mathcal{E}) \cap \mathcal{J}^n \mathcal{F} \subseteq \varphi(\mathcal{J}^{n-c} \mathcal{E}), \quad \forall n \geq c.$$

When \mathcal{X} is a scheme, $(\mathrm{AR})_c$ holds for all sufficiently large c by the Artin–Rees lemma. If $\pi: U \rightarrow \mathcal{X}$ is a flat presentation, then $(\mathrm{AR})_c$ holds for φ along \mathcal{Z} if and only if $(\mathrm{AR})_c$ holds for $\pi^*\varphi: \pi^*\mathcal{E} \rightarrow \pi^*\mathcal{F}$ along $\pi^{-1}(\mathcal{Z})$. In particular $(\mathrm{AR})_c$ holds for φ along \mathcal{Z} for all sufficiently large c . If $f: \mathcal{E}' \rightarrow \mathcal{E}$ is a surjective homomorphism, then $(\mathrm{AR})_c$ for φ holds if and only if $(\mathrm{AR})_c$ for $\varphi \circ f$ holds.

In the following section, we will only use the case when $|\mathcal{Z}|$ is a closed point.

Theorem A.11. Let $\mathcal{E}_2 \xrightarrow{\alpha} \mathcal{E}_1 \xrightarrow{\beta} \mathcal{E}_0$ and $\mathcal{E}'_2 \xrightarrow{\alpha'} \mathcal{E}'_1 \xrightarrow{\beta'} \mathcal{E}'_0$ be two complexes of coherent sheaves on a noetherian algebraic stack \mathcal{X} . Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack defined by the ideal $\mathcal{J} \subseteq \mathcal{O}_{\mathcal{X}}$. Let c be a positive integer. Assume that

- (1) $\mathcal{E}_0, \mathcal{E}'_0, \mathcal{E}_1, \mathcal{E}'_1$ are vector bundles,
- (2) the sequences are isomorphic after tensoring with $\mathcal{O}_{\mathcal{X}}/\mathcal{J}^c$,
- (3) the first sequence is exact, and
- (4) $(\mathrm{AR})_c$ holds for α and β along \mathcal{Z} .

Then

- (a) the second sequence is exact in a neighborhood of \mathcal{Z} ;
- (b) $(\mathrm{AR})_c$ holds for β' along \mathcal{Z} ; and
- (c) given an isomorphism $\varphi: \mathcal{E}_0 \rightarrow \mathcal{E}'_0$, there exists a unique isomorphism ψ of $\mathrm{Gr}_{\mathcal{J}}(\mathcal{O}_{\mathcal{X}})$ -modules in the diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\mathcal{J}}(\mathcal{E}_0) & \xrightarrow{\mathrm{Gr}(\gamma)} & \mathrm{Gr}_{\mathcal{J}}(\mathrm{coker} \beta) \\ \cong \downarrow \mathrm{Gr}(\varphi) & & \cong \downarrow \psi \\ \mathrm{Gr}_{\mathcal{J}}(\mathcal{E}'_0) & \xrightarrow{\mathrm{Gr}(\gamma')} & \mathrm{Gr}_{\mathcal{J}}(\mathrm{coker} \beta') \end{array}$$

where $\gamma: \mathcal{E}_0 \rightarrow \mathrm{coker} \beta$ and $\gamma': \mathcal{E}'_0 \rightarrow \mathrm{coker} \beta'$ denote the induced maps.

Proof. Note that there exists an isomorphism ψ if and only if $\ker \mathrm{Gr}(\gamma) = \ker \mathrm{Gr}(\gamma')$. All three statements can thus be checked after pulling back to a presentation $U \rightarrow \mathcal{X}$. We may also localize and assume that $\mathcal{X} = U = \mathrm{Spec} A$ where A is a local ring. Then all vector bundles are free and we may choose isomorphisms $\mathcal{E}_i \cong \mathcal{E}'_i$ for $i = 0, 1$ such that $\beta = \beta'$ modulo \mathcal{J}^c . We can also choose a surjection $\epsilon': \mathcal{O}_U^n \rightarrow \mathcal{E}'_2$ and a lift $\epsilon: \mathcal{O}_U^n \rightarrow \mathcal{E}_2$ modulo \mathcal{J}^c , so that $\alpha \circ \epsilon = \alpha' \circ \epsilon'$ modulo \mathcal{J}^c . Thus, we may assume that $\mathcal{E}_i = \mathcal{E}'_i$ for $i = 0, 1, 2$ are free. The result then follows from [CJ02, Lem. 3.1 and Thm. 3.2] or [Stacks, Tags 07VE and 07VF]. \square

A.4. Equivariant algebraization.

Theorem A.12. *Let S be an excellent scheme and let T be a noetherian algebraic space over S . Let \mathcal{Z} be an algebraic stack of finite presentation over T and let $z \in |\mathcal{Z}|$ be a closed point such that $\mathcal{G}_z \rightarrow S$ is of finite type. Let $t \in T$ be the image of z . Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be categories fibered in groupoids over S , locally of finite presentation. Let $\eta: \mathcal{Z} \rightarrow \mathcal{X} = \mathcal{X}_1 \times_S \cdots \times_S \mathcal{X}_n$ be a morphism. Fix an integer $N \geq 0$. Then there exists*

- (1) *an affine scheme S' of finite type over S and a point $s' \in S'$ mapping to the same point in S as $t \in T$;*
- (2) *an algebraic stack $\mathcal{W} \rightarrow S'$ of finite type;*
- (3) *a closed point $w \in |\mathcal{W}|$ over s' ;*
- (4) *a morphism $\xi: \mathcal{W} \rightarrow \mathcal{X}$;*
- (5) *an isomorphism $\mathcal{Z} \times_T T^{[N]} \cong \mathcal{W} \times_{S'} S'^{[N]}$ over \mathcal{X} ; in particular, there is an isomorphism $\mathcal{Z}^{[N]} \cong \mathcal{W}^{[N]}$ over \mathcal{X} ; and*
- (6) *an isomorphism $\mathrm{Gr}_{\mathfrak{m}_z} \mathcal{O}_{\mathcal{Z}} \cong \mathrm{Gr}_{\mathfrak{m}_w} \mathcal{O}_{\mathcal{W}}$ of graded algebras over $\mathcal{Z}^{[0]} \cong \mathcal{W}^{[0]}$.*

Moreover, if \mathcal{X}_i is a quasi-compact algebraic stack and $\eta_i: \mathcal{Z} \rightarrow \mathcal{X}_i$ is affine for some i , then it can be arranged so that $\xi_i: \mathcal{W} \rightarrow \mathcal{X}_i$ is affine.

Proof. We may assume that $S = \mathrm{Spec} A$ is affine. Let $t \in T$ be the image of z . By replacing T with the completion $\widehat{T} = \mathrm{Spec} \widehat{\mathcal{O}}_{T,t}$ and \mathcal{Z} with $\mathcal{Z} \times_T \widehat{T}$, we may assume that $T = \widehat{T} = \mathrm{Spec} B$ where B is a complete local ring. By standard limit methods, we have an affine scheme $S_0 = \mathrm{Spec} B_0$ and an algebraic stack $\mathcal{Z}_0 \rightarrow S_0$ of finite presentation and a commutative diagram

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{\quad} & \mathcal{Z}_0 & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow & \square & \downarrow & & \downarrow \\ T & \xrightarrow{\quad} & S_0 & \xrightarrow{\quad} & S \end{array}$$

If \mathcal{X}_i is algebraic and quasi-compact and $\mathcal{Z} \rightarrow \mathcal{X}_i$ is affine for some i , we may also arrange so that $\mathcal{Z}_0 \rightarrow \mathcal{X}_i$ is affine [Ryd15, Thm. C].

Since $\mathcal{G}_z \rightarrow S$ is of finite type, so is $\mathrm{Spec} \kappa(t) \rightarrow S$. We may thus choose a factorization $T \rightarrow S_1 = \mathbb{A}_{S_0}^n \rightarrow S_0$, such that $T \rightarrow \widehat{S}_1 = \mathrm{Spec} \widehat{\mathcal{O}}_{S_1, s_1}$ is a closed immersion; here $s_1 \in S_1$ denotes the image of $t \in T$. Let $\mathcal{Z}_1 = \mathcal{Z}_0 \times_{S_0} S_1$ and $\widehat{\mathcal{Z}}_1 = \mathcal{Z}_1 \times_{S_1} \widehat{S}_1$. Consider the functor $F: \mathrm{Sch}/S_1 \rightarrow \mathrm{Sets}$ where $F(U \rightarrow S_1)$ is the set of isomorphism classes of complexes

$$\mathcal{E}_2 \xrightarrow{\alpha} \mathcal{E}_1 \xrightarrow{\beta} \mathcal{O}_{\mathcal{Z}_1 \times_{S_1} U}$$

of finitely presented quasi-coherent $\mathcal{O}_{\mathcal{Z}_1 \times_{S_1} U}$ -modules with \mathcal{E}_1 locally free. By standard limit arguments, F is locally of finite presentation.

We have an element $(\alpha, \beta) \in F(\widehat{S}_1)$ such that $\mathrm{im}(\beta)$ defines $\mathcal{Z} \hookrightarrow \widehat{\mathcal{Z}}_1$. Indeed, choose a resolution

$$\widehat{\mathcal{O}}_{S_1, s_1}^{\oplus r} \xrightarrow{\widetilde{\beta}} \widehat{\mathcal{O}}_{S_1, s_1} \rightarrow B.$$

After pulling back $\widetilde{\beta}$ to $\widehat{\mathcal{Z}}_1$, we obtain a resolution

$$\ker(\beta) \xrightarrow{\alpha} \mathcal{O}_{\widehat{\mathcal{Z}}_1}^{\oplus r} \xrightarrow{\beta} \mathcal{O}_{\widehat{\mathcal{Z}}_1} \rightarrow \mathcal{O}_{\mathcal{Z}}.$$

After increasing N , we may assume that $(\mathrm{AR})_N$ holds for α and β at $\widehat{\mathcal{Z}}_1^{[0]}$.

We now apply Néron–Popescu approximation [Pop86, Thm. 1.3] to the regular morphism $\widehat{S}_1 \rightarrow S_1$ and obtain an étale neighborhood $(S', s') \rightarrow (S_1, s_1)$ and an element $(\alpha', \beta') \in F(S')$ such that $(\alpha, \beta) = (\alpha', \beta')$ in $F(S_1^{[N]})$. We let $\mathcal{W} \hookrightarrow \mathcal{Z}_1 \times_{S_1} S'$ be the closed substack defined by $\mathrm{im}(\beta')$. Then $\mathcal{Z} \times_T T^{[N]}$ and $\mathcal{W} \times_{S'} S'^{[N]}$

are equal as closed substacks of $\mathcal{Z}_1 \times_{S_1} S_1^{[n]}$ and (1)–(5) follows. Finally (6) follows from Theorem A.11. \square

Theorem A.13. *Let $S, T, \mathcal{Z}, \eta, N, \mathcal{W}$ and ξ be as in Theorem A.12. If $\eta_1: \mathcal{Z} \rightarrow \mathcal{X}_1$ is formally versal, then there are compatible isomorphisms $\varphi_n: \mathcal{W}^{[n]} \xrightarrow{\sim} \mathcal{Z}^{[n]}$ over \mathcal{X}_1 for all $n \geq 0$. For $n \leq N$, the isomorphism φ_n is also compatible with η and ξ .*

Proof. We can assume that $N \geq 1$. By Theorem A.12, we have an isomorphism $\varphi_N: \mathcal{W}^{[N]} \rightarrow \mathcal{Z}^{[N]}$ over \mathcal{X} . By formal versality and induction on $n \geq N$, we can extend φ_N to compatible morphisms $\varphi'_n: \mathcal{W}^{[n]} \rightarrow \mathcal{Z}$ over \mathcal{X}_1 . Indeed, formal versality allows us to find a dotted arrow such the diagram

$$\begin{array}{ccc} \mathcal{W}^{[n]} & \xrightarrow{\varphi'_n} & \mathcal{Z} \\ \downarrow & \nearrow \varphi'_{n+1} & \downarrow \eta \\ \mathcal{W}^{[n+1]} & \xrightarrow{\xi_1|_{\mathcal{W}^{[n+1]}}} & \mathcal{X}_1 \end{array}$$

is 2-commutative. By Proposition A.5, φ'_n induces an isomorphism $\varphi_n: \mathcal{W}^{[n]} \rightarrow \mathcal{Z}^{[n]}$. \square

We now formulate the theorem above in a manner which is transparently an equivariant analogue of Artin algebraization [Art69b, Thm. 1.6]. It is this formulation that is directly applied to prove Theorem 1.2.

Corollary A.14. *Let H be a linearly reductive affine group scheme over an algebraically closed field k . Let \mathcal{X} be a category fibered in groupoids over k that is locally of finite presentation. Let $\mathcal{Z} = [\mathrm{Spec} A/H]$ be a noetherian algebraic stack over k . Suppose that A^H is a complete local k -algebra. Assume that \mathcal{Z} is of finite type over an algebraic space T (or equivalently that $\mathcal{Z} \rightarrow \mathrm{Spec} A^H$ is of finite type). Let $\eta: \mathcal{Z} \rightarrow \mathcal{X}$ be a morphism that is formally versal at $z \in |\mathcal{Z}|$. Let $N \geq 0$. Then there exists*

- (1) an algebraic stack $\mathcal{W} = [\mathrm{Spec} B/H]$ of finite type over k ;
- (2) a closed point $w \in |\mathcal{W}|$;
- (3) a morphism $\xi: \mathcal{W} \rightarrow \mathcal{X}$; and
- (4) an isomorphism $\varphi: \widehat{\mathcal{W}} \rightarrow \mathcal{Z}$ over \mathcal{X} , where $\widehat{\mathcal{W}}$ is the coherent completion of \mathcal{W} at w (i.e., $\widehat{\mathcal{W}} = \mathcal{W} \times_{\mathcal{W}} \widehat{\mathrm{Spec} \mathcal{O}_{\mathcal{W}, w_0}}$ where $W = \mathrm{Spec} B^H$ and $w_0 \in W$ is the image of w under $\mathcal{W} \rightarrow W$).

Moreover, it can be arranged that $\varphi^{[N]}: \mathcal{W}^{[N]} \rightarrow \mathcal{Z}^{[N]}$ is an isomorphism over BH .

Proof. If we apply Theorem A.13 with $S = \mathrm{Spec} k$, $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = BH$, then we immediately obtain (1)–(3) together with isomorphisms $\varphi_n: \mathcal{W}^{[n]} \rightarrow \mathcal{Z}^{[n]}$ that are compatible over \mathcal{X} for all n and compatible over BH for all $n \leq N$. Since \mathcal{Z} and $\widehat{\mathcal{W}}$ are coherently complete, by the Tannakian formalism the isomorphisms φ_n yields an isomorphism $\varphi: \widehat{\mathcal{W}} \rightarrow \mathcal{Z}$ (see Corollary 3.6). \square

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