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A LYAPUNOV-TYPE STABILITY CRITERION USING L^α NORMS

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ABSTRACT. Let $q(t)$ be a T-periodic potential such that $\int_0^T q(t)\,dt < 0$. The classical Lyapunov criterion to stability of Hill's equation $-\ddot{x} + q(t)x = 0$ is $\|q_-\|_1 = \int_0^T |q_-(t)|dt \le 4/T$, where q_- is the negative part of q. In this paper, we will use a relation between the (anti-)periodic and the Dirichlet eigenvalues to establish some lower bounds for the first anti-periodic eigenvalue. As a result, we will find the best Lyapunov-type stability criterion using L^α norms of q_-, $1 \le \alpha \le \infty$. The numerical simulation to Mathieu's equation shows that the new criterion approximates the first stability region very well.

1. Introduction and main result

Let $q(t)$ be a periodic function of period $T > 0$ such that $q \in L^1(0,T)$. Recall that Hill's equation

$$-\ddot{x} + q(t)x = 0 \tag{1}$$

is stable (in the sense of Lyapunov) if any solution $x(t)$ to (1) satisfies

$$\sup_{t\in\mathbb{R}}(|x(t)| + |\dot{x}(t)|) < \infty.$$

The stability of Hill's equation is a basic and an important problem in the theory of ordinary differential equations. Research on it goes back to the time of Lyapunov (see, e.g., [3]). Many theoretical results concerning this problem can be found in textbooks such as [5, 8]. Theoretically, the stability of (1) can be completely described using the periodic and the anti-periodic eigenvalues; see [8, Theorem 2.1].

A classical stability criterion given by Lyapunov, Krein and Borg (see [8, p. 46]) is as follows. Suppose that $q(t) \le 0$ for a.e. $t \in \mathbb{R}$ and $q(t) < 0$ on a subset of positive measure. If

$$\|q\|_1 = \int_0^T |q(t)|dt \le \frac{4}{T}, \tag{2}$$

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then (1) is stable. This can be shown using a Poincaré inequality. Condition (2) is the simplest criterion for the first stability interval. It is also best possible in the sense that for any $\varepsilon > 0$, there is some q such that

$$\|q\|_1 < \frac{4}{T} + \varepsilon,$$

while (1) is instable. Condition (2) has been generalized in several ways; see [8]. There are also many recent works on this problem and related ones such as determining the length of the stability intervals. See [4, 6, 7, 10].

In this paper, we will use certain Sobolev constants given by Talenti [9] and a relation between the (anti-)periodic and the Dirichlet eigenvalues to establish some lower bounds for the first anti-periodic values. Then we will give a stability criterion to (1) using the L^α $(1 \le \alpha \le \infty)$ norms of the potential $q(t)$. The main result of this paper follows:

Theorem 1. *Let q be a T-periodic function such that $\int_0^T q(t)dt < 0$. Assume that $q \in L^\alpha(0,T)$ for some $1 \le \alpha \le \infty$. Then* (1) *is stable when*

$$\|q_-\|_\alpha < K(2\alpha^*), \qquad \text{if} \quad 1 < \alpha \le \infty, \tag{3}$$

or

$$\|q_-\|_\alpha \le K(\infty) = 4/T, \qquad \text{if} \quad \alpha = 1. \tag{4}$$

Here $\alpha^* = \alpha/(\alpha - 1)$ and $K(\cdot)$ are certain Sobolev constants which will be given explicitly in (8)*, and $q_-(t) = \max\{-q(t), 0\}$ is the negative part of $q(t)$, $\|\cdot\|_\alpha$ denotes the L^α norm on the interval $[0,T]$. Furthermore, the upper bounds $K(2\alpha^*)$ for $\|q_-\|_\alpha$ in* (3) *are best possible.*

When the first stability region of the parametrized Mathieu equation

$$\ddot{x} + \lambda(1 + \varepsilon \cos t)x = 0 \qquad (\lambda > 0,\ \varepsilon \in [-1, 1])$$

is considered, a suitable choice of α in (3) depending on ε shows that the stability condition (3) strongly approximates the stability region for all $\varepsilon \in [-1, 1]$.

2. Proofs

Let $q(t)$ be a periodic function of period $T > 0$ such that $q \in L^1(0,T)$. Consider the eigenvalue problems of

$$Lx = -\ddot{x} + q(t)x = \lambda x \tag{5}$$

subject to the periodic boundary condition

$$x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \tag{P}$$

or, to the anti-periodic boundary condition

$$x(0) + x(T) = \dot{x}(0) + \dot{x}(T) = 0. \tag{A}$$

The following is a well-known result concerning eigenvalues and stability of (5).

Theorem 2 ([8, Theorem 2.1])**.** *There exist*

$$\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \le \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \le \overline{\lambda}_2(q) < \cdots < \underline{\lambda}_k(q) \le \overline{\lambda}_k(q) < \cdots$$

such that

(i) *λ is an eigenvalue of* (5)+(P) *if and only if $\lambda = \underline{\lambda}_k(q)$ or $\overline{\lambda}_k(q)$ for $k = 0, 2, 4, \cdots$;*

(ii) *λ is an eigenvalue of* (5)+(A) *if and only if* $\lambda = \underline{\lambda}_k(q)$ *or* $\overline{\lambda}_k(q)$ *for* $k = 1, 3, 5, \cdots$;

(iii) *Equation* (5) *is stable if λ is in the intervals*

$$(\overline{\lambda}_0(q), \underline{\lambda}_1(q)), \quad (\overline{\lambda}_1(q), \underline{\lambda}_2(q)), \quad \cdots, \quad (\overline{\lambda}_k(q), \underline{\lambda}_{k+1}(q)), \quad \cdots;$$

(iv) *Equation* (5) *is unstable if λ is in the intervals*

$$(-\infty, \overline{\lambda}_0(q)), \quad (\underline{\lambda}_1(q), \overline{\lambda}_1(q)), \quad \cdots, \quad (\underline{\lambda}_k(q), \overline{\lambda}_k(q)), \quad \cdots.$$

Our main theorem is proved using this theorem by showing that 0 is in the first stability interval $(\overline{\lambda}_0(q), \underline{\lambda}_1(q))$. To this end, we will establish a lower bound for the first anti-periodic eigenvalue $\underline{\lambda}_1(q)$ of (5)+(A).

Let us introduce some notation. For $1 \le \alpha \le \infty$, we use $\|\cdot\|_\alpha$ to denote the L^α norm in the Lebesque space $L^\alpha(0, T)$.

Consider also the eigenvalues of (5) subject to the Dirichlet boundary condition

$$x(0) = x(T) = 0. \tag{D}$$

Then (5)+(D) has a sequence of eigenvalues

$$\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_k^D(q) < \cdots.$$

It is well-known that the periodic and anti-periodic eigenvalues can be realized using the Dirichlet eigenvalues in the following way: For any $k \in \mathbb{N}$,

$$\underline{\lambda}_k(q) = \min\{\lambda_k^D(q_s) : s \in \mathbb{R}\}, \qquad \overline{\lambda}_k(q) = \max\{\lambda_k^D(q_s) : s \in \mathbb{R}\}, \tag{6}$$

where $q_s(\cdot)$ are translations of $q(\cdot)$: $q_s(t) \equiv q(t+s)$. Such a relation has also been generalized to the one-dimensional p-Laplacian; see [12, Theorem 4.3].

We need also certain Sobolev constants. For any $1 \le \alpha \le \infty$, let $K(\alpha)$ be the best Sobolev constant in the inequality

$$C\|u\|_\alpha^2 \le \|\dot{u}\|_2^2 \qquad \text{for all } u \in \mathcal{H} := H_0^1(0, T),$$

i.e.,

$$K(\alpha) = \inf_{u \in \mathcal{H}\setminus\{0\}} \frac{\|\dot{u}\|_2^2}{\|u\|_\alpha^2}. \tag{7}$$

Proposition 3. (i) *The constants $K(\alpha)$ are given by*

$$K(\alpha) = \begin{cases} \frac{2\pi}{\alpha T^{1+2/\alpha}} \left(\frac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\alpha}\right)}\right)^2, & \text{if } \ 1 \le \alpha < \infty, \\ \frac{4}{T}, & \text{if } \ \alpha = \infty. \end{cases} \tag{8}$$

(ii) *Let $1 \le \alpha < \infty$. Then the infimum in* (7) *can only be attained by functions $u = c\, u_\alpha(t)$, where $c \ne 0$ and $u_\alpha(t)$ is*

$$u_\alpha(t) = \begin{cases} F_\alpha^{-1}\big(2F_\alpha(1)t/T\big), & \text{if } \ t \in [0, T/2], \\ F_\alpha^{-1}\big(2F_\alpha(1)(1 - t/T)\big), & \text{if } \ t \in [T/2, T], \end{cases} \tag{9}$$

where $F_\alpha : [0,1] \to \mathbb{R}$ is given by

$$F_\alpha(u) = \int_0^u \frac{du}{(1-u^\alpha)^{1/2}}.$$

Proof. These results are given in [9]. See also [11] for some generalizations. □

Now we establish the following lower bound for the first Dirichlet eigenvalue $\lambda_1^D(q)$ of (5).

Theorem 4. *Let q be as before. Suppose that $q \in L^\alpha(0,T)$ for some $1 \le \alpha \le \infty$ and $\theta = 1 - \|q_-\|_\alpha / K(2\alpha^*) \ge 0$. Then*

$$\lambda_1^D(q) \ge \theta(\pi/T)^2. \tag{10}$$

Proof. Let us introduce the quadratic form in $\mathcal{H}$:

$$Q(u) = \int_0^T (\dot{u}^2 + q(t)u^2)dt, \qquad u \in \mathcal{H}.$$

The following is a standard result.

$$\lambda_1^D(q) = \inf_{u \in \mathcal{H}\setminus\{0\}} \frac{Q(u)}{\|u\|_2^2}. \tag{11}$$

Moreover, equality is attained if and only if u is an eigenfunction for $\lambda_1^D(q)$.

Now let $u \in \mathcal{H}$ and $u \ne 0$. Then

$$\begin{aligned} Q(u) &= \int_0^T (\dot{u}^2 + q(t)u^2)dt \\ &\ge \int_0^T \dot{u}^2 dt - \int_0^T q_-(t)u^2 dt \\ &\ge \|\dot{u}\|_2^2 - \|q_-\|_\alpha \|u^2\|_{\alpha^*} \\ &= \|\dot{u}\|_2^2 - \|q_-\|_\alpha \|u\|_{2\alpha^*}^2 \\ &\ge \|\dot{u}\|_2^2 - \frac{\|q_-\|_\alpha}{K(2\alpha^*)}\|\dot{u}\|_2^2 = \theta\|\dot{u}\|_2^2, \end{aligned} \tag{12}$$

where the Hölder inequality and (7) are used in the proof. From these estimates, one has

$$\frac{Q(u)}{\|u\|_2^2} \ge \theta \frac{\|\dot{u}\|_2^2}{\|u\|_2^2} \ge \theta(\pi/T)^2.$$

Thus (10) follows the characterization (11) on $\lambda_1^D(q)$. ☐

Remark. By the relation (6), the first anti-periodic eigenvalue $\underline{\lambda}_1(q)$ can be realized by $\lambda_D^1(q_{s_0})$ for some s_0. Note that $\|(q_{s_0})_-\|_\alpha = \|q_-\|_\alpha$. Thus, under the assumption of Theorem 4, one has

$$\underline{\lambda}_1(q) \ge \theta(\pi/T)^2 = \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|q_-\|_\alpha}{K(2\alpha^*)}\right). \tag{13}$$

Proof of Theorem 1. First, it is well-known that the zeroth periodic eigenvalue $\overline{\lambda}_0(q) \le T^{-1}\int_0^T q(t)dt$; see [8, Theorem 4.4]. By the assumption of Theorem 1, $\overline{\lambda}_0(q) < 0$ in this case. On the other hand, if (3) holds, then $\theta > 0$. By (13) the first anti-periodic eigenvalue $\underline{\lambda}_1(q) \ge \theta(\pi/T)^2 > 0$. Thus 0 is inside the interval $(\overline{\lambda}_0(q), \underline{\lambda}_1(q))$. Now Theorem 2 shows that equation (1), which corresponds to (5) with $\lambda = 0$, is stable. If (4) holds, we know from (13) that $\underline{\lambda}_1(q) \ge \theta(\pi/T)^2 \ge 0$. We assert that $\underline{\lambda}_1(q)$ is always positive even when $\theta = 0$. Let us simply prove that $\lambda_1^D(q) > 0$ in this case. Suppose that u_0 is an eigenfunction of (5)+(D) associated with $\lambda_1^D(q)$. Proceeding as in the proof of (12), we have

$$\lambda_1^D(q)\|u_0\|_2^2 \ge \|\dot{u}_0\|_2^2 - \|q_-\|_1\|u_0\|_\infty^2 > (4/T - \|q_-\|_1)\|u_0\|_\infty^2. \tag{14}$$

Note that the last inequality must be strict since $u_0 \in \mathcal{H}$ cannot be the extremal of the inequality

$$(4/T)\|y\|_\infty^2 \le \|\dot{y}\|_2^2$$

which (up to multiplication by constants) is the linear spline

$$y = u_\infty(t) = \begin{cases} (2/T)t, & t \in [0, T/2], \\ -(2/T)t + 2, & t \in [T/2, T], \end{cases}$$

It then follows from (4) and from (14) that $\lambda_1^D(q) > 0$. Similarly, $\underline{\lambda}_1(q) > 0$ in this case.

Finally, we prove the last statement in Theorem 1. Suppose that $1 < \alpha \le \infty$. Let

$$u(t) = u_{2\alpha^*}(t) \in \mathcal{H}$$

and

$$q(t) = q^\eta(t) := -\eta u_{2\alpha^*}^{2\alpha^*/\alpha}(t),$$

where $u_\beta(t)$ is given by (9) and η is a positive parameter. By Proposition 3, one has

$$\|u\|_{2\alpha^*}^2 = \frac{1}{K(2\alpha^*)}\|\dot{u}\|_2^2.$$

It is easy to check that

$$\|q_-\|_\alpha = \eta\|u\|_{2\alpha^*}^{2\alpha^*/\alpha}$$

and

$$\int_0^T q(t)u^2 dt = -\eta\|u\|_{2\alpha^*}^{2\alpha^*} = -\|q_-\|_\alpha\|u\|_{2\alpha^*}^2.$$

Thus

$$Q(u) = \int_0^T (\dot{u}^2 + q(t)u^2)dt = (1 - \|q_-\|_\alpha/K(2\alpha^*))\|\dot{u}\|_2^2. \tag{15}$$

Let $\eta = \eta_0$ be such that $\theta = 1 - \|q_-\|_\alpha/K(2\alpha^*) = 0$, i.e.,

$$\eta_0 = K(2\alpha^*)/\|u_{2\alpha^*}\|_{2\alpha^*}^{2\alpha^*/\alpha}.$$

For this $q = q^{\eta_0}$, (13) implies that $\underline{\lambda}_1(q) \ge 0$. On the other hand, it follows from (15) that

$$\underline{\lambda}_1(q) \le \lambda_1^D(q) = \inf_{y \in \mathcal{H}\setminus\{0\}} \frac{Q(y)}{\|y\|_2^2} \le 0.$$

Thus, $\underline{\lambda}_1(q) = 0$ in this case. Consequently, if one takes $\eta > 0$ a little bit bigger than η_0, then $q = q^\eta$ does not satisfy (3), i.e., $\|q_-\|_\alpha > K(2\alpha^*)$. It follows again from (15) that $\underline{\lambda}_1(q) < 0$. This means that 0 is out of the stable interval $(\overline{\lambda}_0(q),\ \underline{\lambda}_1(q))$ and (1) will be unstable. □

Assume that $q(t) = -w(t)$ in (1), where $w(t)$ is T-periodic and satisfies $w(t) \ge 0$ for a.e. t, $w(t) > 0$ on a subset of positive measure. (We write this as $w \succ 0$.) Instead of making use of eigenvalues of (5), one may consider the weighted (anti-) periodic eigenvalues of

$$-\ddot{x} = \lambda w(t)x \tag{16}$$

subject to (P) or (A). Let us use $\{\overline{\mu}_n(w) : n \in \mathbb{Z}^+\}$ and $\{\underline{\mu}_n(w) : n \in \mathbb{N}\}$ to denote the complete sequence of all weighted periodic and anti-periodic eigenvalues of (16). Then $\overline{\mu}_0(w) = 0$. Like the estimate (13), we have the following lower bound on the first weighted anti-periodic eigenvalue.

Theorem 5. *Assume that $w \succ 0$. If $w \in L^\alpha(0,T)$ for some $1 \le \alpha \le \infty$, then*

$$\underline{\mu}_1(w) \ge \frac{K(2\alpha^*)}{\|w\|_\alpha} \qquad \text{if } 1 < \alpha \le \infty, \tag{17}$$

$$\underline{\mu}_1(w) > \frac{K(\infty)}{\|w\|_1} \qquad \text{if } \alpha = 1. \tag{18}$$

Since equation (1) is stable if $1 \in (0, \underline{\mu}_1(w))$, one can use (17) and (18) to obtain the same stability conditions (3) and (4) when $q = -w$, $w \succ 0$.

3. An Example

In this section we give an example to illustrate our stability criterion.

Consider the first stability region of Mathieu's equation

$$\ddot{x} + \lambda(1 + \varepsilon \cos t)x = 0, \tag{19}$$

where $\varepsilon \in [-1, 1]$ and $\lambda > 0$. Let $w^\varepsilon(t) = 1 + \varepsilon \cos t$, $T = 2\pi$. The first stability region of (19) is

$$S_1 := \left\{(\lambda, \varepsilon) : 0 < \lambda < \underline{\mu}_1(w^\varepsilon)\right\}.$$

Let us approximate S_1 by several stability criteria. The classical stability result (4) yields the stability of (19) only when

$$0 < \lambda \le H_0(\varepsilon) := \frac{4}{2\pi\|w^\varepsilon\|_1} = \frac{1}{\pi^2} \approx 0.10132 \tag{20}$$

for all $\varepsilon \in [-1, 1]$. Such a result is not satisfactory because when $\varepsilon = 0$, for example, the first stability interval is $0 < \lambda < 1/4$.

Now we use the criterion (3) by choosing α depending upon ε. Then equation (19) is stable when λ satisfies

$$0 < \lambda < \frac{K(2\alpha^*)}{\|w^\varepsilon\|_\alpha} =: H_1(\varepsilon, \alpha)$$

for *some* $\alpha \in [1, +\infty]$. Thus (19) is stable when

$$0 < \lambda < H_1(\varepsilon) := \sup_{1 \le \alpha \le \infty} H_1(\varepsilon, \alpha). \tag{21}$$

A numerical evaluation shows that (21) is a very good approximation to S_1 for all $\varepsilon \in [-1, 1]$. See Figure 1.

Let us recall a lower bound result in [11] on the first weighted Dirichlet eigenvalue $\mu_1^D(w)$ of (16)+(D). This is based essentially on the L^2 norm of the primitive $W(t) = \int^t w(t)dt$ and is proved using Opial's inequality in [1, 2].

Theorem 6 ([11, Theorem 4.4]). *Let $w \succ 0$ and $W(t)$ be a primitive of $w(t)$. Define*

$$\kappa_1(\nu) = \left(2\int_0^{T/2} t\,(W(t)-\nu)^2 dt\right)^{1/2},$$

$$\kappa_2(\nu) = \left(2\int_{T/2}^{T} (T-t)(W(t)-\nu)^2 dt\right)^{1/2}.$$

Then the first weighted Dirichlet eigenvalue has the following lower bound:

$$\mu_1^D(w) \geq \frac{1}{\min_{\nu\in\mathbb{R}} \max\{\kappa_1(\nu),\ \kappa_2(\nu)\}}.$$

Now we apply Theorem 6 to derive another stability result of (19). Note that $w_s^\varepsilon(t) = 1+\varepsilon\cos(t+s)$. Take a primitive of $w^\varepsilon(t)$ as $W_s^\varepsilon(t) = t+\varepsilon\sin(t+s)$. Then

$$\begin{aligned}
\kappa_1^2(\nu) = &\left[\frac{\pi^4+\pi^2\varepsilon^2}{2} - 4\varepsilon(4-\pi^2)\cos s - 8\pi\varepsilon\sin s - \frac{\pi\varepsilon^2\sin 2s}{2}\right]\\
&+\left[-\frac{4\pi^3}{3} - 4\pi\varepsilon\cos s + 8\varepsilon\sin s\right]\nu + \pi^2\nu^2,\\
\kappa_2^2(\nu) = &\left[\frac{11\pi^4+3\pi^2\varepsilon^2}{6} - 4\varepsilon(4+\pi^2)\cos s - 8\pi\varepsilon\sin s + \frac{\pi\varepsilon^2\sin 2s}{2}\right]\\
&+\left[-\frac{8\pi^3}{3} + 4\pi\varepsilon\cos s + 8\varepsilon\sin s\right]\nu + \pi^2\nu^2.
\end{aligned}$$

Let

$$\nu_0 = \frac{4\pi^3 - 24\pi\varepsilon\cos s + 3\varepsilon^2\sin 2s}{4\pi^2 - 24\varepsilon\cos s}.$$

By Theorem 6, we have the following lower bound on $\mu_1^D(w_s^\varepsilon)$:

$$\begin{aligned}
\mu_1^D(w_s^\varepsilon) \geq &\frac{1}{\min_{\nu\in\mathbb{R}} \max\{\kappa_1(\nu),\ \kappa_2(\nu)\}} = \frac{1}{\kappa_1(\nu_0)}\\
&= 4\sqrt{3}\,(\pi^2-6\varepsilon\cos s)\Big/\Big[(8\pi^8+24\pi^6\varepsilon^2) - 96\pi^4\varepsilon(8+\pi^2+3\varepsilon^2)\cos s\\
&\quad +288\pi^2\varepsilon^2(32+\pi^2+3\varepsilon^2)\cos^2 s - 27648\varepsilon^3\cos^3 s\\
&\quad +288\pi^2\varepsilon^3\sin s\sin 2s - 27\varepsilon^4(32-\pi^2)\sin^2 2s\Big]^{1/2} =: H_2(\varepsilon, s).
\end{aligned}$$

Hence the first weighted anti-periodic has the following lower bound:

$$\begin{aligned}
\underline{\mu}_1(w^\varepsilon) &= \min_s \mu_1^D(w_s^\varepsilon) \geq \min_s H_2(\varepsilon, s)\\
&= \left(\frac{6}{\pi^4+96|\varepsilon|+3\pi^2\varepsilon^2}\right)^{1/2} =: H_2(\varepsilon).
\end{aligned}$$

By Theorem 5, equation (19) is stable if

$$0 < \lambda < H_2(\varepsilon), \qquad \varepsilon\in[-1,1]. \tag{22}$$

This estimate also strongly approximates S_1. For example, if $\varepsilon = 0$, then $H_2(0) \doteq 0.2481 \approx 1/4$.

Using the Sobolev constants in (8), one can numerically evaluate the function $H_1(\varepsilon)$. In Figure 1, we have plotted, from the left to right, the curves $\lambda = H_0(\varepsilon) \equiv$

$1/\pi^2$, $\lambda = H_2(\varepsilon)$, $\lambda = H_1(\varepsilon)$, and $\lambda = \underline{\mu}_1(w^\varepsilon)$. It can be seen that the estimates given in (21) and (22) are almost the same as the first stability region S_1.

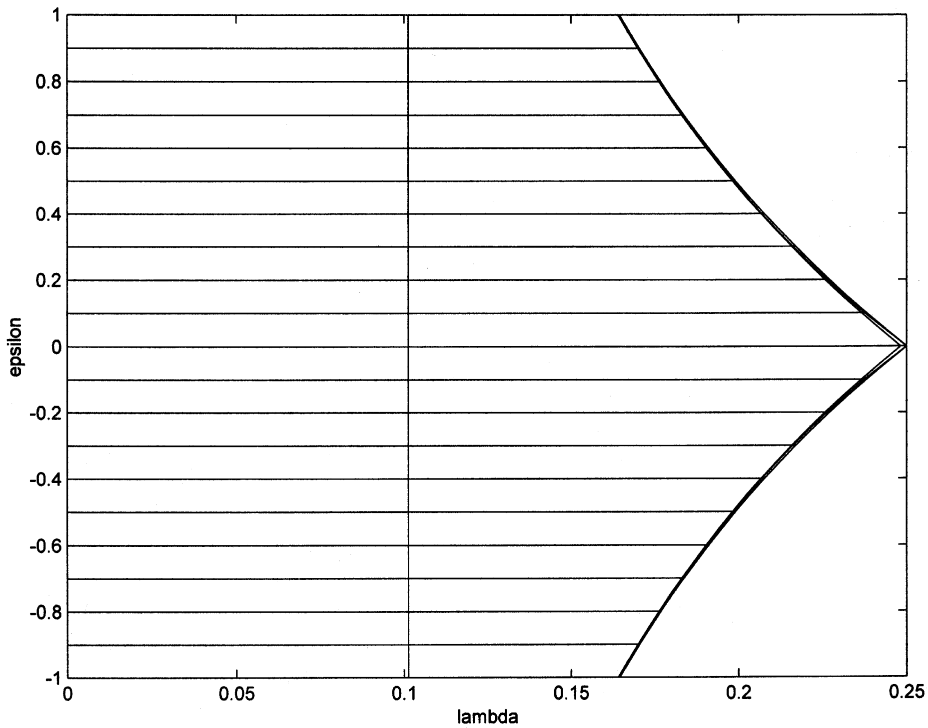

FIGURE 1. The first stability region of (19).

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