

## A mapping of Riemannian manifolds which preserves harmonic functions

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In the theory of harmonic spaces, C. Constantinescu and A. Cornea [3] introduced the concept of harmonic mapping between harmonic spaces as a generalization of analytic mapping between Riemannian surfaces. A harmonic space is a locally compact Hausdorff space on which it is given a sheaf of continuous functions, called harmonic functions, satisfying Brelot's axioms [2], and roughly speaking a harmonic mapping is a mapping of harmonic spaces which preserves harmonic functions.

As a Riemannian manifold with usual harmonic functions is a harmonic space, we can define a harmonic mapping of Riemannian manifolds. H. Imai [10] studied the value distribution of harmonic mappings between Riemannian manifolds of the same dimension. The main purpose of this article is to give a characterization of harmonic mappings between Riemannian manifolds of the same dimension or different ones from the differential geometric point of view.

Later on in the paper, a harmonic mapping of C. Constantinescu and A. Cornea will be called a mapping which preserves harmonic functions, for we would like to use the term "harmonic mapping" in the sense of Eells and Sampson [4], and the concept of harmonic mapping in this sense plays an important role in the present paper.

Let  $\Delta_M$  (resp.  $\Delta_N$ ) be the Laplacian on a Riemannian manifold  $M$  (resp.  $N$ ). S. Helgason [9] proved that a diffeomorphism  $f: M \rightarrow N$  is an isometry iff it is a Laplacian commuting mapping, i.e.,  $f^* \Delta_N = \Delta_M f^*$ . B. Watson [13] extended this result to the following:  $f: M \rightarrow N$  is a Laplacian commuting mapping iff it is a harmonic Riemannian submersion. S. I. Goldberg and T. Ishihara [8] gave a more generalization. Thus a harmonic Riemannian submersion is a mapping which preserves harmonic functions. Moreover it is evident that a Laplacian commuting mapping preserves not only harmonic functions but also all of eigenspaces of the Laplacian. Hence it may be conjectured that a set of mappings which preserve harmonic functions is much larger than that of Laplacian commuting mappings, but our main result (Theorem 5.1) asserts it is not so. In fact  $f: M \rightarrow N$  is a mapping which preserves harmonic functions iff it is a constant mapping or its restriction to the open

submanifold  $M' = \{p \in M, (f_*)_p \neq 0\}$  of  $M$  is a Riemannian submersion after some conformal change of the natural Riemannian metric on  $M'$ .

In § 1, we will review briefly the basic facts on harmonic mappings and Riemannian submersions for the later use. Our problems are described explicitly in § 2. It will be proved in § 3 that a mapping preserves convex functions iff it is totally geodesic (Theorem 3.2). To prove this we have to construct local convex functions satisfying certain conditions (Lemma 3.1). The lemma leads us to the following characterization of harmonic mapping:  $f: M \rightarrow N$  is a harmonic mapping iff  $f^*$  maps every local convex function in  $N$  into a local subharmonic function in  $M$ . In order to deal with a mapping which preserves harmonic functions, we need the fact that there are sufficiently many local harmonic functions, which is shown in § 4 as a generalization of Lemma 5.1 in L. Bers [1]. The last section is devoted to the proof of our main results, Theorems 5.1, 5.2 and 5.7.

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## § 1. Harmonic mappings and Riemannian submersions

Let  $M$  and  $N$  be smooth Riemannian manifolds of dimension  $m$  and  $n$  respectively. The Riemannian metrics of  $M$  and  $N$  are denoted by  $ds_M^2$  and  $ds_N^2$  respectively which are written locally as

$$(1.1) \quad ds_M^2 = \omega_1^2 + \cdots + \omega_m^2, \quad ds_N^2 = \omega_1^{*2} + \cdots + \omega_n^{*2},$$

where  $\omega_a (a=1, \dots, m)$  are local 1-forms in  $M$  and  $\omega_i^* (i=1, \dots, n)$  are local 1-forms in  $N$ . The structure equations in  $M$  are

$$(1.2) \quad \begin{cases} d\omega_a = \sum_{b=1}^m \omega_b \wedge \omega_{ba}, \\ d\omega_{ab} = \sum_{c=1}^m \omega_{ac} \wedge \omega_{cb} - \frac{1}{2} \sum_{c,d} R_{abcd} \omega_c \wedge \omega_d, \end{cases}$$

where the  $\omega_{ab}$  are the components of the connection form of the Riemannian metric  $ds_M^2$  and the  $R_{abcd}$  are the components of its curvature tensor. Similar equations are valid in  $N$  and we will denote the corresponding quantities in the same notations with asterisks. (Throughout the paper, indices  $a, b, c, \dots$  run from 1 to  $m$ , and  $i, j, k, \dots$  from 1 to  $n$ ).

Let  $f: M \rightarrow N$  be a smooth mapping of  $M$  into  $N$  and

$$(1.3) \quad f^*(\omega_i^*) = \sum_{a=1}^m f_{ia} \omega_a, \quad i=1, \dots, n.$$

Let  $f^{-1}T(N)$  denote the bundle induced by  $f$  from the tangent bundle  $T(N)$  over  $N$ . The differential  $f_*$  of  $f$  is regarded as a  $f^{-1}T(N)$ -valued 1-form on  $M$ . The bundle  $f^{-1}T(N)$  has the covariant differential operator compatible with the metric deduced

naturally from the Riemannian metric  $ds_N^2$ . The components  $f_{iab}$  of the covariant differential of  $f_*$  are given by

$$(1.4) \quad \sum_{b=1}^m f_{iab} \omega_b = df_{ia} + \sum_{j=1}^n f_{ja} f^* \omega^*_{ji} + \sum_{b=1}^m f_{ib} \omega_{ba}.$$

From (1.2) and (1.3), it follows that  $f_{iab} = f_{iba}$ . The mapping  $f$  is called *harmonic* (resp. *totally geodesic*) iff  $\sum_{a=1}^m f_{iaa} = 0$  (resp.  $f_{iab} = 0$ ) [6].

In the sequel of this section, we assume  $n \leq m$ . A mapping  $f: M \rightarrow N$  is said to be a *Riemannian submersion* if it satisfies

$$(1.5) \quad \sum_{a=1}^m f_{ia} f_{ja} = \delta_{ij}.$$

Although a Riemannian submersion is sometimes assumed to be surjective [13], it is not in the paper.

It is known that  $f: M \rightarrow N$  is a Riemannian submersion iff we can choose local forms  $\omega_a$  in  $M$  and  $\omega_i^*$  in  $N$  in (1.1) such that the equations (1.3) are reduced to

$$(1.6) \quad f^* \omega_i^* = \omega_i, \quad i = 1, \dots, n,$$

that is,  $f_{ia} = \delta_{ia}$  [8].

When  $f: M \rightarrow N$  is a Riemannian submersion, we always take local 1-forms  $\omega_a$  and  $\omega_i^*$  satisfying (1.6), which we call canonical bases of the Riemannian submersion. We denote the  $f_{iab}$  with respect to canonical bases by  $F_{iab}$ . Then they satisfy

$$(1.7) \quad \omega_{ij} - f^* \omega_{ji}^* = \sum_{a=1}^m F_{ija} \omega_a, \quad \omega_{i\alpha} = \sum_{a=1}^m F_{i\alpha a} \omega_a,$$

and are called *the components of the structure tensor of the Riemannian submersion*  $f: M \rightarrow N$ . (In the paper, the indices  $\alpha, \beta, \dots$  run from  $n+1$  to  $m$  if  $n < m$ . On the other hand for  $m = n$ , terms with indices  $\alpha, \beta, \dots$ , for example  $F_{ija}, \omega_{i\alpha}$ , vanish.) From (1.7) it follows

$$(1.8) \quad F_{ijk} = 0, \quad F_{ij\alpha} = -F_{j\alpha i}.$$

For any point of  $f(M)$ , its inverse image by  $f$  is said to be a fibre of  $f$ . From the property of canonical bases, it follows that  $\omega_1 = \dots = \omega_n = 0$  on each fibre. Let  $p$  be a point of  $f(M)$ . Then the fibre  $f^{-1}(p)$  is a closed submanifold of  $M$  of dimensions  $m - n$ , when  $m > n$ . The restriction of  $\sum_{\alpha=n+1}^m \omega_\alpha^2$  to the fibre gives the induced Riemannian metric. The restriction of the  $F_{i\alpha\beta}$  to the fibre may be regarded as the components of the second fundamental forms of the submanifold  $f^{-1}(p)$ . Hence, when  $m > n$ , a Riemannian submersion  $f: M \rightarrow N$  is said to be *minimal* (resp. *totally geodesic*) if  $\sum_{\alpha=n+1}^m F_{i\alpha\alpha} = 0$  (resp.  $F_{i\alpha\beta} = 0$ ). Since  $F_{ijk} = 0$ , it holds  $\sum_{\alpha=n+1}^m F_{i\alpha\alpha} = \sum_{q=1}^m F_{i\alpha\alpha}$ . Therefore a Riemannian submersion is harmonic iff it is minimal. *The horizontal distribution*, which is defined by  $\omega_{n+1} = \dots = \omega_m = 0$ , is integrable if  $F_{i\alpha\alpha} = 0$ .

**2. Harmonic, subharmonic and convex functions**

Let  $h$  be a smooth function on  $N$ . Put

$$(2.1) \quad \begin{cases} dh = \sum_{i=1}^n h_i \omega_i^*, \\ \sum_{j=1}^n h_{i,j} \omega_j^* = dh_i + \sum_{j=1}^n h_j \omega_j^*, \end{cases}$$

that is,  $h_i$  (resp.  $h_{i,j}$ ) are the components of the first (resp. second) covariant derivative of the function  $h$ . The Laplacian on the functions on  $N$  is defined to be

$$(2.2) \quad \Delta_N h = \sum_{i=1}^n h_{i,i}.$$

$\Delta_M$  also denotes the Laplacian on  $M$ . A function  $h$  is said to be harmonic (resp. subharmonic) if  $\Delta_N h = 0$  (resp.  $\Delta_N h \geq 0$ ).

A function  $h$  on  $N$  is called (geodesically) convex if for every geodesic  $C(t)$  parametrized by arc-length and defined for all  $t \in [t_1, t_2]$ , it holds

$$(2.3) \quad h(C(\lambda t_1 + (1 - \lambda)t_2)) \leq \lambda h(C(t_1)) + (1 - \lambda)h(C(t_2))$$

for all  $\lambda \in [0, 1]$ . If the right hand side of (2.3) is always greater than the left,  $h$  is called *strictly convex*. It is well known that a  $C^2$ -function  $h$  is convex (resp. strictly convex) iff the second covariant derivative  $(h_{i,j})$  is non-negative (resp. positive) definite. All of the functions in the paper are smooth.

Let  $h$  be a local function in  $N$ , that is, a function on an open subset  $U$  of  $N$ . We call  $h$  a *local harmonic function* iff it is harmonic on the open Riemannian submanifold  $U$  of  $N$ . Similarly we can define *local subharmonic functions and local convex functions*.

As it is described in the preface, we would like to study mappings which preserves harmonic functions. Hence we define the set

$$\Omega_H(M, N) = \left\{ \begin{array}{l} f: M \rightarrow N \mid f \text{ is a smooth mapping such that } f^*h \text{ is} \\ \text{a local harmonic function in } M \text{ for every local} \\ \text{harmonic function in } N. \end{array} \right\}$$

Similarly,  $\Omega_{sH}(M, N)$ ,  $\Omega_C(M, N)$  and  $\Omega_{sC}(M, N)$  denote the sets of mappings which preserve subharmonic, convex and strictly convex functions, respectively.

Let  $f: M \rightarrow N$  be a smooth mapping. The components of  $f^*$  are  $f_{i\alpha}$  as in (1.3). For a local function  $h$  on an open set  $U$  of  $N$ , we have on  $f^{-1}(U)$

$$(f^*h)_\alpha = \sum_{i=1}^n h_i f_{i\alpha},$$

where the  $h_i$  are the components of the local 1-form  $dh$ . It follows from (2.1)

$$\sum_{b=1}^m (f^*h)_{\alpha b} \omega_b = \sum_{i=1}^n f_{i\alpha} \left( dh_i + \sum_{j=1}^n h_j f^* \omega_j^* \right) + \sum_{i=1}^n h_i \left( df_{i\alpha} + \sum_{j=1}^n f_{j\alpha} f^* \omega_j^* + \sum_{b=1}^m f_{i b} \omega_{b\alpha} \right).$$

Thus using (1.4), we obtain the following fundamental relations

**Lemma 2.1.** *Let  $f: M \rightarrow N$  be a smooth mapping. For any function  $h$  on an open set of  $N$ ,*

$$(2.4) \quad (fh)_{ab} = \sum_{i,j=1}^n h_{ij} f_{ia} f_{jb} + \sum_{i=1}^n h_i f_{iab},$$

$$(2.5) \quad \Delta_M f^* h = \sum_{i,k=1}^n \sum_{a=1}^m h_{ij} f_{ia} f_{ja} + \sum_{i=1}^n \sum_{a=1}^m h_i f_{iaa}.$$

If  $f: M \rightarrow N$  is a harmonic Riemannian submersion, that is,  $\sum_{a=1}^m f_{iaa} = 0$  and  $\sum_{a=1}^m f_{ia} f_{ja} = \delta_{ij}$ , (2.5) is reduced to  $\Delta_M f^* h = f^* \Delta_N h$ . Moreover it is known that  $\Delta_M f^* = f^* \Delta_N$  iff  $f$  is a harmonic Riemannian submersion [9]. Hence we get

**Proposition 2.2.** *If  $f: M \rightarrow N$  is a harmonic Riemannian submersion, it is contained in  $\Omega_H(M, N)$ .*

### § 3. Mappings which preserve convex functions

Using Lemma 2.1, we will investigate the set  $\Omega_c(M, N)$  and  $\Omega_{sc}(M, N)$  in the section. We need to prove there are sufficiently many local convex functions. In fact we have

**Lemma 3.1.** *Let  $C_i$  and  $C_{ij}$  be any constants such that  $(C_{ij})$  is symmetric and positive definite. For any point  $q$  of  $N$ , there exists a strictly convex function  $h$  defined near  $q$  whose covariant derivatives satisfy*

$$(3.1) \quad h_i(q) = C_i, \quad h_{ij}(q) = C_{ij}.$$

*Proof.* Fix an arbitrary point  $q \in N$ . We use a normal coordinate system  $\{U, y^1, \dots, y^n\}$  about  $q$ . Let  $h$  be any smooth function defined near  $q$ . Put

$$(3.2) \quad \tilde{h}_i = \frac{\partial h}{\partial y^i}, \quad \tilde{h}_{ij} = \frac{\partial^2 h}{\partial y^i \partial y^j} - \sum_{k=1}^n \frac{\partial h}{\partial y^k} \Gamma_{ij}^k,$$

where  $\Gamma_{ij}^k$  are the components of the Riemannian connection on  $N$  with respect to  $\{U, y^i\}$ . We will show that for any constants  $C_i, C_{ij}$  with  $(C_{ij}) > 0$ , there exists a strictly convex function  $h$  defined near  $q$  and satisfying

$$(3.3) \quad \tilde{h}_i(q) = C_i, \quad \tilde{h}_{ij}(q) = C_{ij}.$$

Now there is a neighborhood  $V$  of  $q$  contained in  $U$ , where  $\Gamma_{jk}^i$  are expanded in the following way (for example, see [11]);

$$(3.4) \quad \Gamma_{jk}^i = \sum_{l=1}^n A_{jkl}^i y^l + \sum_{l,h=1}^n A_{jklh}^i y^l y^h,$$

where  $A_{jkl}^i$  are constants and  $A_{jklh}^i$  smooth functions on  $V$ . Defined a function  $h$  on  $V$  by

$$h = \sum_{i=1}^n C_i y^i + \frac{1}{2} \sum_{i,j=1}^n C_{ij} y^i y^j.$$

It is easy to check that the function  $h$  satisfies (3.3). Using (3.2) and (3.4), we may expand  $\tilde{h}_{ij}$  as follows

$$\tilde{h}_{ij} = C_{ij} - \sum_{l,k=1}^n C_k A_{ijl}^k y^l + \frac{1}{2} \sum_{k,l=1}^n B_{ijkl} y^k y^l,$$

where  $B_{ijkl}$  are smooth functions on  $V$ . Since  $(C_{ij})$  is positive definite, it is evident that  $(\tilde{h}_{ij})$  is positive in some neighborhood of  $q$ .

**Theorem 3.2.**  *$f$  is an element of  $\Omega_c(M, N)$  iff it is a totally geodesic mapping.*

*Proof.* Assume that  $f$  is totally geodesic. For any local smooth function  $h$ , it follows from (2.4)

$$(f^*h)_{ab} = \sum_{i,j=1}^n h_{ij} f_{ia} f_{jb}.$$

If  $h$  is convex, that is,  $(h_{ij})$  is non-negative,  $((f^*h)_{ab})$  is non-negative.

Conversely, let  $f \in \Omega_c(M, N)$ . Suppose for some point  $p$  of  $M$  and some integer  $i_0 (1 \leq i_0 \leq n)$ , the matrix  $(f_{i_0 ab}(p))$  is non-zero. Then for some ordered  $m$ -tuple  $X = \{X^1, \dots, X^m\}$  of real numbers,  $\sum_{a,b=1}^m f_{i_0 ab} X^a X^b$  is non-zero. Put

$$\lambda = \sum_{a,b=1}^m f_{i_0 ab} X^a X^b, \quad \mu = \sum_{i=1}^n \left( \sum_{a=1}^m f_{ia} X^a \right)^2.$$

Applying Lemma 3.1 in the case  $C_{i_0} = -(\mu + 1)/\lambda$ ,  $C_i = 0 (i \neq i_0)$ ,  $C_{ij} = \delta_{ij}$ , we get a strictly convex function  $h$  defined near  $p$  and satisfying

$$(f^*h)_{ab}(p) = \sum_{i=1}^n f_{ia}(p) f_{ib}(p) - \frac{\mu + 1}{\lambda} f_{i_0 ab}(p).$$

Hence we have

$$\sum_{a,b=1}^m (f^*h)_{ab}(p) X^a X^b = -1.$$

Thus  $((f^*h)_{ab})$  is not non-negative. But this is contrary to the fact that  $f \in \Omega_c(M, N)$  and  $h$  is strictly convex. Therefore we proved  $f_{i_0 ab} = 0$  on  $M$ .

**Theorem 3.3.** *When  $m > n$ ,  $\Omega_{sc}(M, N) = \emptyset$ . When  $m \leq n$ , a mapping  $f$  is contained in  $\Omega_{sc}(M, N)$  iff it is a totally geodesic immersion.*

*Proof.* If  $f$  is a totally geodesic immersion, it is easy to prove that for any strictly convex function  $h$ ,  $((f^*h)_{ab}) = (\sum_{i,j=1}^n h_{ij} f_{ia} f_{jb})$  is positive.

We will prove the converse. Applying Lemma 3.1 when  $C_i = 0$  and  $C_{ij} = \delta_{ij}$ , for any point  $p$  of  $M$ , we have a local strictly convex function  $h$  such that  $(f^*h)_{ab}(p) = \sum_{i=1}^n f_{ia}(p) f_{ib}(p)$ . As  $((f^*h)_{ab}(p))$  is positive,  $(f_*)_p$  should be injective. This is

true at any point of  $M$ . Hence if  $m > n$ ,  $\Omega_{sc}(M, N) = \emptyset$ , and if  $m \leq n$ ,  $f$  is an immersion. On the other hand, we have already proved in the proof of Theorem 3.2 that  $f_{iab} = 0$  on  $M$ .

Another application of Lemma 3.1 is given in the following theorem which is a characterization of harmonic mappings.

**Theorem 3.4.**  $f: M \rightarrow N$  is a harmonic mapping iff for any open subset  $U$  of  $N$ , it maps any convex function defined on  $U$  into a subharmonic function on  $f^{-1}(U)$ .

*Proof.* If  $f$  is harmonic, for any convex function  $h$  on an open set  $U$ , it holds  $\Delta_M f^* h = \sum_{i,j,a} h_{ij} f_{ia} f_{ja}$ . Thus we get  $\Delta_M f^* h \geq 0$  on  $f^{-1}(U)$ , for  $(h_{ij})$  is non-negative.

The converse is given without difficulty. If for some point  $p \in M$  and some  $i_0 (1 \leq i_0 \leq n)$ ,  $\sum_{a=1}^m f_{i_0 a a}(p)$  is non-zero, we put

$$\lambda = \sum_{a=1}^m f_{i_0 a a}(p), \quad \mu = \sum_{i=1}^n \sum_{a=1}^m (f_{ia})^2.$$

We use Lemma 3.1 when  $C_{i_0} = -(\mu + 1)/\lambda$ ,  $C_i = 0 (i \neq i_0)$ ,  $C_{ij} = \delta_{ij}$ , and get a strictly convex function  $h$  such that

$$(\Delta_M f^* h)(p) = \sum_{i=1}^n \sum_{a=1}^m (f_{ia})^2 - \frac{\mu + 1}{\lambda} \sum_{a=1}^m f_{i_0 a a}(p) = -1.$$

Thus we proved  $\sum_{a=1}^m f_{i_0 a a}(p) = 0$  for any  $i$  and any point  $p \in M$ .

#### § 4. The existence of local harmonic functions

In order to study  $\Omega_H(M, N)$  we need the following

**Lemma 4.1.** For any point  $q \in N$  and any constants  $C_i, C_{ij}$  with  $C_{ij} = C_{ji}$  and  $\sum_{i=1}^n C_{ii} = 0$ , there exists a harmonic function  $h$  on a neighborhood of  $q$  which satisfies

$$h_i(q) = C_i, \quad h_{ij}(q) = C_{ij}.$$

*Proof.* Take a normal coordinate system  $\{U, y^1, \dots, y^n\}$  about  $q$  such that  $y^1(q) = \dots = y^n(q) = 0$ . A function  $h$  on  $U$  is harmonic iff

$$(4.1) \quad \sum_{i,j} g^{ij} \left( \frac{\partial^2 h}{\partial y^i \partial y^j} - \sum_{k=1}^n \frac{\partial h}{\partial y^k} \Gamma_{ij}^k \right) = 0,$$

where  $g^{ij}$  are the contravariant components of the Riemannian metric on  $N$  and  $\Gamma_{jk}^i$  are the components of the Riemannian connection with respect to the normal coordinate  $\{y^i\}$ . We will prove the following; For any constants  $C_i, C_{ij}$  satisfying  $C_{ij} = C_{ji}$  and  $\sum_{i=1}^n C_{ii} = 0$ , there is a solution  $h$  of the equation (4.1) which is defined near  $y = 0$  and satisfies

$$\frac{\partial h}{\partial y^i}(0) = C_i, \quad \frac{\partial^2 h}{\partial y^i \partial y^j}(0) = C_{ij}.$$

But this is true because for  $n \geq 2$ , it is a special case of Lemma 4.2 given below, and for  $n = 1$ , the equation (4.1) is an ordinary differential equation.

Before describing Lemma 4.2, we must make some preparations. From now on in the section, we use the same notations as in [1]. Compare it for details. Let  $x = (x_1, \dots, x_n)$  be a point of Euclidean  $n$ -space ( $n \geq 2$ ). Let  $\phi(x)$  be a function defined for  $|x| < R$ . We set, for  $0 < \alpha < 1$ ,

$$H_\alpha^R(\phi) = \text{l.u.b.}_{|x'|, |x''| < R} \{|\phi(x') - \phi(x'')| |x' - x''|^\alpha\},$$

$$\|D^\nu \phi\|_0^R = \text{maximum}_{i_1 + \dots + i_n = \nu} \left\{ \text{l.u.b.}_{|x| < R} \left| \frac{\partial^\nu \phi(x)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \right| \right\},$$

if the derivatives exist. For an integer  $l \geq 0$  and for  $0 < \alpha < 1$ , we set

$$\|\phi\|_{l+\alpha}^R = \sum_{\nu=0}^l \frac{R^\nu}{\nu!} \|D^\nu \phi\|_0^R + \frac{R^{l+\alpha}}{l!(l+\alpha)} \text{maximum}_{i_1 + \dots + i_n = l} \left\{ H_\alpha^R \left( \frac{\partial^l \phi}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \right) \right\}.$$

Let  $B_{l+\alpha}^{(R)}$  denote the set of functions  $\phi$  for which the norm  $\|\phi\|_{l+\alpha}^R$  is defined and finite.  $B_{l+\alpha}^{(R)}$  is a banach space with respect to the norm.

Let

$$(4.2) \quad L\phi(x) = \sum_{\nu=0}^l \sum_{i_1 + \dots + i_n = \nu} a_{i_1, \dots, i_n} \frac{\partial^\nu \phi}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = 0$$

be a linear partial differential equation of order  $l (\geq 2)$  and of elliptic type which is defined near the origin  $x=0$ . Now we can give a slight extension of Lemma 5.1 in [1].

**Lemma 4.2.** *Assume that the coefficients in (4.2) belong to  $B_\alpha^{(R_0)}$  for some  $\alpha (0 < \alpha < 1)$  and  $R_0 (> 0)$ . For any constants  $C_{i_1, \dots, i_n} (0 \leq i_1 + \dots + i_n \leq l)$  which are symmetric in  $i_1, \dots, i_n$  and satisfy*

$$(4.3) \quad \sum_{0 \leq i_1 + \dots + i_n \leq l} a_{i_1, \dots, i_n}(0) C_{i_1, \dots, i_n} = 0,$$

there exists a solution  $\phi$  of (4.2) defined near the origin and satisfying

$$(4.4) \quad \frac{\partial^\nu \phi}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(0) = C_{i_1, \dots, i_n}, \quad 0 \leq i_1 + \dots + i_n \leq l.$$

*Proof.* We will follow substantially the method used for the proof of Lemma 5.1 by L. Bers. Let  $\lambda(x)$  be a smooth function such that  $0 \leq \lambda(x) \leq 1$  for all  $x$ ,  $\lambda(x) = 1$  for  $|x| \leq 1/4$ ,  $\lambda(x) = 0$  for  $|x| \geq 1/2$ . For  $0 < R \leq R_0$  consider the linear mapping  $\psi = T\phi$  of  $B_{l+\alpha}^{(R)}$  into itself defined by the relation



$$\psi(x) = - \int_{|\xi| < R} J(x - \xi) \lambda(\xi/R) [L_0 \phi(\xi) - L\phi(\xi)] d\xi,$$

where  $J(x)$  is the fundamental solution of the osculating equation

$$(4.5) \quad L_0 \phi(x) = \sum_{i_1 + \dots + i_n = l} a_{i_1 \dots i_n}(0) \frac{\partial^l \phi}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = 0.$$

From the fact  $a_{i_1 \dots i_n} \in B_{\alpha}^{(R)}$ , it follows that there is a constant  $A_1$ , such that

$$(4.6) \quad \|T\| < A_1 R^{\alpha},$$

where  $\|T\| = \sup \|T\phi\|_{l+\alpha}^R / \|\phi\|_{l+\alpha}^R$ . Let  $R$  be so small that  $\|T\| < 1$ . For every  $h(x) \in B_{l+\alpha}^{(R)}$  the equation

$$(4.7) \quad \phi = h - T\phi$$

has a unique solution  $\phi_h = h - Th + T^2h - \dots + (-1)^m T^m h + \dots$ . The property of the fundamental solution  $J(x)$  of the elliptic equation  $L_0 \phi(x) = 0$  implies

$$L_0 \phi_h(x) = L_0 h(x) + \lambda(x/R) [L_0 \phi(x) - L\phi(x)].$$

Hence if  $L_0 h(x) = 0$ , it holds

$$L\phi_h(x) = 0, \quad |x| < R/4.$$

It follows from (4.5) that

$$(4.8) \quad \|\phi_h - h\|_{l+\alpha}^R \leq \frac{\|T\| \|T\phi\|_{l+\alpha}^R}{1 - \|T\|}.$$

For  $0 \leq i_1 + \dots + i_n \leq l$ , it holds

$$(4.9) \quad \|x_1^{i_1} \dots x_n^{i_n}\|_{l+\alpha}^R \leq (2R)^{i_1 + \dots + i_n}.$$

Let  $V$  be the vector space generated by  $x_1^{i_1} \dots x_n^{i_n}$ ,  $0 \leq i_1 + \dots + i_n \leq l$  over the real number. The dimension of  $V$  is given by

$$d = 1 + \sum_{r=1}^l \frac{(n+1) \dots (n+r-1)}{r}.$$

Put

$$V_1 = \left\{ \sum_{0 \leq i_1 + \dots + i_n \leq l} C_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in V, \sum_{i_1 + \dots + i_n = l} a_{i_1 \dots i_n}(0) C_{i_1 \dots i_n} = 0 \right\},$$

$$V_2 = \left\{ \sum_{0 \leq i_1 + \dots + i_n \leq l} C_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in V, \sum_{0 \leq i_1 + \dots + i_n \leq l} a_{i_1 \dots i_n}(0) C_{i_1 \dots i_n} = 0 \right\}.$$

Then,  $V_1$  and  $V_2$  are  $(d-1)$ -dimensional vector subspaces of  $V$ .

Let  $h = \sum_{0 \leq i_1 + \dots + i_n \leq l} C_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$  be an elements of  $V_1$ . It is a solution of the equation (4.5). Hence the corresponding solution  $\phi_h$  of (4.7) is a solution of the

equation (4.2) for  $|x| < R/4$ . From (4.6) (4.8) and (4.9) we get

$$(4.10) \quad \|\phi_h(x) - h(x)\| < A_2 R^\alpha \sum_{0 \leq i_1 + \dots + i_n \leq l} |C_{i_1, \dots, i_n}| R^{i_1 + \dots + i_n},$$

where  $A_2$  is a positive constant.

Let  $\Phi: V_1 \rightarrow V_2$  be a linear mapping defined by

$$\Phi(h) = \sum_{0 \leq i_1 + \dots + i_n \leq l} \frac{\partial^\nu \phi_h}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(0) x_1^{i_1} \dots x_n^{i_n}.$$

Assume  $\Phi(h) = 0$ , i.e.,

$$\frac{\partial^\nu \phi_h}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(0) = 0, \quad 0 \leq i_1 + \dots + i_n \leq l.$$

Let  $A_3, A_4$  be some constants. From (4.10) we have for  $0 \leq j_1 + \dots + j_n \leq l$

$$\frac{R^{j_1 + \dots + j_n}}{(j_1 + \dots + j_n)!} j_1! \dots j_n! |C_{j_1, \dots, j_n}| \leq A_3 R^\alpha \sum_{0 \leq j_1 + \dots + j_n \leq l} |C_{i_1, \dots, i_n}| R^{i_1 + \dots + i_n}.$$

Combining this with (4.10), we get

$$\sum |C_{j_1, \dots, j_n}| R^{j_1 + \dots + j_n} \leq A_4 R^\alpha \sum |C_{i_1, \dots, i_n}| R^{i_1 + \dots + i_n}.$$

If  $R$  is so small that  $A_4 R^\alpha < 1$ , this implies that all of the  $C_{i_1, \dots, i_n}$  vanish. Thus we have shown that  $\Phi: V_1 \rightarrow V_2$  is bijective.

**§ 5. Mappings which preserve harmonic functions**

Let  $f$  belong to  $\mathcal{O}_H(M, N)$ . Firstly we will prove that it is a harmonic mapping. In fact, for any point  $p \in M$  and any integer  $i_0 (1 \leq i_0 \leq n)$ , Lemma 4.1 in the case where  $C_{i_0} = 1, C_i = 0 (i \neq i_0)$  and  $C_{ij} = 0$  assures that there is a harmonic function  $h$  defined near  $f(p)$  such that

$$\Delta_M(f^*h)(p) = \sum_{\alpha=1}^m f_{i_0 \alpha \alpha}(p) = 0.$$

This is true for any  $i_0$  and any  $p \in M$ .

Next, for any  $p \in M$  and any constants  $C_{ij}$  with  $C_{ij} = C_{ji}$  and  $\sum_{i=1}^n C_{ii} = 0$ , there is a harmonic function  $h$  defined near  $f(p)$  such that

$$\Delta_M(f^*h)(p) = \sum_{i,j=1}^n C_{ij} X_{ij}(p) = 0,$$

where  $X_{ij} = \sum_{\alpha=1}^m f_{i \alpha} f_{j \alpha}$ . Thus, for any constants  $C_{ij}$  satisfying  $C_{ij} = C_{ji}$  and  $\sum_{i=1}^n C_{ii} = 0$ , we get

$$\sum_{i \neq j}^n C_{ij} X_{ij}(p) + \sum_{i=1}^n (X_{ii}(p) - X_{11}(p)) C_{ii} = 0,$$

so that

$$X_{11}(p) = X_{22}(p) = \dots = X_{nn}(p), \quad X_{ij}(p) = 0 \quad (i \neq j).$$

Put  $\lambda(p) = X_{11}(p)$ . Then we have

$$(5.1) \quad \sum_{a=1}^m f_{ia}(p)f_{ja}(p) = \lambda(p)\delta_{ij}.$$

This gives

$$(5.2) \quad \lambda(p) = \frac{1}{n} \sum_{i=1}^n \sum_{a=1}^m (f_{ia}(p))^2.$$

At any  $p \in M$ , (5.2) is valid. Hence  $\lambda$  is a non-negative function globally defined on  $M$ . If there is a point  $p \in M$  with  $\lambda(p) \neq 0$ , (5.1) implies

$$\text{rank}(f_{ia}) \geq \text{rank}(\delta_{ij})$$

at  $p$ . Hence we have  $m \geq n$ .

We are now in the position to give the following definition.

**Definition.** Assume that  $m \geq n$ . Let  $f: M \rightarrow N$  be a smooth mapping.  $f$  is called a *conformal submersion* (resp. *pseudo-submersion*) with function  $\lambda$  if it satisfies (5.1) at every point  $p$  of  $M$  for some positive (resp. non-negative) function  $\lambda$  on  $M$ .

Remark that every smooth mapping  $f: M \rightarrow N$  is a conformal pseudo-submersion when  $n = 1$ .

From Lemma 2.1 it follows easily that a harmonic conformal pseudo-submersion  $f: M \rightarrow N$  preserves local harmonic functions. Thus, we have obtained

**Theorem 5.1.** When  $m \geq n$ , a mapping  $f$  belongs to  $\Omega_H(M, N)$  iff it is a harmonic conformal pseudo-submersion. In particular, for  $n = 1$ ,  $\Omega_H(M, N)$  is the set of all harmonic mappings  $f: M \rightarrow N$ . When  $m < n$ ,  $\Omega_H(M, N)$  consists of constant mappings of  $M$  into  $N$ .

Similarly we can get

**Theorem 5.2.**  $\Omega_{SH}(M, N) = \Omega_H(M, N)$ .

*Proof.* We will prove  $\Omega_H(M, N) \subset \Omega_{SH}(M, N)$ . This is trivial when  $m < n$ . Hence we assume that  $m \geq n$ . If  $f: M \rightarrow N$  is a harmonic conformal pseudo-submersion, for any local subharmonic function  $h$  we have  $\Delta_M(f^*h) = \lambda \sum_{i=1}^n h_{ii} \geq 0$ .

Conversely, let  $f \in \Omega_{SH}(M, N)$ . For any point  $p \in M$ , any integer  $i_0 (1 \leq i_0 \leq n)$  and any constant  $C$ , applying Lemma 4.1 in the case when  $C_{i_0} = C$ ,  $C_i = 0 (i \neq i_0)$  and  $C_{ij} = 0$ , we have a local harmonic function  $h$  satisfying

$$\Delta_M(f^*h)(p) = C \sum_{a=1}^m f_{i_0 a a}(p) \geq 0.$$

This is true for any constant  $C$ . Thus  $f$  is harmonic.

For any point  $p \in M$  and any constants  $C_{ij}$  with  $C_{ij} = C_{ji}$  and  $\sum_{i=1}^n C_{ii} = 0$ , we apply Lemma 4.1 and get

$$\sum_{i,j=1}^n C_{ij} X_{ij}(p) \geq 0,$$

where  $X_{ij} = \sum_{a=1}^m f_{ia} f_{ja}$  as in the proof of Theorem 5.1. This also gives (5.1). Thus  $f \in \Omega_{SH}(M, N)$ .

In the argument above, we have practically proved

**Proposition 5.3.** *If  $f: M \rightarrow N$  maps every local harmonic function into a local subharmonic function, it is contained in  $\Omega_H(M, N)$ .*

As it was described in § 2, all of minimal Riemannian submersions are harmonic conformal submersions. Let  $U$  be an open subset in complex Euclidean 2-space  $\mathbb{C}^2 = \{(z_1, z_2)\}$ .  $f$  denotes a holomorphic function on  $U$ . Take the natural metrics on  $U$  and complex Euclidean space  $\mathbb{C}$ . Then  $f: U \rightarrow \mathbb{C}$  is a harmonic conformal pseudo-submersion with function  $\lambda = |\partial f / \partial z_1|^2 + |\partial f / \partial z_2|^2$ .

We proceed to study relations between Riemannian submersions and conformal submersions. To begin with, let  $f: M \rightarrow N$  be a conformal pseudo-submersion with function  $\lambda$ . Then the restriction of  $f$  to the open submanifold  $M' = \{p \in M, \lambda(p) > 0\}$  is a conformal submersion. Hence we assume that  $f$  is a conformal submersion originally. Let  $\tilde{M}$  denote the Riemannian manifold with Riemannian metric  $d\tilde{s}_M^2 = \lambda ds_M^2$ . Then  $f: \tilde{M} \rightarrow N$  is a Riemannian submersion. We call this *the Riemannian submersion corresponding to  $f: M \rightarrow N$* . Let Riemannian metrics  $ds_M^2$  and  $ds_N^2$  be written locally as in (1.1). We may assume that local 1-forms  $\omega_a, \omega_i^*$  satisfy

$$(5.3) \quad f^*(\omega_i^*) = e^\rho \sum_{a=1}^m \delta_{ia} \omega_a,$$

where  $\rho = \frac{1}{2} \log \lambda$ . Then  $d\tilde{s}_M^2$  is given locally by  $d\tilde{s}_M^2 = \sum_{a=1}^m \tilde{\omega}_a^2$ . Here we set  $\tilde{\omega}_a = e^\rho \omega_a$ . Let  $\tilde{\omega}_{ab}$  be the connection forms of the Riemannian metric  $d\tilde{s}_M^2$ . Then

$$\tilde{\omega}_{ab} = \omega_{ab} + \rho_a \omega_b - \rho_b \omega_a,$$

where we put  $d\rho = \sum_{a=1}^m \rho_a \omega_a$ . Let  $F_{iab}$  be the components of the structure tensor of the Riemannian submersion  $f: \tilde{M} \rightarrow N$ . Then we know

**Lemma 5.4** [6].

$$(5.4) \quad f_{iab} = e^{2\rho} F_{iab} - e^\rho (\delta_{ab} \rho_i - \delta_{ia} \rho_b - \delta_{ib} \rho_a).$$

This yields

**Lemma 5.5.** *Let  $f: M \rightarrow N$  be a conformal submersion. It is harmonic iff*

$$(5.5) \quad \sum_{\alpha=n+1}^m F_{i\alpha\alpha} = (m-2)e^{-\rho} \rho_i.$$

For a function  $h$  on  $M$ , put  $dh = d_H h + d_V h$ , where

$$d_H h = \sum_{i=1}^n h_i \omega_i, \quad d_V h = \sum_{a=n+1}^m h_a \omega_a.$$

It is evident that this definition is independent of the choice of local forms  $\omega_a$  satisfying (5.3). (Compare  $d_H$  with Vaisman's foliated exterior derivative [12].) Now we get

**Proposition 5.6.** *Let  $f: M \rightarrow N$  be a conformal submersion with function  $\lambda$ . Assume that the corresponding Riemannian submersion is harmonic. Then  $f$  is harmonic iff  $d_H \lambda = 0$ .*

Let  $R^2 = \{(y_1, y_2)\}$  be Euclidean 2-space with standard metric  $ds_2^2 = \sum_{i=1}^2 (dy_i)^2$  and  $R^4 = \{(x_1, x_2, x_3, x_4)\}$  be Euclidean 4-space with metric  $ds_4^2 = \lambda(x_3, x_4) \sum_{a=1}^4 (dx_a)^2$ , where  $\lambda(x_3, x_4)$  is a smooth positive function depending only on variables  $x_3, x_4$ . Let  $P: R^4 \rightarrow R^2$  be the natural projection, i.e.,  $P(x_1, x_2, x_3, x_4) = (x_1, x_2)$ . From Proposition 5.6 it follows that  $P: R^4 \rightarrow R^2$  is an example of harmonic conformal submersion which are not Riemannian submersions.

Let  $\nu_C: S^3 \rightarrow S^2$  be the Hopf mapping, where  $S^3 = \{(x_1, x_2, x_3, x_4) \in R^4, \sum_{a=1}^4 x_a^2 = 1\}$  and  $S^2 = \{(y_1, y_2, y_3) \in R^3, \sum_{i=1}^3 y_i^2 = (1/2)^2\}$ . We introduce a local coordinate system  $\phi_1, \phi_2, \phi_3$  in  $S^3$  such that

$$x_1 = \cos \phi_1 \cos \phi_2, \quad x_2 = \cos \phi_1 \sin \phi_2, \quad x_3 = \sin \phi_1 \cos \phi_3, \quad x_4 = \sin \phi_1 \sin \phi_3$$

and a local coordinate system  $\psi_1, \psi_2$  in  $S^2$  such that

$$y_1 = \frac{1}{2} \cos \psi_1, \quad y_2 = \frac{1}{2} \sin \psi_1 \cos \psi_2, \quad y_3 = \frac{1}{2} \sin \psi_1 \sin \psi_2.$$

Then  $\nu_C$  is given locally by  $\psi_1 = 2\phi_1, \psi_2 = \phi_3 - \phi_2$ . Let  $ds_3^2$  and  $ds_2^2$  be the standard Riemannian metrics on  $S^3$  and  $S^2$  of constant curvatures 1 and 4 respectively. Put

$$\begin{aligned} \omega_1 &= d\phi_1, & \omega_2 &= \frac{1}{2} \sin \phi_1 (d\phi_3 - d\phi_2), & \omega_3 &= \sin^2 \phi_1 d\phi_3 + \cos^2 \phi_1 d\phi_2, \\ \omega_1^* &= \frac{1}{2} d\psi_1, & \omega_2^* &= \frac{1}{2} \sin \psi_1 d\psi_2. \end{aligned}$$

Then it holds locally  $ds_3^2 = \sum_{a=1}^3 \omega_a^2, ds_2^2 = \sum_{i=1}^2 \omega_i^{*2}$  and  $\nu_C(\omega_i^*) = \sum_{a=1}^3 \delta_{ia} \omega_a$ . It is well known that  $\nu_C: S^3 \rightarrow S^2$  is a harmonic Riemannian submersion. Now let  $d\bar{s}^2$  be a Riemannian metric on  $S^3$  conformally related to  $ds_3^2$ , i.e.,  $d\bar{s}^2 = \lambda ds_3^2$ . Then we have

$$d_H \lambda = \frac{\partial \lambda}{\partial \phi_1} \omega_1 + \left( \cot \phi_1 \frac{\partial \lambda}{\partial \phi_3} - \tan \phi_1 \frac{\partial \lambda}{\partial \phi_2} \right) \omega_2.$$

Hence  $d_H \lambda = 0$  iff  $\lambda$  is constant. Thus  $\nu_C: S^3 \rightarrow S^2$  is an only harmonic conformal submersion with the corresponding Riemannian submersion  $\nu_C: S^3 \rightarrow S^2$  up to a homothetic change of the metrics on  $S^3$ .

Lastly we would like to consider  $\Omega_H(M, N)$  in the equidimensional case. H. Imai states in Lemma 4 [10] that a harmonic mapping (of C. Constantinescu and A.

Cornea) between Riemannian manifolds of the same dimension is a local isometry on its non-singular points. It seems however there is a gap in his proof of the lemma. But the following theorem asserts that his lemma is true up to homothetic changes of metrics, and moreover that a mapping of this kind has no singular point if it is not a constant mapping.

**Theorem 5.7.** *Assume that  $\dim M = \dim N \geq 3$ , and  $M$  is connected. Let  $f: M \rightarrow N$  be a harmonic, conformal pseudo-submersion with function  $\lambda$ . Then it is a constant mapping or  $f: M \rightarrow N'$  is a Riemannian covering after some homothetic change of the metric on  $M$ , where  $N' = f(M)$ .*

*Proof.* Let  $M' = \{p \in M, \lambda(p) > 0\}$ . If  $M' = \emptyset$ ,  $f$  is a constant mapping. Assume  $M' \neq \emptyset$ . Then it is an open submanifold of  $M$ . Applying Lemma 5.5 to the restriction of  $f$  to  $M'$ , since in this case the left hand side of (5.5) vanishes, we get

$$d\lambda = 2e^{2\rho} \sum_{i=1}^n \rho_i \omega_i = 0.$$

Hence  $\lambda$  is a positive constant function on  $M'$ . Since  $\lambda$  is continuous on  $M$ , we obtain  $M' = M$ .

When  $\dim M = \dim N = 2$ , it follows from Lemma 5.5 that all conformal pseudo-submersions are harmonic. Moreover if  $M$  and  $N$  are oriented, they are regarded naturally as complex manifolds, and  $f: M \rightarrow N$  is a conformal pseudo-submersion iff it is holomorphic or antiholomorphic.

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**Added in proof.**

After submitting this paper, the author was noticed that the theorems 5.1 and 5.5 had been shown independently by Bent Fuglede by a different method (*Ann. Inst. Fourier, Grenoble* **28**, 107–144, 1978).