A MAPPING PROPERTY OF THE BERGMAN PROJECTION ON CERTAIN PSEUDOCONVEX DOMAINS

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Abstract. We show that the Bergman kernel function, associated to pseudoconvex domains of finite type with the property that the Levi form of the boundary has at most one degenerate eigenvalue, is a standard kernel of Calderón-Zygmund type with respect to the Lebesgue measure. As an application, we show that the Bergman projection on these domains preserves some of the Lebesgue classes.

1. Introduction.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. The Bergman projection P on Ω is the orthogonal projection

$$P : L^2(\Omega) \longrightarrow H(\Omega) \cap L^2(\Omega) = A^2(\Omega),$$

where $H(\Omega)$ denotes the set of holomorphic functions on Ω. There is a corresponding kernel function $K_\Omega(z, w)$, the Bergman kernel function, such that

$$Pf(z) = \int_\Omega K_\Omega(z, w) f(w) dw.$$

Let a triple (S, d, μ) be a space of homogeneous type, that is, S is a set, d is a pseudometric on S and μ is a positive measure on S; more precisely, $d : S \times S \to [0, \infty)$ satisfies

(a) $d(x, y) = 0 \Longleftrightarrow x = y$,
(b) $C_1^{-1} d(y, x) \le d(x, y) \le C_1 d(y, x)$,
(c) $d(x, y) \le C_2(d(x, z) + d(z, y)) \quad \text{for } x, y, z \in S$,

for independent constants C_1, C_2; and for all $x \in S$ and small $\delta > 0$, there is an independent constant C_3 such that

(i) $\mu(P(x, \delta)) < \infty$;
(ii) $\mu(P(x, 2\delta)) \le C_3 \mu(P(x, \delta))$,

where

$$P(x, \delta) = \{y \in S : d(x, y) < \delta\}.$$

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Definition 1.1. A kernel $K : S \times S - \{x = y\} \to \mathbb{C}$ is called a standard kernel if there exist independent constants $T > 0$ and $C < \infty$ such that for all $x \neq y \in S$,

$$|K(x,y)| \leq \frac{C}{\mu(P(x, d(x,y))}$$

and for all $x, z \in S$,

$$\int_{d(x,y)>Td(x,z)} |K(x,y) - K(z,y)| dy \leq C.$$

In all that follows, we assume that Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r. We also assume that all the points of $b\Omega$ are of finite type in the sense of D'Angelo [4], and the Levi form $\partial\overline{\partial}r(z)$ of $b\Omega$ has at least $(n-2)$-positive eigenvalues at every point $z \in b\Omega$.

Theorem 1.2. *Let Ω be as above. Then the Bergman kernel $K_\Omega(z', z)$ is a standard kernel with respect to a pseudometric d and the Lebesgue measure μ.*

Here d is a pseudometric to be determined explicitly. As an application, we prove:

Theorem 1.3. *Let Ω be as above. Then the Bergman projection P is bounded on $L^p(\Omega)$, $1 < p < \infty$.*

For geometrically convex domains of finite type in $\mathbb{C}^n$, McNeal [5] showed that the Bergman kernel is a standard kernel and is bounded in $L^p(\Omega)$, $1 < p < \infty$. He also mentioned that the same results hold for pseudoconvex domains of finite type in $\mathbb{C}^2$ and for decoupled pseudoconvex domains of finite type in $\mathbb{C}^n$. The main technical difficulties in proving these theorems are to construct a suitable pseudometric d on Ω with "doubling property" of the balls, and to estimate $|K(z', z) - K(w, z)|$ whenever z satisfies $d(z', z) > Td(z', w)$ for some large T. The "doubling property" in our case is proved in Section 2 (Proposition 2.5). To estimate $|K(z', z) - K(w, z)|$, we will use the estimates of the Bergman kernel and its derivatives (cf. [1], [2]) of the domain we are considering.

2. Estimates on the Bergman kernel.

Let Ω be the domain in $\mathbb{C}^n$ considered in Section 1. In this section, we will analyze the local geometry of the domain Ω near $z_0 \in b\Omega$. We may assume that there are coordinate functions $z_1, \ldots, z_n$ defined near z_0 such that $|(\partial r/\partial z_1)(z)| \geq c$ for all z in a neighborhood U of z_0, for some $c > 0$. Let us fix $z' \in U$ for a moment. After an affine transformation for the coordinates $z_2, \ldots, z_{n-1}$, we have coordinate functions $w_1, w_2, \ldots, w_n$ such that $\partial\overline{\partial}r(z')(\partial/\partial w_i, \partial/\overline{w}_j)$, $2 \leq i, j \leq n-1$, is an identity matrix. Then the following special coordinates can be defined by a biholomorphic map $\Phi_{z'}$.

Proposition 2.1 [1, Proposition 2.2]. *For each $z' \in U$ and positive even integer m, there is a biholomorphic map $\Phi_{z'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\Phi_{z'}^{-1}(z') = 0$, $\Phi_{z'}^{-1}(z) =$*

$(\zeta_1, \dots, \zeta_n)$ *such that*

$$r(\Phi_{z'}(\zeta)) = r(z') + \mathrm{Re}\zeta_1 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\le m/2 \\ j,k>0}} \mathrm{Re}\left(b^{\alpha}_{j,k}(z')\zeta_n^j \overline{\zeta}_n^k \zeta_\alpha\right)$$

$$(2.1)\qquad + \sum_{\substack{j+k\le m \\ j,k>0}} a_{j,k}(z')\zeta_n^j\overline{\zeta}_n^k + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2$$

$$+ \mathcal{O}(|\zeta_1||\zeta| + |\zeta''|^2|\zeta| + |\zeta''||\zeta_n|^{m/2+1} + |\zeta_n|^{m+1}).$$

Set $\rho(\zeta) = r \circ \Phi_{z'}(\zeta)$, and set

$$A_l(z') = \max\{|\frac{\partial^l \rho}{\partial\zeta_n^j \partial\overline{\zeta}_n^k}(0)|; j+k=l\},\ 2\le l\le m, \quad \text{and}$$

$$B_{l'}(z') = \max\{|\frac{\partial^{l+1} \rho}{\partial\zeta_n^j \partial\overline{\zeta}_n^k\partial\zeta_\alpha}(0)|; j+k=l'\},\ 2\le l'\le m/2.$$

For each $\delta > 0$, we define $\tau(z',\delta)$ as follows

$$(2.2)\qquad \tau(z',\delta) = \min_{\substack{2\le l\le m \\ 2\le l'\le m/2}} \{(\frac{\delta}{A_l(z')})^{1/l},\ (\frac{\delta^{1/2}}{B_{l'}(z')})^{1/l'}\}.$$

In [1], it was shown that $(\delta^{1/2}/B_{l'}(z'))^{1/l'} \gg \tau(z',\delta)$ whenever $\delta > 0$ is sufficiently small. Hence the terms mixed with ζ_n and ζ_α, $\alpha = 2, \dots, n-1$, would not be an important ones in (2.1) and hence

$$(2.3)\qquad \tau(z',\delta) = \min\{(\frac{\delta}{A_l(z')})^{1/l} :\ 2\le l\le m\}.$$

Since $A_m(z_0) \ge c > 0$, it follows that $A_m(z') \ge c' > 0$ for all $z' \in U$ if U is sufficiently small. This gives the inequality

$$\delta^{1/2} \lesssim \tau(z',\delta) \lesssim \delta^{1/m}, \quad z' \in U,$$

and the definition of $\tau(z',\delta)$ easily implies that if $\delta' < \delta''$, then

$$(2.4)\qquad (\delta'/\delta'')^{1/2}\tau(z',\delta'') \le \tau(z',\delta') \le (\delta'/\delta'')^{1/m}\tau(z',\delta'').$$

Now set $\tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{1/2},\ \tau_n = \tau(z',\delta) = \tau$ and define

$$(2.5)\qquad \begin{aligned} R_\delta(z') &= \{\zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k,\ k = 1,2,\dots,n\}, \text{ and} \\ Q_\delta(z') &= \{\Phi_{z'}(\zeta); \zeta \in R_\delta(z')\}. \end{aligned}$$

In the sequal we denote any partial derivative operator of the form $\partial^{\mu+\nu}/\partial\zeta_k^\mu \partial\overline{\zeta}_k^\nu$ by D_k^l, where $\mu+\nu = l$, $k = 1,2,\dots,n$. By the definitions of τ_k, $k \ge 1$, one has the following useful derivative estimates for the function $\rho = r\circ\Phi_{z'}$.

Proposition 2.2 [1, Proposition 2.3]. *Let $z' \in U$. Then the function $\rho = r \circ \Phi_{z'}(\zeta)$ satisfies*

$$
\begin{aligned}
&|\rho(\zeta) - \rho(0)| \lesssim \delta, \quad \zeta \in R_\delta(z'), \text{ and} \\
&|D_k^i D_n^l \rho(\zeta)| \lesssim \delta \tau_n^{-l} \tau_k^{-i}, \ \zeta \in R_\delta(z'),
\end{aligned}
$$

for $l + im/2 \le m$, $i = 0, 1$, $k = 2, \dots, n-1$.

In [1], the author proved that for $z \in Q_\delta(z')$

$$
\tau(z', \delta) \approx \tau(z, \delta). \tag{2.6}
$$

Now let us study how the polydiscs $Q_\delta(z')$ and $Q_\delta(z'')$ are related. Let $\Phi_{z'}$ be the map as in Proposition 2.1, and set $\Phi_{z'}(\zeta'') = z''$. If we apply Proposition 2.1 at the point ζ'' with r replaced by $\rho = r \circ \Phi_{z'}$, then we obtain a map $\Psi_{z''} : \mathbb{C}^n \to \mathbb{C}^n$. By virtue of the proof of Proposition 2.1 ([1, Proposition 2.2]), we see that $\Psi_{z''} = \phi^1 \circ \phi^2 \circ \dots \circ \phi^m$, where for $l \ge 2$ and $\rho_l = \rho \circ \phi^1 \circ \dots \circ \phi^{l-1}$,

$$
\phi^l(u) = (\phi_1^l(u), \dots, \phi_n^l(u)) = (\zeta_1, \dots, \zeta_n)
$$

is a biholomorphic map on $\mathbb{C}^n$ given by

$$
\begin{aligned}
u_1 &= z_1 + \frac{2}{l!}\frac{\partial^l \rho_l(0)}{\partial z_n^l} z_n^l + \frac{2}{l!} \sum_{\alpha=2}^{n-1} \frac{\partial^{l+1} \rho_l(0)}{\partial z_\alpha \partial z_n^l} z_\alpha z_n^l, \\
u_j &= z_j, \quad j = 2, \dots, n,
\end{aligned}
$$

followed by the coordinate change

$$
z_1 = \zeta_1, \quad z_n = \zeta_n, \quad z_\alpha = \zeta_\alpha + \frac{\partial^{l+1} \rho(0)}{\partial \bar{\zeta}_\alpha \partial \zeta_n^l} \zeta^l,
$$

and ϕ^1 is an affine transformation which is uniformly non-singular in U. From Proposition 2.2, ϕ^2 satisfies, for $l + im/2 \le m$, $i = 0, 1$, $k = 2, \dots, n-1$, that

$$
\begin{aligned}
&|D_k^i D_n^l \phi_1^2(0)| \lesssim \delta \tau_n^{-l} \tau_k^{-i} \text{ and} \\
&|D_k^i D_n^l \phi_\alpha^2(0)| \lesssim \delta^{1/2} \tau_n^{-l} \tau_k^{-i}, \ \alpha = 2, \dots, n-1.
\end{aligned} \tag{2.7}
$$

By induction, one can show that the same estimates hold for the components of ϕ^l. Since $\Psi_{z''} = (\psi_1, \dots, \psi_n)$ is a composite of ϕ^l, $l = 1, \dots, m$, and since each ϕ^l satisfies an analog of (2.7), we have the following estimates for the component functions ψ_k of $\Psi_{z''}$.

Lemma 2.3. *For $l + im/2 \le m$, $i = 0, 1$, $k = 2, \dots, n-1$, one has*

$$
\begin{aligned}
&|D_k^i D_n^l \psi_1(0)| \lesssim \delta \tau_n^{-l} \tau_k^{-i} \quad \text{and} \\
&|D_k^i D_n^l \psi_\alpha(0)| \lesssim \delta^{1/2} \tau_n^{-l} \tau_k^{-i}, \ \alpha = 2, \dots, n-1.
\end{aligned} \tag{2.8}
$$

Remark 2.4. *Since the component functions of $\Psi_{z''}^{-1}$ have expressions similar to those of $\Psi_{z''}$, they satisfy the same estimates as (2.8).*

Proposition 2.5. *There exists a constant C such that if $z'' \in Q_\delta(z')$, then*

$$Q_\delta(z'') \subset Q_{C\delta}(z') \quad \text{and} \tag{2.9}$$

$$Q_\delta(z') \subset Q_{C\delta}(z'') \tag{2.10}$$

Proof. Define $S_\delta(z'') = \{\Psi_{z''}(u); u \in R_\delta(z'')\}$. From (2.5) we see that to prove (2.9), it suffices to show that

$$S_\delta(z'') \subset R_{C\delta}(z').$$

Note that (2.6) implies that $\tau(z'', \delta) \lesssim \tau(z', \delta)$. Since $\zeta'' = (\Phi_{z'})^{-1}(z'') \in R_\delta(z')$, it follows that if $\zeta \in S_\delta(z'')$, then

$$\begin{aligned} |\zeta_n| &= |\zeta_n'' + u_n| < |\zeta_n''| + \tau(z'', \delta) \\ &\lesssim \tau(z', \delta) + \tau(z'', \delta) \lesssim \tau(z', \delta), \end{aligned}$$

where we have used the fact that $u \in R_\delta(z'')$, and hence that $|u_n| \lesssim \tau(z'', \delta)$. Also by Lemma 2.3 and by the Taylor series expansion theorem,

$$|\zeta_\alpha| = |\psi_\alpha(u)| = |\zeta_\alpha'' + \psi_\alpha(u)| \lesssim \delta^{1/2} + \sum_{1 \le k \le m/2} \delta^{1/2} \tau_n^{-k} |u_n|^k \lesssim \delta^{1/2},$$

for $\alpha = 2, \dots, n-1$, and

$$|\zeta_1| = |\zeta_1'' + \psi_1(u)| \lesssim \delta + |\psi_1(u)| \lesssim \delta + \delta \lesssim \delta.$$

This shows that $\zeta \in R_{C\delta}(z')$ and proves (2.9). To prove (2.10), define $\tilde{R}_\delta(z') = \{\Psi_{z''}^{-1}(\zeta); \zeta \in R_\delta(z')\}$. Then (2.5) also implies that it is sufficient to prove that

$$\tilde{R}_\delta(z') \subset R_{C\delta}(z'').$$

Since each component function of $\Psi_{z''}^{-1}$ also satiefies the same estimates as that of $\Psi_{z''}$ and since $\tau(z', \delta) \lesssim \tau(z'', \delta)$, we may apply the same method as above to prove $\tilde{R}_\delta(z') \subset R_{C\delta}(z'')$. □

For z^1 and z^2 in $U \cap \Omega$, let Φ_{z^1} be the biholomorphic map as in Proposition 2.1 associated with z^1 and set $0 = \zeta^1 = \Phi_{z^1}^{-1}(z^1)$, $\zeta^2 = \Phi_{z^1}^{-1}(z^2)$. Then we define

$$d_1(z^1, z^2) = \inf\{\eta > 0; z^2 \in Q_\eta(z^1)\}, \tag{2.11}$$

and set

$$M_1(z^1, z^2) = |\zeta_1^1 - \zeta_1^2| + \sum_{j=2}^{n-1} |\zeta_j^1 - \zeta_j^2|^2 + \sum_{l=2}^{m} A_l(z^1)|\zeta_n^1 - \zeta_n^2|^l = M(\zeta^1, \zeta^2). \tag{2.12}$$

Then from the definitions (2.2), (2.3), (2.5), and by virtue of the proof of Proposition 2.1, we have

$$d_1(z^1, z^2) \approx M_1(z^1, z^2). \tag{2.13}$$

Proposition 2.6. *$d_1(z^1, z^2)$ is a pseudometric on $U \cap \Omega$.*

Proof. Suppose $d_1(z^1, z^2) \neq 0$ and choose $\alpha > d_1(z^1, z^2)$. Then $Q_\alpha(z^1) \cap Q_\alpha(z^2) \neq \emptyset$ and by Proposition 2.5, $Q_\alpha(z^1) \subset Q_{C\alpha}(z^2)$ for an independent constant C. Thus $z^1 \in Q_{C\alpha}(z^2)$ for all $\alpha > d_1(z^1, z^2)$. It follows that $d_1(z^1, z^2) \leq C d_1(z^2, z^1)$. Let $z^1, z^2, z^3 \in U \cap \Omega$ and set $\beta = \max\{d_1(z^1, z^2), d_1(z^3, z^2)\}$. Then $Q_\beta(z^1) \cap Q_\beta(z^3) \neq \emptyset$, and hence Proposition 2.5 implies that $Q_\beta(z^3) \subset Q_{C\beta}(z^1)$ for an independent constant C. Thus it follows that

$$d_1(z^1, z^3) \leq C\beta \leq C(d_1(z^1, z^2) + d_1(z^3, z^2)) \leq C^2(d_1(z^1, z^2) + d_1(z^2, z^3)).$$

□

We recall the estimates on the Bergman kernel function and its derivatives for the domain Ω obtained in [1], [2].

Theorem 2.7. *Let Ω and $z_0 \in b\Omega$ be as above. For $z^1, z^2 \in U \cap \Omega$, set $\zeta^i = \Phi_{z^1}^{-1}(z^i)$, $i = 1, 2$. Then there exist a neighborhood U of z_0 and constants $C_{\alpha,\beta}$, independent of $z^1, z^2 \in U \cap \Omega$, such that*

$$|D_{\zeta^1}^{\alpha} \overline{D}_{\zeta^2}^{\beta} K_{\Omega_{z^1}}(\zeta^1, \zeta^2)| \leq C_{\alpha,\beta} \delta^{-n-\alpha_1-\beta_1-(\alpha_2+\beta_2+\cdots+\alpha_{n-1}+\beta_{n-1})/2} \tau(z^1, \delta)^{-2-\alpha_n-\beta_n}$$

where $\delta = |\rho(\zeta^1)| + |\rho(\zeta^2)| + M(\zeta^1, \zeta^2)$, and $\rho = r \circ \Phi_{z^1}$.

3. L^p-boundedness of the Bergman projection.

Now we construct a global pseudometric d based simply on patching together the local pseudometric $d_1(z', z)$. Let $B_j = B(a^j; \epsilon/2)$, $j = 1, \cdots, N$, be a minimal open covering of $b\Omega$ by ordinary Euclidean balls with centers $a^j \in b\Omega$ and radius $\epsilon/2 > 0$ such that $B(a^j; 2\epsilon)$, $j = 1, 2, \ldots, N$, are the set of the neighborhoods given by Theorem 2.7 and $B(a^j; \epsilon/4) \cap B(a^k; \epsilon/4) = \emptyset$ for all $j \neq k$. Set $B_0 = \Omega - (\cup_{j=1}^N B_j)$ and let $d_j(z', z)$ be defined on $B(a^j; 2\epsilon)$ by (2.11). Choose $\phi_j \in C_0^\infty(B(a^j; 2\epsilon))$, $\phi_j \geq 0$, $j = 1, \cdots, N$, with $\phi_j(z) = 1$ if $z \in B(a^j; 3\epsilon/2)$, and set

$$d_0(z', z) = \sum_{j=1}^{N} \phi_j(z')\phi_j(z) d_j(z', z).$$

Then d_0 is well-defined by the compatability of the functions $d_j(z', z)$ on the overlaps of the covering, that is, $d_j(z', z) \approx d_k(z', z)$ if $z', z \in B(a^j; \epsilon/2) \cap B(a^k; \epsilon/2)$. To obtain a global pseudometric on Ω, set

$$d(z', z) = \begin{cases} d_0(z', z), & |z' - z| < \epsilon \\ |z' - z|, & \text{otherwise} . \end{cases}$$

Then it is easy to show that d is a pseudometric on Ω.

Lemma 3.1. *Let μ be the Lebesgue measure on Ω. Then the triple (Ω, d, μ) is a space of homogeneous type.*

Proof.

$$\mu(Q_\delta(z')) \approx \prod_{j=1}^{n} \tau_j(z', \delta)^2 = \delta^n \tau(z', \delta)^2 < \infty$$

and

$$\mu(Q_{2\delta}(z')) \approx \prod_{j=1}^{n} \tau_j(z',2\delta)^2 = (2\delta)^n \tau(z',2\delta)^2 \lesssim 2^{n+1}\delta^n \tau(z',\delta)^2 = 2^{n+1}\mu(Q_\delta(z',)).$$

For small δ and z near $b\Omega$, the volume of the balls $P(z',\delta) = \{z : d(z',z) < \delta\}$ are comparable with those of the polydiscs $Q_\delta(z')$. Thus it follows that

$$\mu(P(z',\delta)) < \infty$$

and

$$\mu(P(z',2\delta)) \le C\mu(P(z',\delta)),$$

where C is an independent constant. Thus (Ω, d, μ) is a space of homogeneous type. □

Now assume that $z', z, w \in B(a_j;\epsilon)\cap\Omega$ for some j and consider the biholomorphic map $\Phi_{z'}$ as in Proposition 2.1 and set $\zeta' = 0 = \Phi_{z'}^{-1}(z')$, $\zeta = \Phi_{z'}^{-1}(z)$, $\xi = \Phi_{z'}^{-1}(w)$. Note that $d(z',z) \approx M(\zeta',\zeta) \approx d_j(z',z)$ in this case.

Lemma 3.2. *Let ζ', ζ, ξ be given as above. Then there are $\nu > 0$, and $T > 0$ such that*

$$|K(\zeta',\zeta) - K(\xi,\zeta)| \lesssim \left(\frac{M(\zeta',\xi)}{M(\zeta',\zeta)}\right)^{\nu} \frac{1}{\mathrm{Vol}(P_{M(\zeta',\zeta)}(\zeta'))}, \tag{3.1}$$

for $M(\zeta',\zeta) > TM(\zeta',\xi)$. Here $K = K_{\Omega_{z'}}$ and $P_\delta(\zeta') = \{\zeta : M(\zeta',\zeta) < \delta\}$.

Proof. From the definitions of (2.11) and (2.12), we have $\xi \in P_{2M(\zeta',\xi)}(\zeta')$. If we apply Proposition 2.1 at the point ξ' with r replaced by $\rho = r \circ \Phi_{z'}$ and by virtue of Theorem 2.7, it follows that

$$\begin{aligned} |K(\zeta',\zeta) - K(\xi,\zeta)| &\lesssim \sum_{j=1}^{n} |\frac{\partial}{\partial z_j} K(\xi',\zeta)||\zeta'_j - \xi_j| \\ &\lesssim (\sum_{j=1}^{n} \frac{|\zeta'_j - \xi_j|}{\tau_j(\xi', M(\xi',\zeta))}) \cdot \frac{1}{\mathrm{Vol}(P_{M(\xi',\zeta)}(\xi'))}, \end{aligned} \tag{3.2}$$

for some $\xi' \in P_{CM(\zeta',\xi)}(\zeta') \cap \Omega$. It follows from the definition of $M(\zeta^1,\zeta^2)$ and (2.6) that $M(\xi',\zeta) \approx M(\zeta',\zeta)$ for ζ satisfying $TM(\zeta',\xi) < M(\zeta',\zeta)$, provided T is sufficiently large. Note that $P_\delta(\zeta')$ and $R_\delta(z')$ (as in (2.5)) are comparable in the sense that $P_{\delta/C}(\zeta') \subset R_\delta(z') \subset P_{C\delta}(\zeta')$ for an independent constant C. Thus $R_{C^2M(\zeta',\zeta)}(\xi') \cap R_{C^2M(\zeta',\zeta)}(\zeta') \neq \emptyset$ and hence $R_{C^2M(\zeta',\zeta)}(\zeta') \subset R_{C^3M(\zeta',\zeta)}(\xi')$ by Proposition 2.5 for an independent constant C. Therefore

$$\mathrm{Vol}(P_{M(\zeta',\zeta)}(\zeta')) \lesssim \mathrm{Vol}(P_{M(\zeta',\zeta)}(\xi')) \approx \mathrm{Vol}(P_{M(\xi',\zeta)}(\xi')). \tag{3.3}$$

Set $\kappa = \min\{k : (M(\zeta',\zeta)/A_k(\zeta'))^{1/k} = \tau_n(\zeta', M(\zeta',\zeta))\}$ where τ_n is defined as in (2.3) at ζ' with r replaced by $r \circ \Phi_{z'}$. Since $\xi', \zeta' \in P_{CM(\xi',\zeta')}(\xi') \subset R_{C^2M(\xi',z')}(\xi')$, it follows from (2.4) and (2.6) that

$$\tau_n(\xi', M(\xi',\zeta)) \approx \tau_n(\xi', M(\zeta',\zeta)) \approx \tau_n(\zeta', M(\zeta',\zeta)). \tag{3.4}$$

By virtue of the definitions of τ_i, $i = 1, 2, \dots, n-1$, we also have

$$\tag{3.5} \begin{aligned} &\tau_1(\xi', M(\xi', \zeta)) \approx M(\xi', \zeta) \approx M(\zeta', \zeta), \text{and} \\ &\tau_j(\xi', M(\xi', \zeta)) \approx M(\xi', \zeta)^{1/2} \approx M(\zeta', \zeta)^{1/2}. \end{aligned}$$

With (3.4) and the definitions of κ and $M(\zeta', \xi)$, we have

$$\tag{3.6} \frac{|\zeta_n' - \xi_n|}{\tau_n(\xi', M(\xi', \zeta))} \approx \frac{|\zeta_n' - \xi_n|}{\tau_n(\zeta', M(\zeta', \zeta))} \approx \frac{A_\kappa(z')^{1/\kappa}|\zeta_n' - \xi_n|}{M(\zeta', \zeta)^{1/\kappa}} \lesssim \left(\frac{M(\zeta', \xi)}{M(\zeta', \zeta)}\right)^{1/\kappa}.$$

Because $M(z', \xi)/M(z', \zeta) \le 1$ and $\kappa \ge 2$, we also have from (3.5) that

$$\tag{3.7} \frac{|\zeta_j' - \xi_j|}{\tau_j(\xi', M(\xi', \zeta))} \le \left(\frac{M(\zeta', \xi)}{M(\zeta', \zeta)}\right)^{1/\kappa}, \quad \text{for } j = 1, 2, \dots, n-1.$$

We get (3.1) if we combine (3.2), (3.3), (3.6) and (3.7). □

Theorem 3.3. *The Bergman kernel function $K_\Omega(z', z)$ associated to the domain Ω is a standard kernel with respect to the metric d and the Lebesgue measure.*

Proof. Let μ denote the Lebesgue measure. Minimally cover $\overline{\Omega}$ by open balls $B(a^j; \epsilon/2), j = 0, 1, \cdots, N$, as in the begining of this section. Since $K_\Omega(z', z)$ is smooth away from the boundary diagonal, there is a constant C such that $|K(z', z)| \le C$ if $z', z \in B_0$ or if $z' \in B(a^j; \epsilon/4)$ and $z \in B(a^k; \epsilon/4)$ for some $j \ne k$. Also, in this case, $(\mu(P(z', d(z', z)))^{-1} \le C'$. Thus it follows that

$$|K(z', z)| \lesssim \frac{1}{\mu(P(z', d(z', z)))}.$$

Now assume that $z', z \in B(a^j; \epsilon)$. Then, by Theorem 2.7 and the transformation formula for the Bergman kernel function, it follows that

$$\begin{aligned} |K(z', z)| &\lesssim (d(z', z))^{-n}(\tau(z', d(z', z)))^{-2} \\ &= \prod_{j=1}^{n} \tau_j(z', d(z', z))^{-2} \approx \frac{1}{\mu(P(z', d(z', z)))}, \end{aligned}$$

where $d(z', z)$ is associated to $B(a^j; \epsilon)$. Thus for any $z', z \in \Omega$,

$$|K(z', z)| \lesssim \frac{1}{\mu(P(z', d(z'z)))}.$$

Assume now that $z', w \in \Omega$ with $z' \ne w$. Then

$$\begin{aligned} \int_{d(z',z) > T d(z',w)} |K(z', z) - K(w, z)| dz &= \int_{A_\epsilon^T} |K(z', z) - K(w, z)| dz \\ &\quad + \int_{B_\epsilon^T} |K(z', z) - K(w, z)| dz \\ &:= \mathrm{I} + \mathrm{II}, \end{aligned}$$

where $A_\epsilon^T = \{z : d(z', z) > Td(z', w),\ |z' - z| \geq \epsilon/2\} \cap \Omega$ and $B_\epsilon^T = \{z : d(z', z) > Td(z', w),\ |z' - z| \leq \epsilon/2\} \cap \Omega$. Note that $|z - w| \geq \epsilon/4$ for $z \in A_\epsilon^T$ provided T is sufficiently large. Since $K(\cdot, \cdot)$ is smooth away from the boundary diagonal, it follows that I $\lesssim 1$. To estimate II, assume that $z' \in B(a^j; \epsilon/2)$ for some j. Thus $z', z, w \in B(a^j; \epsilon)$ for $z \in B_\epsilon^T$. Let $\Phi_{z'}$ be the biholomorphic map associated with z' as in Proposition 2.1 and set $\zeta = \Phi_{z'}^{-1}(z)$, $\xi = \Phi_{z'}^{-1}(w)$. Then

$$\begin{aligned} \mathrm{II} &= \int_{B(a^j;\epsilon)\cap B_\epsilon^T} |K_\Omega(z', z) - K_\Omega(w, z)| dz \\ &\lesssim \int_{\{\zeta : M(0,\zeta) > TM(0,\xi)\}} |K_{\Omega_{z'}}(0, \zeta) - K_{\Omega_{z'}}(\xi, \zeta)| d\zeta \ := \mathrm{III}. \end{aligned}$$

Set $\delta = M(0, \xi)$ and define the dyadic rings $D_k = \{\zeta : 2^k(T\delta) < M(0, \zeta) < 2^{k+1}(T\delta)\}$. Recall that $R_\delta(z') \approx P(\zeta', \delta) = \{\zeta : M(\zeta', \zeta) < \delta\}$. Thus it follows from Lemma 3.2 and the "doubling property" of the ball that

$$\begin{aligned} \mathrm{III} &\lesssim \int_{\cup D_k} \left(\frac{M(0, \xi)}{M(0, \zeta)} \right)^\nu \cdot \frac{1}{\mathrm{Vol}(P(0, M(0, \zeta)))} d\zeta \\ &= \delta^\nu \sum_{k=0}^\infty \int_{D_k} M(0, \zeta)^{-\nu} \frac{1}{\mathrm{Vol}(P(0, M(0, \zeta)))} \\ &\leq \sum_{k=0}^\infty \frac{\delta^\nu}{(2^k T\delta)^\nu} \cdot \frac{\mathrm{Vol}(P(0, 2^{k+1}T\delta))}{\mathrm{Vol}(P(0, 2^k T\delta))} \lesssim 1. \end{aligned}$$

□

Definition 3.4. Let (X, μ) be a measure space. An operator $T : L^p(X, \mu) \to$ { measurable functions on X} is said to be of weak type (p, p), $0 < p < \infty$, if

$$\mu\{x; |Tf(x)| > \lambda\} \leq C \frac{\|f\|_{L^p}^p}{\lambda^p}, \text{ all } f \in L^p,\ \lambda > 0,$$

where C is a constant independent of f and λ.

If we use Theorem 3.3 and the Calderón- Zygmund decomposition of Ω in terms of balls $P(z', \delta)$ (this is analogous to the Calderón-Zygmund decomposition of $\mathbb{R}^n$ in terms of standard cubes (see [3])), it is a routine matter to show that the Bergman kernel is of weak type (1,1) on Ω (cf. [3],[5]), and we get the following corollary:

Corollary 3.5. *The Bergman projection P is of weak type (1,1) on Ω.*

If we combine Theorem 3.3, Corollary 3.5 and the L^2-boundedness of P, it follows that the following theorem holds.

Theorem 3.6. *Let P be the Bergman projection associated to the domain Ω in Section 1. Then P maps $L^p(\Omega)$ to $L^p(\Omega)$, boundedly, for all $1 < p < \infty$.*

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