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# A MARKOV'S THEOREM FOR EXTENDED WELDED BRAIDS AND LINKS

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**Abstract**

Extended welded links are a generalization of Fenn, Rimányi, and Rourke's welded links. Their braided counterpart are extended welded braids, which are closely related to ribbon braids and loop braids. In this paper we prove versions of Alexander and Markov's theorems for extended welded braids and links, following Kamada's approach to the case of welded objects.

## 1. Introduction

**State of the art.** *Welded links* were introduced by Fenn-Rimányi-Rourke [8] as equivalence classes of link diagrams in the 2-dimensional space. They can be considered as virtual links up to additional Reidemeister moves called *forbidden moves of type* ($F1$). Satoh [18] considered the relation between welded links and *ribbon torus-links*. He extended a construction of Yajima [22], defining a surjective map $Tube$ from welded knots to ribbon torus-knots, which allows to associate to any ribbon torus-knot a welded knot. This fact suggested that ribbon torus-knots could be the topological counterparts of welded knots. However, the $Tube$ map is not injective: for instance, it is invariant under the *horizontal mirror image* on welded diagrams [11, Proposition 3.3] (see also [21, 18]), while welded links in general are not equivalent to their horizontal mirror images.

The braided counterparts of welded links are *welded braids*, also introduced in [8]. Welded braid groups can be seen as quotients of virtual braid groups. Fenn-Rimányi-Rourke proved that these groups are isomorphic to the groups of *braid-permutation automorphisms* of the free groups. They are also isomorphic to groups appearing in many other contexts, see for instance the surveys [7, 4]. In particular, they are isomorphic to the groups of *ribbon braids*, which are the braided counterparts of ribbon torus-links. In fact, on braided objects, the $Tube$ map is an isomorphism [1].

For welded braids and links we have versions of Alexander's and Markov's theorems, due to Kauffman-Lambropoulou [15] and to Kamada [12]. The isomorphism between welded braid groups and ribbon braid groups guarantees that Alexander's theorem for welded objects holds when passing to ribbon braids and ribbon torus-links. However, the lack of a bijection between welded links and ribbon torus-links impedes us to translate Markov's theorem for welded objects to ribbon braids and ribbon torus-links.

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**Motivation and contribution.** The aim of this note is to make a step towards a Markov's theorem for ribbon braids and ribbon torus-links. This is done by studying a class of objects that appear as suitable candidates to be a diagrammatical representation of ribbon torus-links, while remaining in the domain of usual link diagrams. In fact ribbon torus-links can be also represented by chord-diagrams, see for instance [14]. The objects we consider are an enhanced version of welded links: they are called *extended welded links* and have been introduced in [7]. In a certain sense, they can be seen as a quotient of welded links: in fact each extended welded link is equivalent to a welded link, but the introduction of certain marks on the diagrams, called *wen marks*, adds some moves to the generalized Reidemeister relations, making it possible that two non-equivalent welded links, when considered as extended welded links, become equivalent through moves involving wen marks.

There are two reasons that point to extended welded links as promising candidates to being diagrammatical representations for ribbon torus-links. The first reason is that extended welded link diagrams are equivalent to their sign reversal, and in consequence to their horizontal mirror image (Proposition 5.1). As for the second reason, let us consider *extended ribbon braids* by allowing *wens*, which are embeddings in the 4-dimensional space of a Klein bottle cut along a meridional circle, on the braided annuli that compose ribbon braids. The groups of extended ribbon braids appear when looking for a version of Markov theorem for ribbon braids and torus-links in $B^3 \times S^1$. In fact it can be proven that taken a pair of ribbon braids, their closures are isotopic as ribbon torus-links in $B^3 \times S^1$ if and only if they are conjugate as *extended* ribbon braids [6]. However, if one considers extended ribbon braids to begin with, the statement is the exact analogue of the usual case in dimension 3: taken a pair of extended ribbon braids, their closures are isotopic as ribbon torus-links in $B^3 \times S^1$ if and only if they are conjugate as extended ribbon braids. This is relevant to this paper because the groups of extended ribbon braids are isomorphic to the groups of *extended welded braids*, which are the braided counterpart of extended welded links [7, Theorem 6.12].

The main result of this paper is the following:

**Theorem 4.1.** *Two extended welded braid diagrams that admit closure have equivalent closures as extended welded link diagrams if and only if they are related by a finite sequence of the following moves:*
($M0$) *isotopy of* $\mathbb{R}^2$ *and generalized Reidemeister moves;*
($M1$) *conjugation in the extended welded braid group;*
($M2$) *a right stabilization of positive, negative or welded type, and its inverse operation.*

**Structure of the paper.** In Section 2 we introduce extended welded braid diagrams and the extended welded braid groups. We give a presentation for them and describe their relation with virtual and welded braids. In Section 3 we discuss extended welded links and give a combinatorial description for them in terms of Gauss data. We state a version of Alexander's Theorem for extended welded objects (Proposition 3.3) and state some results that allow us to use Gauss data to describe extended welded links. In Section 4 we prove the main result (Theorem 4.1). Finally, in Section 5 we show that extended welded knots are equivalent to their horizontal mirror images (Proposition 5.1).

## 2. Extended welded braids

An *extended welded braid diagram*, or *EW braid diagram* on $n$ strings is a planar diagram composed by a set of $n$ oriented and monotone 1-manifolds immersed in $\mathbb{R}^2$ starting from $n$ points on a horizontal line at the top of the diagram down to a similar set of $n$ points at the bottom of the diagram. The 1-manifolds are allowed to cross in transverse double points, which will be decorated in three kinds of ways, as shown in Figure 1. Depending on the decoration, double points will be called: *classical positive* crossings, *classical negative* crossings and *welded* crossings. On each 1-manifold there can possibly be marks as in Figure 2, which we will call *wen marks*. Remark that the inclination of wen marks is arbitrary and bears no meaning. Double points and wen marks are required to occur at different $y$-coordinates.

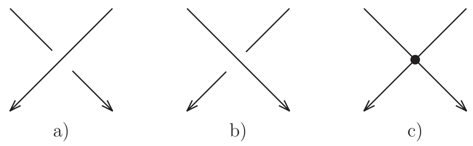


Fig. 1. a) Classical positive crossing, b) Classical negative crossing, c) Welded crossing.

Fig. 2. A wen mark on a strand.

An EW braid diagram determines a word in the elementary diagrams illustrated in Figure 3. We call $\sigma_i$ the elementary diagram representing the $(i + 1)$-th strand passing over the $i$-th strand, $\rho_i$ the welded crossing of the strands $i$ and $(i + 1)$, and $\tau_i$ the wen mark diagram.

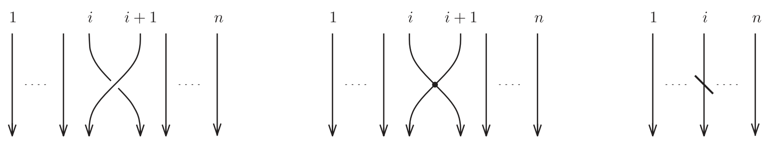


Fig. 3. Elementary diagrams $\sigma_i$, $\rho_i$, and $\tau_i$.

Definition 2.1. An *extended welded braid*, or *EW braid* to keep notation short, is an equivalence class of EW braid diagrams under the equivalence relation given by isotopy of $\mathbb{R}^2$ and the following moves:

- classical Reidemester moves (Figure 4);
- virtual Reidemeister moves (Figure 5);
- mixed Reidemeister moves (Figure 6);
- welded Reidemeister moves (Figure 7);
- extended Reidemester moves (Figure 8).

This equivalence relation is called *(braid) generalized Reidemeister equivalence*. For $n \geq 1$, the *extended welded braid group* on $n$ strands $WB_n^{ext}$ is the group of equivalence classes of EW braid diagrams by generalized Reidemeister equivalence. The group structure on these objects is given by: stacking and rescaling as product, braid mirror image as inverse, and the trivial diagram as identity.

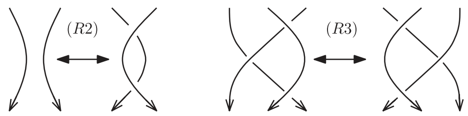


Fig. 4. Classical Reidemeister moves for braid-like objects.

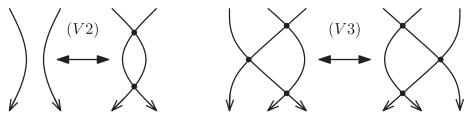


Fig. 5. Virtual Reidemeister moves for braid-like objects.

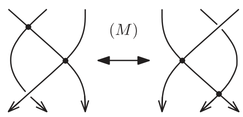


Fig. 6. Mixed Reidemeister moves.

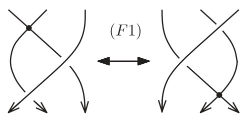


Fig. 7. Welded Reidemeister moves.

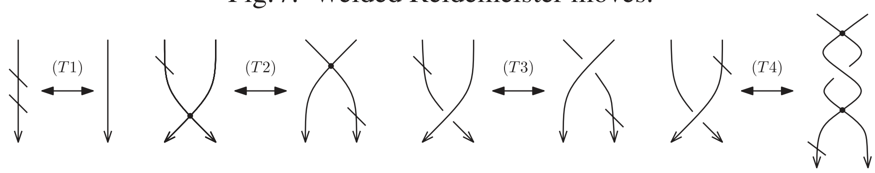


Fig. 8. Extended Reidemeister moves.

Remark 2.2. If wen marks were not allowed, the group defined would be the group of *welded braids $WB_n$*, introduced by Fenn, Rimányi and Rourke in [8]. This group is isomorphic to *loop braid groups $LB_n$*, *ribbon braid groups $rB_n$*, and many others. More precisely, we can see welded braids as the diagrams describing loop braids. For a survey on these groups, see [7]. For more details about loop braid groups seen in different contexts, this is a non-exhaustive list of references: [5, 3, 9, 16, 17, 19, 20].

EW braid groups $WB_n^{ext}$ are isomorphic to *ring groups* $R_n$ discussed in [3] (see [7] for a proof of the equivalence). Therefore, they admit a presentation given by the sets of generators $\{\sigma_i, \rho_i \mid i = 1, \ldots, n-1\}$ and $\{\tau_i \mid i = 1, \ldots, n\}$, subject to the following relations:

$$
\begin{cases}
\sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \ldots, n-2 \\
\rho_i\rho_j = \rho_j\rho_i & \text{for } |i-j| > 1 \\
\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1} & \text{for } i = 1, \ldots, n-2 \\
\rho_i^2 = 1 & \text{for } i = 1, \ldots, n-1 \\
\rho_i\sigma_j = \sigma_j\rho_i & \text{for } |i-j| > 1 \\
\rho_{i+1}\rho_i\sigma_{i+1} = \sigma_i\rho_{i+1}\rho_i & \text{for } i = 1, \ldots, n-2 \\
\sigma_{i+1}\sigma_i\rho_{i+1} = \rho_i\sigma_{i+1}\sigma_i & \text{for } i = 1, \ldots, n-2 \\
\tau_i\tau_j = \tau_j\tau_i & \text{for } i \neq j \\
\tau_i^2 = 1 & \text{for } i = 1, \ldots, n \\
\sigma_i\tau_j = \tau_j\sigma_i & \text{for } |i-j| > 1 \\
\rho_i\tau_j = \tau_j\rho_i & \text{for } |i-j| > 1 \\
\tau_i\rho_i = \rho_i\tau_{i+1} & \text{for } i = 1, \ldots, n-1 \\
\tau_i\sigma_i = \sigma_i\tau_{i+1} & \text{for } i = 1, \ldots, n-1 \\
\tau_{i+1}\sigma_i = \rho_i\sigma_i^{-1}\rho_i\tau_i & \text{for } i = 1, \ldots, n-1.
\end{cases}
\tag{2.1}
$$

**2.1. Extended Markov moves.** We introduce here some "moves" on EW braid diagrams. Let $b_1$ and $b_2$ be two EW braid diagrams.

- If $b_1$ and $b_2$ represent the same EW braid, then we say that $b_2$ is obtained from $b_1$ by a $(M0)$-*move*.
- If $b_1$ and $b_2$ have the same *degree* (number of strands), then the EW braid diagram $b_1b_2$ is obtained from the diagram $b_2b_1$ by *conjugation*, also called $(M1)$-*move*.
- Suppose $b_1$ has degree $n$. Then $\iota(b_1)$ is the diagram of degree $n+1$ obtained by adding a trivial strand to the right of $b_1$. If $b_2 = \iota(b_1)\sigma_n$, or $\iota(b_1)\sigma_n^{-1}$, or $\iota(b_1)\rho_n$, we say that $b_2$ is obtained from $b_1$ by *positive, negative, or welded stabilization*, also called $(M2)$-*move*.

We introduce also a third type of move, which is a direct consequence of moves $(M0)$ and $(M1)$, as we will prove in Proposition 3.10, but will be used in Section 5 to enlighten an important property that extended welded links have and (non extended) welded links do not.

- If $b_2$ is obtained from $b_1$ by replacing each classical crossing of $b_1$ with a crossing of the opposite sign conjugated by welded crossings as in Figure 9, we say that $b_2$ is obtained by *sign reversal*.

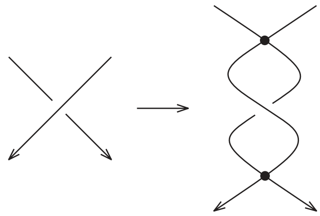

Fig. 9. Sign reversal.

## 3. Extended welded links and Gauss data

**3.1. Extended welded links.** An *extended welded link diagram*, or just *EW link diagram*, is the immersion in $\mathbb{R}^2$ of a collection of disjoint, closed, oriented 1-manifolds such that all multiple points are transverse double points. Double points are decorated with classical positive, classical negative, or welded information as in Figure 1. On each 1-manifod there can possibly be an even number of wen marks as in Figure 2. We assume that EW link diagrams are the same if they are isotopic in $\mathbb{R}^2$. For an EW link diagram $K$, we call *real crossings* its set of classical positive and classical negative crossings.

DEFINITION 3.1. An *extended welded link* or *EW link* is an equivalence class of EW link diagrams under the equivalence relation given by isotopies of $\mathbb{R}^2$, moves from Definition 2.1 with all possible orientations, and by classical and virtual Reidemeister moves $(R1)$ and $(V1)$ as in Figure 10. This equivalence relation is called *generalized Reidemeister equivalence*.

REMARK 3.2. The reason because wen marks on EW links can only appear with even parity on each component of an EW link diagram is motivated by the relation between EW objects and ext. ribbon objects. For more details, see [2, proof of Proposition 2.4].

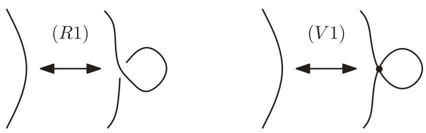


Fig. 10. Reidemeister moves of type I.

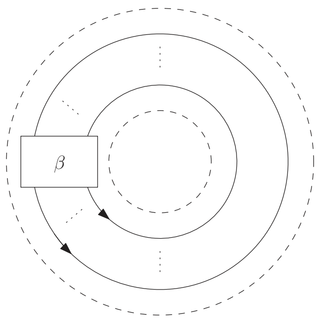


Fig. 11. Closure of an EW braid diagram.

The *closure* of an EW braid diagram is obtained as for usual braid diagrams (see Figure 11), with the condition that EW braids can be closed only when the link obtained has an even number of wen marks on each component.

**Proposition 3.3.** *Any EW link can be described as the closure of an EW braid diagram which is generalized Reidemeister equivalent to a (non-extended) welded braid diagram.*

Proof. Recall that an EW link can have only an even number of wen marks on each component. Then, taken $l$ to be an EW link, and $L$ an EW link diagram representing it, it is always possible to find a diagram $L'$ without wen marks. This is done by making one wen mark slide along the component it belongs to, in order to make it adjacent to another wen mark, and the cancelling the wen marks pairwise. For an example, see Figure 12. Then it is enough to prove that any welded link can be described as the closure of a welded braid. This is done in [12, Proposition 8] and [15, Theorem 1].

Alternatively, this result can be proved directly by giving a braiding algorithm, as the one we present in Subsection 3.2. □

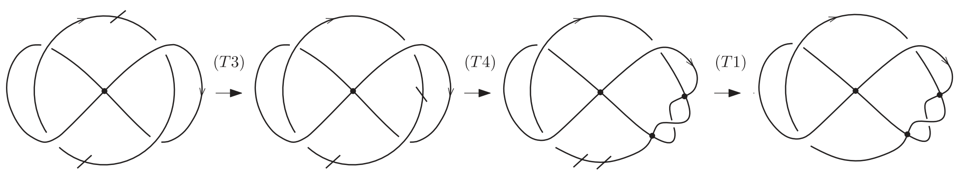


Fig. 12. Sliding wen marks along a diagram in order to obtain a wen marks-free diagram.

**3.2. Gauss data.** We can associate to an EW knot diagram a *Gauss data*. This is a combinatoric description of a Gauss diagram, as intended in the spirit of [10]. In the context of EW diagrams, Gauss data needs to contain the added information for wen marks. We recall and adapt here the description of Gauss data given in [12]. Let $K$ be an EW link diagram. We introduce some notation.

- We denote by $C_K$ the set of positive and negative crossings of $K$, also called the set of *real crossings*.
- We define a sign map $S_K\colon C_K \to \{-1, 1\}$ on the set real crossings, sending positive crossings to 1 and negative crossings to $-1$.
- For a real crossing $c \in C_K$, let $N_c$ be a regular neighbourhood of $c$; we denote by $W_K$ the closure of $\mathbb{R}^2 \setminus \bigcup_{c\in C_K} N_c$, and by $K\,|_{W_K}$ the restriction of $K$ to $W_K$.
- We denote by $c^1, c^2, c^3$ and $c^4$ the four points that compose $\partial N(c) \cap K$, and by $C_K^\partial$ the set $\{c^i \mid c \in C_K, i \in \{1, \ldots, 4\}\}$. See Figure 13.

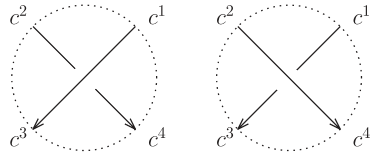


Fig. 13. Intersection points $c^1, c^2, c^3$ and $c^4$ composing $\partial N(c) \cap K$.

Define a subset $G_K \subset C_K^\partial \times C_K^\partial \times \mathbb{Z}/2\mathbb{Z}$ such that $(a, b, n + 2\mathbb{Z}) \in G_K$ if and only if the restriction of $K$ to $W_K$ has an oriented arc starting from $a$ and terminating at $b$, and $n$ is the number of wen marks on it. Elements $(a, b, 0 + 2\mathbb{Z})$ and $(a, b, 1 + 2\mathbb{Z})$ will be respectively denoted by $(a, b)$ and $\overline{(a, b)}$. Finally, we denote by $\mu_K$ the number of components of $K$.

Definition 3.4. The Gauss data of $K$ is the quadruple $G(K) = (C_K, S_K, G_K, \mu_K)$. We say that two EW link diagrams $K$ and $K'$ have the *same Gauss data* if they have the same number of components $\mu_K = \mu_{K'}$ and if there is a bijection $g\colon C_K \to C_{K'}$ such that $g$ preserves the signs of the crossings, and the presence of welded marks. This means that if $(a, b)$ is in $G_K$, then $(g(a), g(b))$ is in $G_{K'}$, and if $\overline{(a, b)}$ is in $G_K$, then $\overline{(g(a), g(b))}$ is in $G_{K'}$.

Example 3.5. Let us compute the Gauss data for link $L_1$ in Figure 14. Its Gauss data is given by:

- $C_{L_1} = \{c_1, c_2, c_3\}$;
- $S_{L_1} = c_1 \longmapsto +1, c_2 \longmapsto +1, c_3 \longmapsto +1$;
- $G_{L_1} = \{(c_2^3, c_1^2), \overline{(c_1^4, c_2^2)}, (c_2^4, c_3^1), (c_3^3, c_1^1), \overline{(c_1^3, c_2^1)}, (c_3^4, c_3^2)\}$;
- $\mu_{L_1} = 2$.

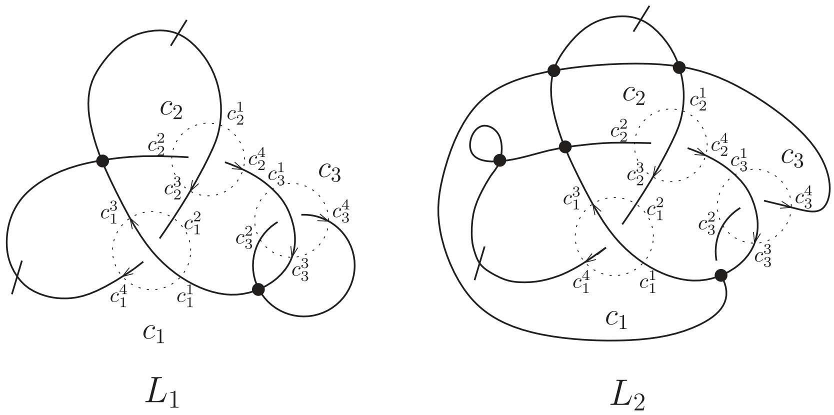


Fig. 14. Two EW link diagram with the same Gauss data.

Let $K$ be an EW link diagram. We say that $K'$ is the EW link diagram obtained from $K$ by *replacing* $K\,|_{W_K}$ if the following conditions are satisfied:

(1) $K$ and $K'$ are equal in $N_c$, for all $c \in C_K$;
(2) $K'$ has no real crossings in $W_{K'}$;
(3) there is a one-to-one correspondence between the arcs of $K\,|_{W_K}$ and the arcs of $K'\,|_{W_{K'}}$, preserving endpoints, orientation and the parity of the numbers of wen marks;
(4) there is a one-to-one correspondence between the loops of $K\,|_{W_K}$ and those of $K'\,|_{W_{K'}}$.

Note that an EW link diagram $K'$ has the same Gauss data as $K$ if and only if $K'$ can be deformed through an isotopy of $\mathbb{R}^2$ such that it is obtained from $K$ by replacing $K\,|_{W_K}$.

**Lemma 3.6.** *Let $K$ and $K'$ be two EW links diagrams with the same Gauss data. Then $K$ and $K'$ are equivalent. Moreover one can be obtained from the other by isotopy of $\mathbb{R}^2$ and a* *finite sequence of moves of type* $(V1)$, $(V2)$, $(V3)$, $(M)$, $(T1)$, *and* $(T2)$.

Proof. The proof is the same as the one given for the virtual case in [12, Lemma 4]. It should be noted that there is an unsaid difference in the fact that in the extended welded context saying that two diagrams have the same Gauss data implies one condition more than in the virtual and welded cases. This is the condition of preserving wen marks information on the elements of $G_K$, which is solved by moves $(T1)$ and $(T2)$. Compare Definition 3.4 and virtual Gauss data definition in [12, Section 4]. Other proofs for the virtual case can be found in [10, 13]. □

We introduce now a particular kind of EW link diagrams.

Definition 3.7. An EW link diagram is said to be *braided* if there exists a point on the plane with respect to which the EW link diagram is braided around. More formally: we can assume the point to be the origin $O$ of $\mathbb{R}^2$, and identify $\mathbb{R}^2 \setminus \{O\}$ with $\mathbb{R}_+ \times S^1$ passing to polar coordinates and considering $S^1$ to be oriented anti-clockwise. Let us consider $\pi_2 \colon \mathbb{R}_+ \times S^1 \to S^1$ to be the standard projection to the second factor. Then $L$ is a *braided EW link diagram* if:

(i) $L$ is contained in $\mathbb{R}_+ \times S^1$;
(ii) each component of $L$ is monotone with respect to the coordinate in $S^1$;
(iii) the restriction $\pi_2 |_{C_L}$ is injective.

A braided EW link diagram can be represented as the closure of an EW braid diagram, and the representation is unique up to conjugation ($(M1)$-move) of EW braid diagrams.

For completeness, we quickly recall here the *braiding process* exposed in [12] for virtual link diagrams, which can be adapted without modification to EW diagrams. Let $L$ be an EW link diagram, and let $N_1, N_2, \ldots, N_n$ be regular neighbourhoods of its real crossings. With an isotopy of $\mathbb{R}^2$ we deform $L$ in such a way so that: all the $N_i$s are in $\mathbb{R}^2 \setminus \{O\}$; $\pi_2(N_i) \cap \pi_2(N_j) = \emptyset$ for $i \neq j$ in $\{1, \ldots, n\}$; each $N_i$ contains two oriented arcs, each of which is mapped to $S^1$ by $\pi_2$ homeomorphically with respect to the orientation of $S^1$. Finally replace $L |_{W_L}$ arbitrarily such that the result is a braided EW link diagram, which is equivalent to $L$ (Lemma 3.6).

In the following we will apply the extended Markov moves defined in Subsection 2.1 to braided EW link diagrams in the natural way. We will still denote them by $(M0)$ and $(M2)$.

**Lemma 3.8.** *Let $K$ and $K'$ be braided EW link diagrams, possibly with a different degree as braids, such that $K'$ is obtained from $K$ by replacing $K |_{W_K}$. Then $K$ and $K'$ are related by a finite number of moves of type* $(M0)$ *and* $(M2)$.

Proof. The proof is the same as the one given for the virtual case in [12, Lemma 5], once more with the hidden difference that arcs or loops of $K |_{W_K}$ and $K' |_{W_{K'}}$ could have wen marks. However, the operation of replacing the restrictions for EW diagrams preserves the presence of wen marks, leaving the proof unchanged. □

Remark 3.9. Moves of type $(M1)$ do not appear in Lemma 3.8 because, as stated before, braided EW link diagrams are defined as closures of EW braid diagrams up to $(M1)$-moves.

**3.3. About the sign reversal move.** Among the extended Markov moves introduced in Subsection 2.1, the most exotic one is the sign reversal move. As noted in [2, Notation 2.6], the sign reversal move on an EW link diagram changes its Gauss data by reversing the signs associated to the crossings and leaving everything else unchanged. We refer to the effect of the sign reversal move on the Gauss data as the *sign reversal of the Gauss data*.

**Proposition 3.10.** *The following statements hold:*

(1) *For EW link diagrams, sign reversal is a consequence of moves* $(T1) - (T4)$.
(2) *For braided EW link diagramns, sign reversal is a consequence of moves of type* $(M0)$.
(3) *For EW braid diagrams, sign reversal is a consequence of moves of type* $(M0)$ *and* $(M1)$.

Proof.

(1) Remark that when move $(T4)$ is applied to a crossing, it locally induces a sign reversal on that crossing. Let $L$ be an EW link. It is always possible to introduce a pair of wen marks on each components through a $(T1)$ move. Then, we make one wen mark of each pair slide along the component until it comes back to the starting point, on the other side of the other wen mark of the pair. In such a way it will have passed once on the overstrand of each crossing in which the component is involved as an overstrand, conjugating the crossings by welded crossings and changing the sign of the crossing. Then we cancel the pairs of wen marks added through an inverse $(T1)$ move. The resulting diagram $L'$ is the sign reversal of $L$, and is equivalent to $L$ through the moves $(T1) - (T4)$.
(2) Same as point 1.
(3) This is a direct consequence of point 2, since moves $(M1)$ allow to pass wen marks from the bottom to the top of a diagram.

□

**Lemma 3.11.** *Two braided EW link diagrams with the same Gauss data are related by a finite sequence of Markov moves of types* $(M0)$ *and* $(M2)$*. If the Gauss data of one braided EW link diagram is the sign reversal of the Gauss data of the other, then the two diagrams are related by a finite number of Markov moves of type* $(M0)$ *and* $(M2)$.

Proof. The first part is the same as in [12, Lemma 6]. We recall it here for completeness. Let $L$ and $L'$ be braided EW link diagrams with the same Gauss data. Let $c_1, \dots, c_n$ be the real crossings of $L$, and $N_1, \dots, N_n$ the relative regular neighbourhoods. In the same way let $c'_1, \dots, c'_n$ be the real crossings of $L'$ and $N'_1, \dots, N'_n$ their neighbourhoods. Let us distinguish two cases, depending on the order of the images of the crossings through $\pi_2$ (Definition 3.7).

(a) Suppose the crossings $\pi_2(N_1), \dots, \pi_2(N_n)$ and $\pi_2(N'_1), \dots, \pi_2(N'_n)$ appear on $S^1$ in the same cyclic order. Then, with an isotopy of $\mathbb{R}^2$ we can deform $L$ keeping the braidedness, in such a way that $N_i$ and $N'_i$ coincide for $i = 1, \dots, n$, and the restrictions of $L$ and $L'$ to these disks are identical. By Lemma 3.8, one can pass from $L$ to $L'$ with a finite number of Markov moves of type $(M0)$ and $(M2)$.
(b) Suppose that $\pi_2(N_1), \dots, \pi_2(N_n)$ and $\pi_2(N'_1), \dots, \pi_2(N'_n)$ do not appear on $S^1$ in the same cyclic order. It is enough to treat the case when only a pair, for example

$\pi_2(N_1)$ and $\pi_2(N_2)$, is exchanged. Then, with a finite sequence of moves $(M_0)$ and $(M_1)$ (details in [12, Lemma 6]) it is possible to move one crossing, preserving at each step the braidedness and the Gauss data, in order to reconduct ourselves to case (a).

Let us now consider the case in which $L$ and $L'$ are braided EW link diagrams with sign reversed Gauss data. We consider two cases as before.

(a) Suppose the crossings $\pi_2(N_1), \ldots, \pi_2(N_n)$ and $\pi_2(N'_1), \ldots, \pi_2(N'_n)$ appear on $S^1$ in the same cyclic order. Again with an isotopy of $\mathbb{R}^2$ we can deform $L$ keeping the braidedness, in such a way that $N_i$ and $N'_i$ coincide for $i = 1, \ldots, n$, but the restrictions of $L$ and $L'$ to these disks present opposite crossings. Let us apply to $L$ a sign reversal. Then we obtain a braided EW diagram $L''$ which is equivalent to $L$ via $(M0)$ moves by Proposition 3.10, whose crossings regular neighbourhoods $N''_1, \ldots, N''_n$ contain a real crossing conjugated by welded crossings. With an isotopy of $\mathbb{R}^2$, push the welded crossings outside of the regular neighbourhood, deforming $L''$ in such a way that $N''_i$ and $N'_i$ coincide for $i = 1, \ldots, n$, and the restrictions of $L''$ and $L'$ to these disks are identical. Then by Lemma 3.8, one can pass from $L''$ to $L'$ with a finite number of Markov moves of type $(M0)$ and $(M2)$.

(b) Suppose the crossings $\pi_2(N_1), \ldots, \pi_2(N_n)$ and $\pi_2(N'_1), \ldots, \pi_2(N'_n)$ appear on $S^1$ in the same cyclic order. Then with the manoeuvre recalled in point 3.3 of the first part, we reconduct this case to point (a) of this part.

□

The following is a corollary of the first part of Lemma 3.11, which is a direct consequence of the fact that the braiding process does not change the Gauss data. In the case of virtual link diagrams, it appears in [12, Corollary 7].

**Corollary 3.12.** *For an EW link diagram $K$, a braided EW link diagram obtained by the braiding process is uniquely determined up to EW Markov moves $(M0)$ and $(M2)$.*

## 4. A Markov theorem for welded extended diagrams

**Theorem 4.1.** *Two EW braid diagrams that admit closure (i.e., the links obtained have an even number of wen marks on each component) have equivalent closures as EW link diagrams if and only if they are related by a finite sequence on the following moves:*
$(M0)$ *isotopy of $\mathbb{R}^2$ and generalized Reidemeister moves;*
$(M1)$ *conjugation in the EW braid group $WB_n^{ext}$;*
$(M2)$ *a right stabilization of positive, negative or welded type, and its inverse operation.*

Proof. Let $b$ and $b'$ be EW braid diagrams that admit closure related by $(M0)$ moves. Then their closures $\widehat{b}$ and $\hat{b}'$ are equivalent EW link diagrams for definition of generalized Reidemeister equivalence. Suppose that $b$ and $b'$ are related by moves of type $(M1)$ and $(M2)$: then also in these cases their closures $\hat{b}$ and $\hat{b}'$ are clearly equivalent as EW link diagrams.

On the other hand, let $K$ and $K'$ be EW link diagrams representing the same exteded welded link. Then there is a finite sequence of EW link diagrams $K = K_0, K_1, \ldots, K_s = K'$ such that $K_i$ is obtained from $K_{i-1}$ by oriented generalized Reidemeister moves of kind

$(R1a)$, $(R1b)$, $(V1)$, $(R2a)$, $(R2b)$, $(R2c)$, $(R2d)$, $(V2a)$, $(V2b)$, $(V2c)$, $(R3)$, $(V3)$, $(M)$, $(F1)$, $(T1)$, $(T2)$, $(T3)$, or $(T4)$, as shown in Figure 15 [12, Proposition 11]. Applying the braiding process to each $K_i$, we obtain a braided EW link $\widetilde{K}_i$ with the same Gauss data as $K_i$. By Corollary 3.12, $\widetilde{K}_i$ is uniquely determined up to moves $(M0)$ and $(M2)$. To prove the statement it is enough to check that for each $i = 1, \dots, s$, $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are equivalent up to moves $(M0) - (M2)$. All the moves except $(T1)$, $(T2)$, $(T3)$ and $(T4)$ are considered in [12, Theorem 2]. So let us consider the remaining cases. Suppose that $K_i$ is obtained by $K_{i-1}$ by a $(T1)$, $(T2)$, $(T3)$ or $(T4)$ move. Then, let $\Delta$ be a 2-disk in $\mathbb{R}^2$ that contains one of these moves, and let $\Delta^c$ be such that $K_i \cap \Delta^c = K_{i-1} \cap \Delta^c$. Deform $K_i$ and $K_{i-1}$ by an isotopy of $\mathbb{R}^2$ in such a way that $K_i \cap \Delta$ and $K_{i-1} \cap \Delta$ satisfy the condition to be a braided EW link diagram. Applying the braiding process to $K_i \cap \Delta^c$ and $K_{i-1} \cap \Delta^c$ we obtain diagram $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ such that:

$$\widetilde{K}'_i \cap \Delta = K_i \cap \Delta \quad \text{and} \quad \widetilde{K}'_{i-1} \cap \Delta = K_{i-1} \cap \Delta \quad \text{and} \quad \widetilde{K}'_i \cap \Delta^c = \widetilde{K}'_{i-1} \cap \Delta^c$$

Then $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ are related by a $(M0)$ move corresponding to $(T1)$, $(T2)$, $(T3)$, or $(T4)$. Since $\widetilde{K}'_i$ has the same Gauss data as $K_i$, it is equivalent by Markov moves to $\widetilde{K}_i$ (Lemma 3.11). Same for $\widetilde{K}'_{i-1}$ and $\widetilde{K}_i$. Therefore $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are equivalent by Markov moves. □

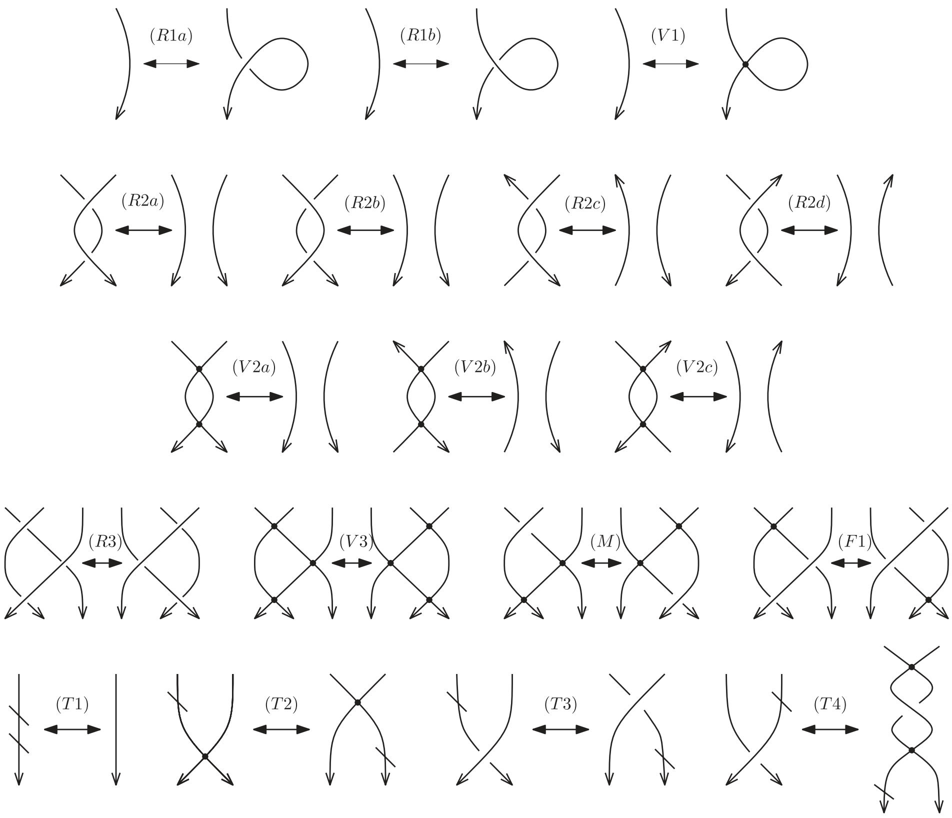


Fig. 15. Oriented generalized Reidemeister moves.

## 5. Extended welded knots and horizontal mirror images

Let $K$ be an EW diagram. Its *horizontal mirror image* $K^\dagger$ is its reflection with respect to a line on the plane of the diagram, as in Figure 16. In this section we show that EW knots are equivalent to their horizontal mirror images. In particular, we show that the horizontal mirror image $K^\dagger$ of an EW knot diagram $K$ is equivalent to the sign reversal of $K$.

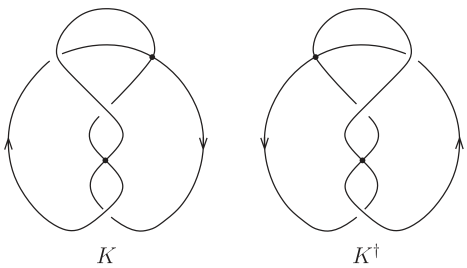


Fig. 16. An extened welded knot diagram $K$ and its horizontal mirror image $K^\dagger$.

**Proposition 5.1.** *Every EW knot diagram is equivalent to its horizontal mirror image.*

Proof. Consider an EW knot diagram $K$. Let us denote by $sK$ the sign reversal of $K$. The Gauss data of $sK$ is obtained by changing all the sign in K's Gauss data. It is easy to see that $K^\dagger$ has the same Gauss data as $sK$ (also shown in [11, Section 2.2]). By Lemma 3.6, $sK$ and $K^\dagger$ are equivalent. Then, $K^\dagger$ is equivalent to $K$ by Proposition 3.10. ◻

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