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# A MASLOV-TYPE INDEX THEORY FOR SYMPLECTIC PATHS

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## 1. Introduction

In this paper, we extend the Maslov-type index theory defined in [7], [15], [10], and [18] to all continuous degenerate symplectic paths, give a topological characterization of this index theory for all continuous symplectic paths, and study its basic properties.

Suppose $\tau > 0$. We consider an $\tau$-periodic symmetric continuous $2n \times 2n$ matrix function $B(t)$, i.e. $B \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$ with $S_\tau = \mathbb{R}/(\tau\mathbb{Z})$, $\mathcal{L}(\mathbb{R}^{2n})$ being the set of all real $2n\times 2n$ matrices, and $\mathcal{L}_s(\mathbb{R}^{2n})$ being the subset of all symmetric matrices. It is well-known that the fundamental solution $\gamma$ of the linear first order Hamiltonian system

$$\dot{y} = JB(t)y, \quad y \in \mathbb{R}^{2n}, \tag{1.1}$$

yields a path in the symplectic group $\mathrm{Sp}(2n) = \{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^T JM = J\}$ starting from the identity matrix, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix on $\mathbb{R}^n$. When there is no confusion we shall omit the subindex $n$ of $I_n$. Define $\mathrm{Sp}(2n)^0 = \{M \in \mathrm{Sp}(2n) \mid \det(M - I) = 0\}$ and $\mathrm{Sp}(2n)^* = \mathrm{Sp}(2n)\backslash\mathrm{Sp}(2n)^0$. In order to study such problems, we introduce the following families of paths in

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Sp$(2n)$. For $\tau > 0$, we define

$$\text{(1.2)}\quad \mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I\},$$

$$\text{(1.3)}\quad \mathcal{P}_\tau^*(2n) = \{\gamma \in \mathcal{P}_\tau(2n)) \mid \gamma(\tau) \in \mathrm{Sp}(2n)^*\}, \quad \mathcal{P}_\tau^0(2n) = \mathcal{P}_\tau(2n)\backslash\mathcal{P}_\tau^*(2n).$$

Paths in $\mathcal{P}_\tau^*(2n)$ and $\mathcal{P}_\tau^0(2n)$ are called *nondegenerate* and *degenerate* respectively. In order to study periodic solutions of nonlinear Hamiltonian systems for $\tau > 0$ we define

$$\text{(1.4)}\quad \widehat{\mathcal{P}}_\tau(2n) = \{\gamma \in C^1([0,\tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I,\ \dot\gamma(1) = \dot\gamma(0)\gamma(1)\},$$

$$\text{(1.5)}\quad \widehat{\mathcal{P}}_\tau^*(2n) = \widehat{\mathcal{P}}_\tau(2n) \cap \mathcal{P}_\tau^*(2n), \quad \widehat{\mathcal{P}}_\tau^*(2n) = \widehat{\mathcal{P}}_\tau(2n)\backslash\widehat{\mathcal{P}}_\tau^*(2n).$$

Note that $B = -J\dot\gamma(\,\cdot\,)\gamma^{-1}(\,\cdot\,) \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$ if and only if $\gamma \in \widehat{\mathcal{P}}_\tau(2n)$, i.e. the path family $\widehat{\mathcal{P}}_\tau(2n)$ is formed by fundamental solutions of all linear Hamiltonian systems (1.1) with continuous symmetric and $\tau$-periodic coefficients.

Basing upon the topological structures of the symplectic group Sp$(2n)$ and its subsets, in this paper we give a complete homotopy classification of all paths in $\mathcal{P}_\tau(2n)$. Basing on this homotopy classification, we define an index theory for all such paths. Specially this index theory assigns to each $\gamma \in \mathcal{P}_\tau(2n)$ a pair of integers $(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, \ldots, 2n\}$. This index theory also gives a finite representation of the Morse index theory of indefinite functionals corresponding to the Hamiltonian systems. To make the concepts of this homotopy classification for these paths and the Maslov-type index precise, we need the following definitions and notations.

Given any two matrices of square block form:

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \qquad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

we define an operation $\diamond$-product of $M_1$ and $M_2$ to be the $2(i+j) \times 2(i+j)$ symplectic matrix $M_1 \diamond M_2$ given by

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

We denote by $M^{\diamond k}$ the $k$-fold $\diamond$-product $M \diamond \ldots \diamond M$. Note that the $\diamond$-multiplication is associative, and the $\diamond$-product of any two symplectic matrices is still symplectic.

For $\tau > 0$ and any two paths $f : [0,\tau] \to \mathrm{Sp}(2n)$ and $g : [0,\tau] \to \mathrm{Sp}(2n)$ with $f(\tau) = g(0)$, we define their joint path by

$$g{*}f(t) = \begin{cases} f(2t) & 0 \le t \le \tau/2, \\ g(2t-\tau) & \tau/2 \le t \le \tau. \end{cases}$$

We define $D(a) = \mathrm{diag}(a, a^{-1})$ for $a \in \mathbb{R}\backslash\{0\}$. For $\theta$ and $b \in \mathbb{R}$ we define

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad N_1(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{1.6}$$

Definition 1.1. For any $\gamma \in \mathcal{P}_\tau(2n)$, let us define

$$\nu_\tau(\gamma) = \dim_{\mathbb{R}} \ker_{\mathbb{R}}(\gamma(\tau) - I). \tag{1.7}$$

In the following we always work on the field $\mathbb{R}$ and shall omit the notation for it. By definition, a path $\gamma \in \mathcal{P}_\tau(2n)$ is nondegenerate if and only if $\nu_\tau(\gamma) = 0$.

Definition 1.2. Given two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$, if there is a map $\delta \in C([0,1]\times[0,\tau], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot)$, $\delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I$, and $\nu_\tau(\delta(s,\cdot))$ is constant for $0 \le s \le 1$, then $\gamma_0$ and $\gamma_1$ are *homotopic on* $[0,\tau]$ *along* $\delta(\cdot,\tau)$ and we write $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $\delta(\cdot,\tau)$. This homotopy possesses *fixed end points* if $\delta(s,\tau) = \gamma_0(\tau)$ for all $s \in [0,1]$.

This topological concept of the homotopy of paths was first introduced in [Lo1]. Its analytical meaning is given by Lemma 4.2 together with Theorem 4.1 and (5.2) below.

The Maslov-type index theory definded for any nondegenerate sympletic paths in $\mathcal{P}_\tau^*(2n)$ by [7] and [15] are well known. Based on these results, our index for degenerate paths is characterized by the following theorem.

Theorem 1.3. *For* $\gamma \in \mathcal{P}_\tau^0(2n)$ *we obtain*

$$\begin{aligned} i_\tau(\gamma) = \inf\{i_\tau(\beta) \mid \beta \in \mathcal{P}_\tau^*(2n) \\ \text{and } \beta \text{ is sufficiently } C^0\text{-close to } \gamma \text{ in } \mathcal{P}_\tau(2n)\}, \end{aligned} \tag{1.8}$$

*where the topology of* $\mathcal{P}_\tau(2n)$ *is the* $C^0$*-topology induced from the topology of* $\mathrm{Sp}(2n)$.

The following theorem characterizes the Maslov-type index on any continuous symplectic paths in $\mathcal{P}_\tau(2n)$.

Theorem 1.4. *The index part of the Maslov-type index theory*

$$i_\tau : \bigcup_{n\in\mathbb{N}} \mathcal{P}_\tau(2n) \to \mathbb{Z},$$

*is uniquely determined by the following five axioms:*

1° (Homotopy invariant) *For* $\gamma_0$ *and* $\gamma_1 \in \mathcal{P}_\tau(2n)$, *if* $\gamma_0 \sim \gamma_1$ *on* $[0,\tau]$, *then*

$$i_\tau(\gamma_0) = i_\tau(\gamma_1). \tag{1.9}$$

2° ($\diamond$-product additivity) *For any* $\gamma_i \in \mathcal{P}_\tau(2n_i)$ *with* $i = 0$ *and* $1$, *we obtain*

$$i_\tau(\gamma_0 \diamond \gamma_1) = i_\tau(\gamma_0) + i_\tau(\gamma_1). \tag{1.10}$$

3° (Clockwise continuity) *For any* $\gamma \in \mathcal{P}_\tau^0(2)$ *with* $\gamma(\tau) = N_1(b)$ *for* $b = \pm 1$ *or* 0 *defined by* (1.6), *there exists a* $\theta_0 > 0$ *such that*

$$i_\tau([\gamma(\tau)\phi_{\tau,-\theta}] * \gamma) = i_\tau(\gamma), \quad \forall\, 0 < \theta \leq \theta_0, \tag{1.11}$$

*where* $\phi_{\tau,\theta}(t) = R(\theta t/\tau)$ *for* $0 \leq t \leq \tau$.

4° (Counterclockwise jumping) *For any* $\gamma \in \mathcal{P}_\tau^0(2)$ *with* $\gamma(\tau) = N_1(b)$ *for* $b = \pm 1$ *defined by* (1.6), *there exists a* $\theta_0 > 0$ *such that*

$$i_\tau([\gamma(\tau)\phi_{\tau,\theta}] * \gamma) = i_\tau(\gamma) + 1, \quad \forall 0 < \theta \leq \theta_0. \tag{1.12}$$

5° (Normality) *For the path* $\widehat{\alpha}_{1,0,\tau}(t) = D((\tau + t)/\tau)$ *with* $0 \leq t \leq \tau$, *there holds*

$$i_\tau(\widehat{\alpha}_{1,0,\tau}) = 0. \tag{1.13}$$

As a direct consequence of our study, for any end point free curve in the symplectic group, we can also define its index as follows.

DEFINITION 1.5. For any curve $f \in C([a,b], \mathrm{Sp}(2n))$, let us choose $\xi \in \mathcal{P}_1(2n)$ so that $\xi(1) = f(a)$. Let us define $\eta(t) = \xi(2t)$ for $t \in [0, 1/2]$, and $\eta(t) = f(a + (2t-1)(b-a))$ for $t \in [1/2, 1]$. Let us define

$$i(f) = i_1(\eta) - i_1(\xi). \tag{1.14}$$

Our studies of the index $i_1$ show that the above definition depends only on $f$ itself, and therefore is well defined. As a direct consequence of Theorem 1.4 we obtain the following characterization of the index $i$, whose proof is omitted.

COROLLARY 1.6. *The index* $i$ *defined above for continuous curves in the symplectic groups is uniquely determined by the following five axioms:*

1° (Homotopy invariant) *Two continuous curves in* $\mathrm{Sp}(2n)$ *with the same initial and end points possess the same index if and only if they can be continuously deformed to each other with the end points fixed.*

2° (Vanishing) $i(f) = 0$ *for any* $f \in C([a,b], \mathrm{Sp}(2n))$ *with* $\dim\ker(f(t) - I) = \text{constant}$.

3° ($\diamond$-product additivity) $i(f_0 \diamond f_1) = i(f_0) + i(f_1)$ *for any curve* $f_i \in C([a,b], \mathrm{Sp}(2n_i))$ *with* $i = 0, 1$.

4° (Catenation) $i(f) = i(f|_{[a,b]}) + i(f|_{[b,c]})$ *for any* $f \in C([a,c], \mathrm{Sp}(2n))$ *with* $a < b < c$.

5° (Normality) $i(f_{r,\theta}|_{[-1,0]}) = 0$ *and* $i(f_{r,\theta}|_{[0,1]}) = 2/r$ *for* $f_{r,\theta}(t) = D(r) \cdot R(\theta + t/2)$ *with* $(r,\theta) = (1,0)$ *or* $(2, \pm\cos^{-1}(4/5))$ *and* $t \in [-1, 1]$.

Historically as far as the author knows, this index theory for nondegenerate continuous paths starting from the identity matrix $I$ in $\mathrm{Sp}(2n)$ was established by C. Conley and E. Zehnder in [7] for $n \geq 2$ in 1984, and by E. Zehnder and the author in [15] in 1990 for $n = 1$. For the degenerate paths which are fundamental solutions of the linear Hamiltonian system with symmetric continuous and $\tau$-periodic coefficients, this index theory was established by the author in [10] and by C. Viterbo in [18] simultaneously in 1990 independently by different methods. In the current paper we generalize these above mentioned results to all continuous symplectic paths including those degenerate ones. This extension is based on our understanding of the structure of and near $\mathrm{Sp}(2n)^0$. This index theory establishes a solid background for the Morse theoritical study of Hamiltonian analysis and symplectic geometry. Note that the index theory $\mu(\Psi) = \mu(\Psi V, V)$ with $V = 0 \times \mathbb{R}^n$ defined and studied by J. Robbin and D. Salamon in [16] for any symplectic path $\Psi : [0, 1] \to \mathrm{Sp}(2n)$ is different from the one discussed in this paper, since they have chosen a singular hypersurface $\overline{\mathrm{Sp}_1}(2n)$ in $\mathrm{Sp}(2n)$ which is different from our $\mathrm{Sp}(2n)^0$. Note that our method is elementary and is different from those of [3], [4], [5], [8], [16], and [18]. We shall further discuss these points in forthcoming papers.

In the following Section 2, we briefly recall the definition of the index theory for the nondegenerate paths, and define it for degenerate paths in $\mathrm{Sp}(2)$. In order to establish the index theory for the degenerate paths in general case, we first construct rotation perturbation paths for given degenerate paths and study their basic properties in Section 3. Then in Section 4, we study the variational properties of these perturbation paths, and define the Maslov-type index theory for fundamental solutions of linear Hamiltonian systems. Basing upon these preparations we are able to establish the index theory for any continuous degenerate paths in the symplectic group in Section 5. Finally, in Section 6, we study various properties of the Maslov-type index theory, and give the proof of the above Theorems 1.3 and 1.4.

## 2. Maslov-type indices for nondegenerate paths

In this section we briefly recall the definition of the Maslov-type index theory for nondegenerate paths in the symplectic group $\mathrm{Sp}(2n)$ with $n \geq 1$ given in [7] and [15], and give detailed proofs of certain properties of this index theory which have not been rigorously proved before.

As it is well known, every $M \in \mathrm{Sp}(2n)$ has its unique polar decomposition $M = AU$, where $A = (MM^T)^{1/2}$ is symmetric positive definite and symplectic, $U$ is orthogonal and symplectic. Therefore $U$ has the form

$$U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix},$$

where $u = u_1 + \sqrt{-1}u_2 \in \mathcal{L}(\mathbb{C}^n)$ is a unitary matrix. So for every path $\gamma$ in $\mathcal{P}_\tau$ we can associate uniquely a path $u(t)$ in the unitary group on $\mathbb{C}^n$ to it. Let $\Delta : [0,\tau] \to \mathbb{R}$ be any continuous real function satisfying

$$\det u(t) = \exp(\sqrt{-1}\Delta(t)), \quad \forall t \in [0,\tau]. \tag{2.1}$$

Then the difference $\Delta(\tau) - \Delta(0)$ depends only on $\gamma$ but not on the choice of the function $\Delta$. Therefore the rotation number of $\gamma$ on $[0,\tau]$ can be defined by

$$\Delta_\tau(\gamma) = \Delta(\tau) - \Delta(0). \tag{2.2}$$

They give invariants of the path $\gamma$ on the corresponding intervals respectively.

The following lemma studies the relation between the usual homotopy and the homotopy which fixes the end points. The proof is left to the readers.

LEMMA 2.1. *Suppose $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$ possess common end point $\gamma_0(\tau) = \gamma_1(\tau)$. Suppose $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ via a homotopy $\delta : [0,1] \times [0,\tau] \to \mathrm{Sp}(2n)$ such that $\delta(\,\cdot\,,\tau)$ is contractible in $\mathrm{Sp}(2n)$. Then the homotopy $\delta$ can be modified to fix the end points all the time, i.e. $\delta(s,\tau) = \gamma_0(\tau)$ for all $0 \le s \le 1$.*

LEMMA 2.2. *If $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$ possess common end point $\gamma_0(\tau) = \gamma_1(\tau)$, then $\Delta_\tau(\gamma_0) = \Delta_\tau(\gamma_1)$ if and only if $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ with fixed end points.*

This lemma is well known and its proof needs Lemma 2.1. One proof can be found in [19].

Let us define

$$M_n^+ = D(2)^{\diamond n} \quad \text{and} \quad M_n^- = D(-2) \diamond D(2)^{\diamond(n-1)}. \tag{2.3}$$

By [7] and [15], $\mathrm{Sp}(2n)^*$ contains precisely two path connected components

$$\mathrm{Sp}(2n)^{\pm} = \{M \in \mathrm{Sp}(2n) \mid \pm(-1)^{n-1}\det(M - I) < 0\}$$

which contains $M_n^{\pm}$ respectively. Thus for any $\gamma \in \mathcal{P}_\tau^*(2n)$, we can connect $\gamma(\tau)$ to $M_n^+$ or $M_n^-$ by a path $\beta$ in $\mathrm{Sp}(2n)^*$ to get a product path $\beta * \gamma$. Then $k \equiv \Delta_\tau(\beta * \gamma)/\pi$ is an integer, and we denote by $\gamma \in \mathcal{P}_{\tau,k}^*(2n)$. Since $\mathrm{Sp}(2n)^*$ is simply connected in $\mathrm{Sp}(2n)$ as proved by [17], this integer $k$ is independent of the choice of the path $\beta$. This integer also satisfies

$$k \in \begin{cases} 2\mathbb{Z}+1 & \text{if } \beta(\tau) = M_n^-, \\ 2\mathbb{Z} & \text{if } \beta(\tau) = M_n^+. \end{cases} \tag{2.4}$$

DEFINITION 2.3. Le us define $i_\tau(\gamma) = k$, if $\gamma \in \mathcal{P}_{\tau,k}^*(2n)$.

Theorem 2.4. *If* $\gamma_0$ *and* $\gamma_1 \in \mathcal{P}_\tau^*(2n)$, *then* $i_\tau(\gamma_0) = i_\tau(\gamma_1)$ *if and only if* $\gamma_0 \sim \gamma_1$ *on* $[0,\tau]$.

Proof. Connect $\gamma_j(\tau)$ to $M_n^+$ or $M_n^-$ by a path $\beta_j : [0,\tau] \to \mathrm{Sp}(2n)^*$ to get a product path $\alpha_j \equiv \beta_j * \gamma_j$ for $j = 0$ and 1. Then we have

$$\gamma_j \sim \alpha_j \qquad \text{on } [0,\tau] \text{ along } \beta_j, \tag{2.5}$$

$$i_\tau(\gamma_j) = i_\tau(\alpha_j), \tag{2.6}$$

for $j = 0$ and 1. Thus $\gamma_0 \sim \gamma_1$ if and only if $\alpha_0 \sim \alpha_1$.

If $\gamma_0 \sim \gamma_1$, by (2.5) we have $\alpha_0 \sim \alpha_1$ on $[0,\tau]$ along a curve $\xi$ in $\mathrm{Sp}(2n)^*$. Therefore $\alpha_0$ and $\alpha_1$ must have the same end points. Thus by Lemma 2.2, we obtain $\Delta_\tau(\alpha_0) = \Delta_\tau(\alpha_1)$, and then $i_\tau(\alpha_0) = i_\tau(\alpha_1)$. By (2.6) this proves $i_\tau(\gamma_0) = i_\tau(\gamma_1)$.

If $i_\tau(\gamma_0) = i_\tau(\gamma_1)$. By (2.6) this implies $i_\tau(\alpha_0) = i_\tau(\alpha_1)$. So $\alpha_0$ and $\alpha_1$ must have the same end points by (2.4). Thus $\Delta_\tau(\alpha_0) = \Delta_\tau(\alpha_1)$. Then by Lemma 2.2 we obtain $\alpha_0 \sim \alpha_1$ on $[0,\tau]$ with fixed end points. By (2.5) we then obtain $\gamma_0 \sim \gamma_1$ on $[0,\tau]$. □

By this theorem, these $\mathcal{P}_{\tau,k}^*(2n)$ give a homotopy classification of $\mathcal{P}_\tau^*(2n)$.

The following proposition gives the $\diamond$-product additivity of the Maslov-type index theory for nondegenerate paths, which has been taciturnly used in many papers without proof.

Proposition 2.5. *Suppose* $\gamma_0 \in \mathcal{P}_\tau^*(2n_0)$ *and* $\gamma_1 \in \mathcal{P}_\tau^*(2n_1)$. *Then* $\gamma_0 \diamond \gamma_1 \in \mathcal{P}_\tau^*(2n_0 + 2n_1)$ *and*

$$i_\tau(\gamma_0 \diamond \gamma_1) = i_\tau(\gamma_0) + i_\tau(\gamma_1). \tag{2.7}$$

Proof. Without loss of generality we suppose $\tau = 1$. We only need to prove (2.7). Let $n = n_0 + n_1$. For $i = 0$ and 1, choose paths $\beta_i : [0,\tau] \to \mathrm{Sp}(2n_i)^*$ to connect $\gamma_i(\tau)$ to $M_{n_i}^+$ or $M_{n_i}^-$ respectively. Note that we obtain

$$(\beta_0 \diamond \beta_1) * (\gamma_0 \diamond \gamma_1) = (\beta_0 * \gamma_0) \diamond (\beta_1 * \gamma_1). \tag{2.8}$$

Then we obtain

$$\begin{aligned} i_1(\gamma_0 \diamond \gamma_1) &= \Delta_1([\beta_0 \diamond \beta_1] * [\gamma_0 \diamond \gamma_1])/\pi = \Delta_1([\beta_0 * \gamma_0] \diamond [\beta_1 * \gamma_1])/\pi \\ &= \Delta_1(\beta_0 * \gamma_0)/\pi + \Delta_1(\beta_1 * \gamma_1)/\pi = i_1(\gamma_1 \diamond \gamma_0). \end{aligned}$$

We continue our study in two cases according to the values of $\beta_0(1)$ and $\beta_1(1)$.

*Case 1.* $\beta_0(1) = M_{n_0}^+$ and $\beta_1(1) = M_{n_1}^-$. In this case, note that

$$\beta_1(1) \diamond \beta_0(1) = M_{n_1}^- \diamond M_{n_0}^+ = M_n^-. \tag{2.10}$$

Thus by (2.8) and (2.9) we have

$$\begin{aligned} i_1(\gamma_0 \diamond \gamma_1) &= i_1(\gamma_1 \diamond \gamma_0) = \Delta_1([\beta_1 \diamond \beta_0] * [\gamma_1 \diamond \gamma_0])/\pi \\ &= \Delta_1([\beta_1 * \gamma_1] \diamond [\beta_0 * \gamma_0])/\pi \\ &= \Delta_1(\beta_1 * \gamma_1)/\pi + \Delta_1(\beta_0 * \gamma_0))/\pi \\ &= i_1(\beta_1 * \gamma_1) + i_1(\beta_0 * \gamma_0) = i_1(\gamma_0) + i_1(\gamma_1). \end{aligned} \tag{2.11}$$

Note that when $\beta_0(1) = M_{n_0}^{\pm}$ and $\beta_1(1) = M_{n_1}^{+}$, (2.7) follows from the fact that $M_{n_0}^{\pm} \diamond M_{n_1}^{+} = M_n^{\pm}$ and a similar argument of (2.11).

*Case 2.* $\beta_0(1) = M_{n_0}^{-}$ and $\beta_1(1) = M_{n_1}^{-}$. Define a path $\xi = (\xi_{i,j})_{1\le i,j\le 2n} : [0,1] \to \mathrm{Sp}(2n)$ by

$$\begin{cases} \xi_{i,i}(t) = \cos(t\pi) & \text{if } i = 1, n_0+1, n+1, n+n_0+1, \\ \xi_{i,j}(t) = -\xi_{j,i}(t) = \sin(t\pi) & \text{if } (i,j) = (n+1, 1) \\ & \text{or } (n+n_0+1, n_0+1), \\ \xi_{i,i}(t) = 1 & \text{for other } i \in [1, 2n], \\ \xi_{i,j}(t) = 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Let us define

$$\eta(t) = [M_{n_0}^{-} \diamond M_{n_1}^{-}]\xi(t), \quad \forall 0 \le t \le 1. \tag{2.13}$$

Then $\eta$ connects $M_{n_0}^{-} \diamond M_{n_1}^{-}$ to $M_n^{+}$ within $\mathrm{Sp}(2n)^*$ and $\Delta_1(\eta) = 0$. Thus we obtain

$$\begin{aligned} i_1(\gamma_0 \diamond \gamma_1) &= \Delta_1(\eta * [\beta_0 \diamond \beta_1] * [\gamma_0 \diamond \gamma_1])/\pi \\ &= \Delta_1(\eta)/\pi + \Delta_1([\beta_0 * \gamma_0] \diamond [\beta_1 * \gamma_1])/\pi \\ &= \Delta_1(\beta_0 * \gamma_0)/\pi + \Delta_1(\beta_1 * \gamma_1)/\pi \\ &= i_1(\gamma_0) + i_1(\gamma_1). \end{aligned} \tag{2.14}$$

The proof is complete. □

Next for any $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\tau > 0$ we define a sequence of *zigzag standard paths* $\widehat{\alpha}_{n,k,\tau}$ in $\mathcal{P}_\tau^*(2n)$ as follows. Using the path $\widehat{\alpha}_{1,0,\tau}$ defined in (1.13), we define

$$\widehat{\alpha}_{n,0,\tau}(t) = (\widehat{\alpha}_{1,0,\tau})^{\diamond n}(t) \quad \forall t \in [0, \tau], \tag{2.15}$$

$$\widehat{\alpha}_{n,k,\tau}(t) = [(D(2)\phi_{\tau,k\pi}) * \widehat{\alpha}_{1,0,\tau}] \diamond (\widehat{\alpha}_{1,0,\tau})^{\diamond(n-1)}(t), \tag{2.16}$$

for all $t \in [0, \tau]$, $k \in \mathbb{Z}\backslash\{0\}$. Then $\widehat{\alpha}_{n,k,\tau} \in \mathcal{P}_\tau^*(2n)$ satisfies

$$i_\tau(\widehat{\alpha}_{n,k,\tau}) = k \text{ and } \widehat{\alpha}_{n,k,\tau}(\tau) = M_n^{\pm} \text{ if } (-1)^k = \pm 1. \tag{2.17}$$

Any path $\gamma_B(t) = \exp(tJB/\tau)$ with $0 \le t \le \tau$ for $B \in \mathcal{L}_s(\mathbb{R}^2)$ is called an *exponential path.* Note that in [15], a family of smooth standard paths $\{\widehat{\beta}_{n,k}\}$ in

$\mathcal{P}_1^*(2n)$ is defined. Then all $\widehat{\beta}_{n,k}$ are exponential paths except when $n = 1$ and $k \in 2\mathbb{Z}\backslash\{0\}$. The following proposition explains why it is so.

Proposition 2.6. *The homotopy class* $\mathcal{P}^*_{1,k}(2)$ *contains an exponential path if and only if* $k \in (2\mathbb{Z}+1) \cup \{0\}$.

Proof. Note that $\gamma_B \in \mathcal{P}_1^*(2)$ if and only if $\det B \neq 4k^2\pi$ for all $k \in \mathbb{Z}$. By direct computation we obtain that $i_1(\gamma_B) = 0$ if $\det B < 0$, and $i_1(\gamma_B) \in 2\mathbb{Z}+1$ if $\det B > 0$. On the other hand we have $\widehat{\beta}_{1,k} \in \mathcal{P}^*_{1,k}(2)$ for $k \in (2\mathbb{Z}+1) \cup \{0\}$. This completes the proof. □

## 3. Rotational perturbations for degenerate paths

At the beginning of this section we define the Maslov-type index theory for any path $\gamma \in \mathcal{P}_\tau(2)$ via the $\mathbb{R}^3$-cylindrical coordinate representation of $\mathrm{Sp}(2)^0$ in $\mathrm{Sp}(2)$ introduced in [11]. By this representation it is clear that for any $M \in \mathrm{Sp}(2)^0$ the infinimum number of the path connected components of the intersection of any small open neighborhood of $M$ in $\mathrm{Sp}(2)$ with $\mathrm{Sp}(2)^*$ is precisely $\dim\ker(M-I)+1$. Therefore for every path $\gamma \in \mathcal{P}_\tau^0(2)$, all paths in $\mathcal{P}_\tau^*(2)$ which are sufficiently close to $\gamma$ split into precisely $\nu_\tau(\gamma)+1$ homotopy classes. So the following definition is well defined.

Definition 3.1. For all $\gamma \in \mathcal{P}_\tau^0(2)$ we define

$$i_\tau(\gamma) = \inf\{i_\tau(\beta) \mid \beta \in \mathcal{P}_\tau^*(2) \text{ is sufficiently } C^0\text{-close to } \gamma\}, \tag{3.1}$$

Note that Definitions 1.1, 2.3, and 3.1 assign to each path $\gamma \in \mathcal{P}_\tau(2)$ a pair of integers $(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z}\times\{0,1,2\}$, which is the Maslov-type index of $\gamma$. Using the $\mathbb{R}^3$-cylindrical coordinate representation of $\mathrm{Sp}(2)$ of [11], it is easy to verify pictorically and rigorously that the index theory defined above satisfies the five axioms listed in the Theorem 1.4 for all paths in $\mathcal{P}_\tau(2)$. Note that for any path $\gamma \in \mathcal{P}_\tau^0(2)$, we can choose $\varepsilon > 0$ small enough so that we obtain

$$\gamma_s(t) \equiv [\gamma(\tau)\phi_{\tau,s\varepsilon}] * \gamma(t) \in \mathrm{Sp}(2)^*, \quad \forall s \in [-1,1]\backslash\{0\},\ t \in (0,1]. \tag{3.2}$$

Then we get a one-parameter family of perturbation paths $\gamma_s$ with $s \in [-1,1]$, which possess the following property

$$i_\tau(\gamma_{-s}) = i_\tau(\gamma) = i_\tau(\gamma_s) - \nu_\tau(\gamma), \quad \forall s \in (0,1]. \tag{3.3}$$

We devote the rest of this section, Sections 4 and 5 to the definition of the Maslov-type index theory for any degenerate paths in $\mathcal{P}_\tau(2n)$.

For $\tau > 0$, fix $\gamma \in \mathcal{P}_\tau^0(2n)$. We apply Theorem 7.3 to $M = \gamma(\tau)$ and obtain (7.8) for some $P \in \mathrm{Sp}(2n)$. Let $\Sigma_0$ be the subset of $\mathrm{Sp}(2n)^0$ which contains all matrices $A$ satisfying

$$\dim\ker(A-I) > \nu_\tau(\gamma) \equiv \dim\ker(\gamma(\tau)-I).$$

Let $\Sigma_1$ be the path connected component of $\mathrm{Sp}(2n)^0\backslash\Sigma_0$ which contains $\gamma(\tau)$. For $\varepsilon > 0$ small enough, let $B_\varepsilon(\gamma(\tau))$ be the open ball in $\mathrm{Sp}(2n)$ centered at $\gamma(\tau)$ with radius $\varepsilon$, here the metric on $\mathrm{Sp}(2n)$ is the one induced from that of $\mathbb{R}^{4n^2}$. Choose $\varepsilon > 0$ to be sufficiently small so that $\overline{B_\varepsilon(\gamma(\tau))}$ is contractible and possesses no intersection with $\mathrm{Sp}(2n)^0\backslash\Sigma_1$.

Let $\theta_0 \in (0, \pi/8n)$ and the integers $\{m_1, \dots, m_{p+2q}\}$ be the numbers defined by (7.9). For $s_i \in [-1, 1]$ with $1 \le i \le p + 2q$, we define

$$Q(s_1, \dots, s_{p+2q}) \equiv \gamma(\tau)P^{-1}R_{m_1}(s_1\theta_0)\dots R_{m_{p+2q}}(s_{p+2q}\theta_0)P. \tag{3.4}$$

Then by Theorem 7.3, for all $s_i \in [-1, 1]\backslash\{0\}$ with $1 \le i \le p + 2q$, we obtain

$$Q(s_1, \dots, s_{p+2q}) \in \mathrm{Sp}(2n)^* \cap B_\varepsilon(\gamma(\tau)), \tag{3.5}$$

$$Q(s_1, \dots, s_{p+2q})P^{-1}R_{m_k}(-s_k\theta_0)P \in \mathrm{Sp}(2n)^0, \quad 1 \le k \le p + 2q, \tag{3.6}$$

$$\dim\ker(Q(s_1, \dots, s_{p+2q})P^{-1}R_{m_k}(-s_k\theta_0)P - I) = c_k, \quad 1 \le k \le p + 2q, \tag{3.7}$$

where the constant $c_k = 1$ or $2$, and $R_k(\theta) = I_{2k-2} \diamond R(\theta) \diamond I_{2n-2k}$.

For $t_0 \in (0, \tau)$, let $\rho \in C^2([0, \tau], [0, 1])$ such that $\rho(t) = 0$ for $0 \le t \le t_0$, $\dot\rho(t) \ge 0$ for $0 \le t \le \tau$, $\rho(\tau) = 1$, and $\dot\rho(\tau) = 0$. Whenever $t_0 \in (0, \tau)$ is sufficiently close to $\tau$, for any $(s, t) \in [-1, 1] \times [0, \tau]$ the paths

$$\gamma_s(t) = \gamma(t)P^{-1}R_{m_1}(s\rho(t)\theta_0)\dots R_{m_{p+2q}}(s\rho(t)\theta_0)P \tag{3.8}$$

satisfy $\gamma_s$ converges to $\gamma$ in $C^1([0, \tau], \mathrm{Sp}(2n))$ as $s \to 0$, and

$$\begin{cases} \gamma_0 = \gamma \\ \gamma_s(t) = \gamma(t) & \forall 0 \le t \le t_0,\ s \in [-1, 1], \\ \gamma_s(t) \in B_\varepsilon(\gamma(\tau)) & \forall t_0 \le t \le 1,\ s \in [-1, 1], \\ \nu_\tau(\gamma_s) = 0 & \forall s \in [-1, 1]\backslash\{0\}, \\ i_\tau(\gamma_s) = i_\tau(\gamma_{s'}) & \forall s, s' \in [-1, 1] \text{ with } ss' > 0. \end{cases} \tag{3.9}$$

Note that the matrix $P$ and the normal form $\diamond$-product $N$ in (7.8) of $\gamma(\tau)$ need not be uniquely determined by the matrix $\gamma(\tau)$. The path $\gamma_s$ depends not only on $\gamma(\tau)$ but also on the choices of $P$ and $N$. Neverthless in the later sections we shall see that $i_\tau(\gamma_s)$ for any $s \neq 0$ is uniquely determined by $\gamma$.

The following is the most important property of these rotation perturbation paths.

THEOREM 3.2. *For any $\gamma \in \mathcal{P}_\tau^0(2n)$ and $0 < s \le 1$, the rotation perturbation paths defined by* (3.8) *satisfy*

$$i_\tau(\gamma_s) - i_\tau(\gamma_{-s}) = \nu_\tau(\gamma). \tag{3.10}$$

Proof. Without loss of generality, we assume $\tau = 1$. Fix $s \in (0,1]$. For $0 \le k \le p+2q$ we define

$$\alpha_k(t) = \gamma(t)P^{-1}R_{m_1}(s\rho(t)\theta_0)\dots R_{m_k}(s\rho(t)\theta_0) \cdot R_{m_{k+1}}(-s\rho(t)\theta_0)\dots R_{m_{p+2q}}(-s\rho(t)\theta_0)P. \tag{3.11}$$

Then by definition we have $\alpha_k \in \mathcal{P}_1^*(2n)$ for $0 \le k \le p+2q$, and $\alpha_0 = \gamma_{-s}$, $\alpha_{p+2q} = \gamma_s$.

For $1 \le k \le p+2q$, we define

$$a_k = \dim\ker(\alpha_k(1)P^{-1}R_{m_k}(-s\theta_0)P - I). \tag{3.12}$$

By (3.7), the constant $a_k$ only takes the value 1 or 2. Define $F_1(s,\theta_0) = I$. For $1 \le k \le p+2q$, we define

$$F_k(s,\theta_0) = P^{-1}R_{m_1}(-s\theta_0)\cdots R_{m_{k-1}}(-s\theta_0)P, \tag{3.13}$$

$$G_k(s,\theta_0) = P^{-1}R_{m_1}(s\theta_0)\cdots R_{m_{k-1}}(s\theta_0) \cdot R_{m_{k+1}}(-s\theta_0)\cdots R_{m_{p+2q}}(-s\theta_0)P. \tag{3.14}$$

By the definition of $\{m_1,\dots,m_{p+2q}\}$ and 4° of Theorem 7.3 we obtain

$$\begin{aligned} \nu_1(\gamma) &= \dim\ker(\gamma(1) - I) \\ &\quad - \dim\ker(\gamma(1)F_{p+2q}(s,\theta_0)P^{-1}R_{m_{p+2q}}(-s\theta_0)P - I) \\ &= \sum_{k=1}^{p+2q}\{\dim\ker(\gamma(1)F_k(s,\theta_0) - I) \\ &\quad - \dim\ker(\gamma(1)F_k(s,\theta_0)P^{-1}R_{m_k}(-s\theta_0)P - I)\} \\ &= \sum_{k=1}^{p+2q}\{\dim\ker(\gamma(1)G_k(s,\theta_0) - I) \\ &\quad - \dim\ker(\gamma(1)G_k(s,\theta_0)P^{-1}R_{m_k}(-s\theta_0)P - I)\} \\ &= \sum_{k=1}^{p+2q}\dim\ker(\alpha_k(1)P^{-1}R_{m_k}(-s\theta_0)P - I) = \sum_{k=1}^{p+2q} a_k. \end{aligned} \tag{3.15}$$

Let us assume that the following equalities hold

$$i_1(\alpha_k) - i_1(\alpha_{k-1}) = a_k, \quad \text{for } 1 \le k \le p+2q. \tag{3.16}$$

Summing (3.16) up from $k=1$ to $k = p+2q$ yields

$$i_1(\gamma_s) - i_1(\gamma_{-s}) = i_1(\alpha_{p+2q}) - i_1(\alpha_0) = \sum_{k=1}^{p+2q} a_k = \nu_1(\gamma).$$

Therefore the proof of (3.10) is reduced to that of (3.16).

Fix a $k \in \{1,\dots,p+2q\}$. Let $\eta(t) = \alpha_{k-1}(1)P^{-1}R_{m_k}(2ts\theta_0)P$ for $t \in [0,1]$. Then by definition we have $\eta(0) = \alpha_{k-1}(1)$, $\eta(1) = \alpha_k(1)$, $\eta(1/2) =$

$\alpha_k(1)P^{-1}R_{m_k}(-s\theta_0)P \in \mathrm{Sp}(2n)^0$, $\dim\ker(\eta(1/2)-I) = a_k$, and $\eta(t) \in \mathrm{Sp}(2n)^*$ for $t \in [0,1]\backslash\{1/2\}$. Note that $B_\varepsilon(\gamma(1))$ is contractible if $\varepsilon > 0$ is small enough. By (3.7) and (3.9) we obtain that $\alpha_k|_{[t_0,1]}$ is homotopic to the joint path of $\alpha_{k-1}|_{[t_0,1]}$ and $\eta$ with fixed end points. Therefore we obtain $\alpha_k \sim \eta * \alpha_{k-1}$ with fixed end points. Then (3.16) becomes

$$i_1(\eta * \alpha_{k-1}) - i_1(\alpha_{k-1}) = a_k. \tag{3.17}$$

The proof of (3.17) contains two cases according to the value of $a_k$.

*Case 1.* $a_k = 2$. We notice that $\dim\ker(\eta(1/2)-I) = a_k = 2$ and $\eta*\alpha_{k-1}(t) = \eta(1/2)P^{-1}R_{m_k}((t-3/4)4s\theta_0)P$ for $1/2 \le t \le 1$. For $0 < |t-3/4| \le 1/4$, we get

$$\dim\ker(\eta(1/2)-I) - \dim\ker(\eta(1/2)P^{-1}R_{m_k}((t-3/4)4s\theta_0)P - I) = 2.$$

Hence the $k$-th normal form matrix is of the first type which is denoted by $H_k$. Since $a_k = 2$, it must hold for $H_k = I_2$. Thus when $0 < \theta < 2\pi$ we have $P\eta(1/2)P^{-1}R_{m_k}(\theta) = Q_1 \diamond R(\theta) \diamond Q_2$ for some symplectic matrices $Q_1$ and $Q_2$ with $1 \notin \sigma(Q_1) \cup \sigma(Q_2)$. Therefore $\eta(1/2)P^{-1}R_{m_k}(\theta)P$ belongs to the same path connected component of $\mathrm{Sp}(2n)^*$ for $\theta \ne 0 \bmod 2\pi$. Then we get

$$\eta * \alpha_{k-1} \sim \beta * \alpha_{k-1}, \tag{3.18}$$

where $\beta(t) = \alpha_{k-1}(1)P^{-1}R_{m_k}(2t\pi)P$ for $0 \le t \le 1$. By the discussion at the end of the section 2, we get $\alpha_{k-1} \sim \widehat{\alpha}_{n,j,1}$ for some zigzag standard path $\widehat{\alpha}_{n,j,1}$ defined there. So there exists a path $\xi \in C([0,1], \mathrm{Sp}(2n)^*)$ such that $\xi(0) = \widehat{\alpha}_{n,j,1}(1)$ and $\xi(1) = \alpha_{k-1}(1)$ hold. Thus $\alpha_{k-1} \sim \xi * \widehat{\alpha}_{n,j,1}$ and $\beta * \alpha_{k-1} \sim \beta * \xi * \widehat{\alpha}_{n,j,1} \sim \xi^{-1} * \beta * \xi * \widehat{\alpha}_{n,j,1} \equiv \zeta$ hold. Here and below because $\beta*(\xi*\widehat{\alpha}_{n,j,1}) \sim (\beta*\xi)*\widehat{\alpha}_{n,j,1}$, we simply use the notation $\beta * \xi * \beta_j$. By direct computation on these standard paths, we obtain $\zeta(1) = \widehat{\alpha}_{n,j,1}(1) = \widehat{\alpha}_{n,j+2,1}(1)$ and

$$\Delta_1(\zeta) = \Delta_1(\widehat{\alpha}_{n,j,1}) + \Delta_1(\beta) = (j+2)\pi = \Delta_1(\widehat{\alpha}_{n,j+2,1}).$$

Therefore from Lemma 2.2, we get that $\zeta \sim \widehat{\alpha}_{n,j+2,1}$. Combining with (3.18) we get $\eta * \alpha_{k-1} \sim \widehat{\alpha}_{n,j+2,1}$, and hence by Theorem 2.4,

$$i(\eta * \alpha_{k-1}) - i(\alpha_{k-1}) = i(\widehat{\alpha}_{n,j+2,1}) - i(\widehat{\alpha}_{n,j,1}) = 2.$$

Thus (3.17) holds in this case.

*Case 2.* $a_k = 1$. If $n = 1$, the equality (3.17) follows from our discussion at the beginning of this section. Next we study the case for $n \ge 2$.

By Theorem 8.2, further requiring $\theta_0 > 0$ to be sufficiently small, there are continuous paths $\sigma : [0,1] \to \mathcal{M}(2n)$, and $\sigma^\pm : [0,1] \to \mathrm{Sp}(2n)^*$ such that

$$\sigma(0) = \eta(1/2), \quad \sigma(1) = N_1(b) \diamond M_0,$$

for $b = 1$ or $-1$ and $M_0 \in \mathrm{Sp}(2n-2)^*$, and

$$\sigma^-(0) = \alpha_{k-1}(1), \qquad \sigma^+(0) = \alpha_k(1),$$
$$\sigma^-(1) = [N_1(b)R(-s\theta_0)] \diamond M_0, \qquad \sigma^+(1) = [N_1(b)R(s\theta_0)] \diamond M_0.$$

Now we choose paths $\xi_0 \in \mathcal{P}_1(2n-2)$ and $\xi \in \mathcal{P}_1(2)$ such that $\xi_0(1) = M_0$, $\xi(1) = N_1(b)$, and $\xi(t) \in \mathrm{Sp}(2)^0$ for all $t \in [0,1]$. We define $\eta_{1/2}(t) = \eta(t/2)$ for $0 \le t \le 1$. Let us define

$$\zeta(t) = \begin{cases} R(4m\pi t) & 0 \le t \le 1/2, \\ \xi(2t-1) & 1/2 \le t \le 1, \end{cases}$$

where $m \in \mathbb{Z}$ is chosen so that

$$\Delta_1(\zeta) + \Delta_1(\xi_0) = \Delta_1(\sigma * \eta_{1/2} * \alpha_{k-1}).$$

Let us define paths $\chi^\pm$ and splitting paths $f^\pm$ and $f$ by

$$\begin{aligned} \chi^\pm &= (N_1(b)\phi_{1,\pm s\theta_0}) * \zeta : [0,1] \to \mathrm{Sp}(2), \\ f^\pm &= \chi^\pm \diamond \xi_0 : [0,1] \to \mathrm{Sp}(2n), \\ f &= \zeta \diamond \xi_0 : [0,1] \to \mathrm{Sp}(2n). \end{aligned}$$

Then we obtain

$$\tag{3.19} \begin{cases} f^\pm &= [(N_1(b)\phi_{1,\pm s\theta_0}) \diamond I_{2n-2}] * f, \\ f^+(1) &= \sigma^+ * \eta * \alpha_{k-1}(1), \\ f^-(1) &= \sigma^- * \alpha_{k-1}(1), \\ \Delta_1(f^+) &= s\theta_0 + \Delta_1(f) \\ &= s\theta_0 + \Delta_1(\sigma * \eta_{1/2} * \alpha_{k-1}) = \Delta_1(\sigma^+ * \eta * \alpha_{k-1}), \\ \Delta_1(f^-) &= -s\theta_0 + \Delta_1(f) \\ &= -s\theta_0 + \Delta_1(\sigma * \eta_{1/2} * \alpha_{k-1}) = \Delta_1(\sigma^- * \alpha_{k-1}). \end{cases}$$

Thus by Lemma 2.2, we obtain

$$f^+ \sim \sigma^+ * \eta * \alpha_{k-1}, \quad f^- \sim \sigma^- * \alpha_{k-1}.$$

Therefore

$$\tag{3.20} \begin{aligned} i_1(\eta * \alpha_{k-1}) - i_1(\alpha_{k-1}) &= i_1(\sigma^+ * \eta * \alpha_{k-1}) - i_1(\sigma^- * \alpha_{k-1}) \\ &= i_1(f^+) - i_1(f^-) \\ &= [i_1(\chi^+) + i_1(\xi_0)] - [i_1(\chi^-) + i_1(\xi_0)] \\ &= i_1(\chi^+) - i_1(\chi^-), \end{aligned}$$

where we have used Proposition 2.5 on $f^\pm$. Now by (3.19) and our study on the case of $n = 1$ at the beginning of this section, the right hand side of (3.20) must be 1. This proves (3.17) in this case. The proof of Theorem 3.2 is complete. □

Remark 3.3. The above proof follows the idea of [10]. Note that (3.15) and the case 2 of (3.17) are claimed in [10] without proofs. Here we have modified the definition of the perturbation paths $\gamma_s$ with $s \in [-1,1]$ in (3.8) by Theorem 7.3 using normal forms to get (3.15), and by Theorem 8.2 to get the proof of the above Case 2 of (3.17).

## 4. Maslov-type indices and Morse indices

In this section we study the relation between the Maslov-type index of linear system (1.1) and the Morse indices of the corresponding functional at its origin obtained via the saddle point reduction.

Fix $\tau > 0$ and $B \in C(S_\tau, \mathcal{L}_s(\mathbb{R}^{2n}))$. Denote by $\gamma$ the fundamental solution of the system (1.1). Denote by $\gamma_s$ with $s \in [-1,1]$ the rotation perturbation paths of $\gamma$ defined by (3.8) via a normal form $\diamond$-product $N$ of $\gamma(\tau)$ defined by (7.10). Define

$$B_s(t) = -J\dot{\gamma}_s(t)\gamma_s(t)^{-1}, \quad \forall t \in [0,\tau],\ s \in [-1,1].$$

Then $B_0 = B$. For $s \in [-1,1]$, on the space $L = L^2(S_\tau, \mathbb{R}^{2n})$ the functional corresponding to the system (1.1) with the coefficient $B_s(t)$ has the form of

$$f_s(x) = \frac{1}{2}\int_0^\tau (Ax\cdot x - B_s(t)x\cdot x)\,dt \quad \forall x \in \operatorname{dom} A \subset L,$$

where $A = -Jd/dt$. In [1] or the section IV.2.1 of [6], by using the saddle point reduction method a functional $a_s$ on a finite dimensional truncated space $Z$ uniformly chosen for all $s \in [-1,1]$ is defined by

$$a_s(z) = f_s(u_s(z)) \quad \forall z \in Z,$$

where $u : Z \to \operatorname{dom} A$ is a $C^\infty$ injection such that Theorems IV.2.1 of [6] holds. That the space $Z$ can be uniformly chosen for all $s \in [-1,1]$ follows from the boundedness of $\|B_s\|_{C^0}$ for all these $s$. In the following we shall establish the relation theorem on the Morse indices of the functional $a_0$ at its critical point $z = 0$ and the Maslov-type indices of the system (1.1) with the coefficient $B(t)$.

Denote by $2d = \dim Z$. For all $s \in [-1,1]$, $x = 0$ is a critical point of $f_s$. So $z = 0$ is a critical point of $a_s$. Denote by $m_s^-$, $m_s^0$, and $m_s^+$ the Morse indices of the functional $a_s$ at $z = 0$, i.e. the total multiplicities of the negative, zero, and positive eigenvalues of the matrix $a_s''(0)$ respectively. We write $m^* = m_0^*$ for $* = -$, $0$, or $+$. The main result of this section is the following

Theorem 4.1. *Under the above assumptions, for any* $s \in (0,1]$ *we obtain*

$$m^- = d + i_\tau(\gamma_{-s}), \quad m^0 = \nu_\tau(\gamma), \quad m^+ = d - i_\tau(\gamma_{-s}) - \nu_\tau(\gamma). \tag{4.1}$$

We need the following lemmas in this section and later.

Lemma 4.2. *Under the above assumptions, suppose*

$$m_s^0 = m_0^0, \quad \forall s \in [0,1]. \tag{4.2}$$

*Then we obtain*

$$m_s^- = m_0^-, \quad m_s^+ = m_0^+, \quad \forall s \in [0,1]. \tag{4.3}$$

Proof. Fix any $r \in [0,1]$. For any $s \in [0,1]$ which is sufficiently close to $r$, we obtain

$$m_r^- \le m_s^-, \quad m_r^+ \le m_s^+.$$

Thus by (4.2) we obtain

$$2d = m_r^- + m_r^0 + m_r^+ \le m_s^- + m_s^0 + m_s^+ = 2d.$$

This proves that $m_s^-$ and $m_s^+$ are locally constants. By the connectedness of $[0,1]$, they are also globally constants, i.e. (4.3) holds. □

Remark 4.3. Note that by Theorem 4.1, Lemma 4.2, and (5.2) below, the homotopy invariance of the Maslov-type index theory becomes natural.

Lemma 4.4. *Suppose $\gamma_0$, $\gamma_1 \in \widehat{\mathcal{P}}_1(2n)$, and $\gamma_0 \sim \gamma_1$. Then this homotopy can be realized in $\widehat{\mathcal{P}}_1(2n)$.*

Proof. Denote the homotopy from $\gamma_0$ to $\gamma_1$ in $\mathcal{P}_1(2n)$ by $\delta : [0,1]^2 \to \mathrm{Sp}(2n)$. Pushing the value of $\delta$ on the boundary of $D = [0,1]^2$ into the interior of this square, without loss of generality, we may assume $\delta$ is $C^1$ in the variable $t$ near the boundary of $D$ and satisfies $\dot{\delta}(s,1) = \dot{\delta}(s,0)\delta(s,1)$ for all $s \in [0,1]$, where $\dot{\delta}$ is the derivative of $\delta$ with respect to $t$. Now $\delta$ is not $C^1$ in the variable $t$ at most on a compact subregion of the interior of the square $D$. Following the standard differential topology argument (for example cf. [8]), we can find an approximation $\eta$ of $\delta$ such that $\eta$ coincides with $\delta$ near the boundary of $[0,1]^2$, is $C^1$ in the variable $t$ for all $s \in [0,1]$, and can be chosen to be as close to $\delta$ as we want. This map $\eta$ gives a homotopy claimed by the lemma. □

Based on these lemmas, we give next

The Proof of Theorem 4.1. Without loss of generality, we suppose $\tau = 1$. Note that when $\nu_1(\gamma) = 0$, Theorem 4.1 has been proved in [7] and [15]. We slightly modify the proof of [10] to further consider the case of $\nu_1(\gamma) > 0$.

Recall that $\gamma$ is the fundamental solution of the system (4.1), and $\gamma_s$ for $s \in [-1,1]$ is the rotation perturbation path of $\gamma$ defined by (3.6). When $\nu_1(\gamma) = \nu_1(B) > 0$, $z = 0$ is a critical point of the functional $a$ on $Z$ corresponding to the system (4.1). Note that here $z = 0$ need not be isolated. Then $\gamma(1) \in \mathrm{Sp}(2n)^0$. By a result of E. Zehnder in [20] we obtain

$$m^0 = \dim\ker a''(0) = \dim\ker(\gamma(1) - I) = \nu_1(\gamma). \tag{4.4}$$

Then $a_s$ converges to $a$ in $C^2$ as $s \to 0$. By [7] and [15], the origin $z = 0$ is a nondegenerate isolated critical point of $a_s$ when $s \neq 0$. Then by the nondegenerate case of Theorem 4.1 we obtain

$$m_s^- = d + i_1(\gamma_s), \quad m_s^0 = 0, \quad m_s^+ = d - i_1(\gamma_s), \quad \text{if } s \in [-1,1]\backslash\{0\}. \tag{4.5}$$

When $|s| > 0$ is sufficiently small, the matrix $a_s''(0)$ is a small perturbation of $a''(0)$. Thus in this case we obtain $m^+ \le m_s^+$ and $m^- \le m_{-s}^-$. Together with (4.5) we obtain that when $0 < s \le 1$ and sufficiently close to 0 we obtain

$$m^+ \le m_s^+ = d - i_1(\gamma_s), \tag{4.6}$$

$$m^- \le m_{-s}^- = d + i_1(\gamma_{-s}). \tag{4.7}$$

By Theorem 3.2, the above (4.6) can be rewritten into

$$m^+ \le d - i_1(\gamma_s) = d - i_1(\gamma_{-s}) - \nu_1(\gamma). \tag{4.8}$$

Note that $\dim Z = 2d$. Together with (4.4) we then obtain

$$d - i_1(\gamma_{-s}) \le 2d - m^- = m^+ + m^0 = m^+ + \nu_1(\gamma) \le d - i_1(\gamma_{-s}). \tag{4.9}$$

Thus in (4.9) equalities must hold and they yield

$$m^- = d + i_1(\gamma_{-s}) \quad \text{and} \quad m^+ = d - i_1(\gamma_{-s}) - \nu_1(\gamma). \tag{4.10}$$

Now (4.4) and (4.10) together with the nondegenerate case contained in [7] and [15] complete the proof of Theorem 4.1. □

PROPOSITION 4.5. *Denote by* $\gamma$ *the fundamental solution of* (1.1). *Suppose* $\nu_\tau(\gamma) > 0$. *Then for any paths* $\alpha$ *and* $\beta \in \widehat{\mathcal{P}}_\tau^*(2n)$ *which are sufficiently* $C^1$*-close to* $\gamma$, *for any* $s \in (0,1]$ *we obtain*

$$i_\tau(\gamma_{-s}) \le i_\tau(\alpha) \le i_\tau(\gamma_s) = i_\tau(\gamma_{-s}) + \nu_\tau(\gamma), \tag{4.11}$$

$$|i_\tau(\beta) - i_\tau(\alpha)| \le \nu_\tau(\gamma). \tag{4.12}$$

PROOF. Without loss of generality we suppose $\tau = 1$. Fix $s \in (0,1]$. We prove the first inequality in (4.11) indirectly by assuming that there exist $\alpha_k \in \widehat{\mathcal{P}}_1^*(2n)$ for $k \in \mathbb{N}$ such that $\alpha_k$ converges to $\gamma$ in $C^1$ as $k \to \infty$, and we obtain

$$i_1(\gamma_{-s}) > i_1(\alpha_k). \tag{4.13}$$

Denote by $B_k(t) = -J\dot{\alpha}_k(t)\alpha_k^{-1}(t)$. Then $B_k \to B$ in $C^0$. Therefore we can choose the truncated space $Z$ to be large enough for all the $B$ and $B_k$ to carry out the saddle point reduction. Let $2d = \dim Z$. Denote by $m_k^-$, $m_k^0$, and $m_k^+$ the Morse indices of the functional $a_k$ on $Z$ corresponding to $\alpha_k$, and by $m^-$, $m^0$, and $m^+$ the Morse indices of the functional $a$ on $Z$ corresponding to $\gamma$. Then whenever $k$ is large enough, by Theorem 4.1 and (4.13) we obtain

$$m^- \le m_k^- = d + i_1(\alpha_k) < d + i_1(\gamma_{-s}).$$

Together with Theorem 4.1, we get the following contradiction

$$d - i_1(\gamma_{-s}) = m^+ + m^0 = 2d - m^- > d - i_1(\gamma_{-s}).$$

This proves the first inequality in (4.11).

The second inequality in (4.11) is followed by a similar proof. Then (4.12) follows from (4.11) and Theorem 4.1. The proof is complete. □

As a direct consequence of Theorem 4.1 and Proposition 4.5, we give the following topological charaterization of the Maslov-type index for rotation perturbation paths of fundamental solutions of degenerate linear Hamiltonian systems as follows.

Corollary 4.6. *For $\tau > 0$, denote by $\gamma$ the fundamental solution of* (4.1)*. Suppose $\nu_\tau(\gamma) > 0$. Denote by $\gamma_s$ for $s \in [-1, 1]$ the rotation perturbation paths defined by* (4.6) *for $\gamma$. Then for any $s \in (0, 1]$ we obtain*

$$i_\tau(\gamma_{-s}) = \inf\{i_\tau(\beta) \mid \beta \in \widehat{\mathcal{P}}^*_\tau(2n) \text{ and } \beta \text{ is } C^1\text{-sufficiently close to } \gamma \text{ in } \widehat{\mathcal{P}}^*_\tau(2n)\}, \tag{4.14}$$

$$i_\tau(\gamma_s) = \sup\{i_\tau(\beta) \mid \beta \in \widehat{\mathcal{P}}^*_\tau(2n) \text{ and } \beta \text{ is } C^1\text{-sufficiently close to } \gamma \text{ in } \widehat{\mathcal{P}}^*_\tau(2n)\}, \tag{4.15}$$

*where the topology of $\widehat{\mathcal{P}}^*_\tau(2n)$ is the $C^1$-topology induced from the topology of* $\mathrm{Sp}(2n)$*.*

## 5. Maslov-type indices for degenerate paths

For $\tau > 0$ and any $\gamma \in \mathcal{P}^0_\tau(2n)$ in this section we define the Maslov-type index for $\gamma$. By Corollary 4.6 the following definition is well defined for fundamental solutions of linear Hamiltonian systems (1.1) with continuous symmetric and $\tau$-periodic coefficients.

Definition 5.1. For any $\gamma \in \widehat{\mathcal{P}}^0_\tau(2n)$, we define

$$i_\tau(\gamma) = i_\tau(\gamma_{-s}) \quad \forall s \in (0, 1], \tag{5.1}$$

where $\gamma_{-s}$ is a rotation perturbation path of $\gamma$ defined by (3.8).

As a direct consequence of this definition, the conclusion (4.1) of Theorem 4.1 can be restated as follows:

$$m^- = d + i_\tau(\gamma), \quad m^0 = \nu_\tau(\gamma), \quad m^+ = d - i_\tau(\gamma) - \nu_\tau(\gamma). \tag{5.2}$$

Now we consider the case for any path $\gamma \in \mathcal{P}^0_\tau(2n)$. From the structure of $\mathrm{Sp}(2n)$ we can find a path $\beta \in C^1([0, \tau], \mathrm{Sp}(2n))$ such that $\beta(0) = I$, $\beta(\tau) = \gamma(\tau)$, and $\beta$ is as $C^0$ close to $\gamma$ as we want. Now by composing $\beta$ with a suitable function $\rho \in C^1([0, \tau], [0, \tau])$ such that $\rho(0) = 0$, $\dot\rho(0) = 0$, $\rho(\tau) = \tau$, $\dot\rho(\tau) = 0$, and

$\dot{\rho}(t) \geq 0$ for $0 \leq t \leq \tau$, we can further assume $\beta \in \widehat{\mathcal{P}}^0_\tau(2n)$. When $\beta$ is sufficiently close to $\gamma$, since they possess the same end points, we obtain

$$\Delta_\tau(\beta) = \Delta_\tau(\gamma). \tag{5.3}$$

Therefore there is an open neighborhood $U(\gamma)$ of $\gamma$ in $\mathcal{P}^0_\tau(2n)$ such that (5.3) holds for any $\beta \in U(\gamma) \cap \widehat{\mathcal{P}}^0_\tau(2n)$ possessing the same end points with $\gamma$. Denote by $U^{\#}(\gamma)$ the set of all such paths.

DEFINITION 5.2. For $\tau > 0$ and any $\gamma \in \mathcal{P}^0_\tau(2n)$, we define

$$i_\tau(\gamma) = i_\tau(\beta) \quad \forall \beta \in U^{\#}(\gamma). \tag{5.4}$$

PROPOSITION 5.3. *The Definition* 5.2 *of* $i_\tau(\gamma)$ *for any* $\gamma \in \mathcal{P}^0_\tau(2n)$ *is well defined.*

PROOF. Without loss of generality, suppose $\tau = 1$. Fix a path $\gamma \in \mathcal{P}^0_\tau(2n)$. Let $\beta \in U^{\#}(\gamma)$. Then we obtain $\beta(1) = \gamma(1)$. By Definition 5.2, we obtain

$$i_1(\beta) = i_1(\beta_{-s}),$$

where $\beta_{-s}$ for $s \in (0,1]$ is the negative rotation perturbation paths of $\beta$ defined by (3.8) via the same normal form $\diamond$-product of $\beta(1) = \gamma(1)$. Define the path

$$\zeta_{-s}(t) = Q(-st, \dots, -st), \quad \forall (s,t) \in [0,1]^2, \tag{5.5}$$

where $Q(s_1, \dots, s_q)$ is given by (3.4) and determined by the normal form $\diamond$-product $N$ of $\gamma(1)$ and the matrix $P$ in (7.10).

Now fix $s \in (0,1]$. Then $\zeta_{-s}(1) = \beta_{-s}(1)$. Denote the path connecting $\zeta_{-s}(1)$ to $M_n^+$ or $M_n^-$ within $\mathrm{Sp}(2n)^*$ in the section 2 by $\psi : [0,1] \to \mathrm{Sp}(2n)^*$. By the definition of $\beta_{-s}$, the path $\beta_{-s}|_{[t_0,1]}$ is homotopic to the joint path of $\gamma|_{[t_0,1]}$ followed by $\zeta_{-s}$ defined above with fixed end points. Thus these two paths possess the same rotation numbers. Then from $i_1(\beta) = i_1(\beta_{-s})$, we obtain

$$i_1(\beta) = \Delta_1(\psi * \zeta_{-s} * \gamma)/\pi = \Delta_1(\psi)/\pi + \Delta_1(\zeta_{-s})/\pi + \Delta_1(\gamma)/\pi. \tag{5.6}$$

This proves that the Definition 5.2 is independent of the choice of $\beta$ in $U^{\#}(\gamma)$, and then is well defined. $\square$

Further topological characterization of $i_\tau(\gamma)$ for $\gamma \in \mathcal{P}_\tau(2n)$ are given in the Theorems 6.6 and 1.4. Finally we give the definition of the Maslov-type index for any path in $\mathcal{P}_\tau(2n)$.

DEFINITION 5.4. For every path $\gamma \in \mathcal{P}_\tau(2n)$, the Definitions 1.1, 2.3, and 5.2 assign a pair of integers

$$(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, \dots, 2n\}$$

to it. This pair of integers is called the *Maslov-type index* of $\gamma$. We call $i_\tau(\gamma)$ the *rotation index* of $\gamma$ and $\nu_\tau(\gamma)$ the *nullity* of $\gamma$.

## 6. Basic properties of the Maslov-type index theory

In this section we study basic proferties of the Maslov-type index theory defined for any path in $\mathcal{P}_\tau(2n)$ by Definition 5.4.

Proposition 6.1. *For any two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$ with $\gamma_0(\tau) = \gamma_1(\tau)$, we obtain $\Delta_\tau(\gamma_0) = \Delta_\tau(\gamma_1)$ if and only if $i_\tau(\gamma_0) = i_\tau(\gamma_1)$.*

Proof. Without loss of generality, suppose $\tau = 1$. If $\nu_1(\gamma_0) = 0$, then this result follows from Lemma 2.2 and Theorem 2.4. Suppose $\nu_1(\gamma_0) > 0$. By Definition 5.2 and (5.6) in the proof of Proposition 5.3, we obtain

$$i_1(\gamma_j) = \Delta_1(\psi)/\pi + \Delta_1(\zeta_{-1})/\pi + \Delta_1(\gamma_j)/\pi, \quad \text{for } j = 0, 1, \tag{6.1}$$

where $\zeta_{-1}$ is defined in (5.5) replacing $\gamma(1)$ by $\gamma_0(1) = \gamma_1(1)$, the path $\psi : [0,1] \to \mathrm{Sp}(2n)^*$ connecting $\zeta_{-1}(1)$ to $M_n^+$ or $M_n^-$ is defined in the proof of Proposition 5.3. From (6.1) the proposition follows. □

Corollary 6.2. *For any two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$ with $\gamma_0(\tau) = \gamma_1(\tau)$, we obtain $i_\tau(\gamma_0) = i_\tau(\gamma_1)$ if and only if $\gamma_0 \sim \gamma_1$ on $[0, \tau]$ with fixed end points.*

Proof. This is a consequence of Lemma 2.1 and Proposition 6.1. □

Theorem 6.3 (Homotopy invariant). *For any two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$, if $\gamma_0 \sim \gamma_1$ on $[0, \tau]$, we obtain*

$$i_\tau(\gamma_0) = i_\tau(\gamma_1) \quad \text{and} \quad \nu_\tau(\gamma_0) = \nu_\tau(\gamma_1). \tag{6.2}$$

Proof. Without loss of generality, we suppose $\tau = 1$. By Corollary 6.2, we only need to consider the case when the end points are not fixed. If $\nu_1(\gamma_0) = 0$, the claim follows from Lemma 2.1 and Definition 2.3.

Suppose $\nu_1(\gamma_0) > 0$. Because $\gamma_0 \sim \gamma_1$ on $[0, 1]$, we obtain $\nu_1(\gamma_0) = \nu_1(\gamma_1)$.

To prove the first equality in (6.2), by Definition 5.2, there are smooth paths $\xi_j \in U^{\#}(\gamma_j)$ sufficiently close to $\gamma_j$ for $j = 0$ and 1 respectively such that we obtain $\xi_j(1) = \gamma_j(1)$ and $\Delta_1(\xi_j) = \Delta_1(\gamma_j)$. Then we obtain $i_1(\xi_j) = i_1(\gamma_j)$ by Proposition 6.1, and $\xi_i \sim \gamma_i$ with fixed end points by Corollary 6.2 for $i = 0, 1$.

Combining the homotopies from $\xi_0$ to $\gamma_0$, from $\gamma_0$ to $\gamma_1$, and from $\gamma_1$ to $\xi_1$ together, we obtain a homotopy $\delta \in C([0,1]^2, \mathrm{Sp}(2n))$ from $\xi_0$ to $\xi_1$ in $\mathcal{P}_1(2n)$ such that we obtain $\dim\ker(\delta_s(1) - I) = $ constant for all $0 \le s \le 1$. By the Lemma 4.4, we can further assume

$$\delta_s \equiv \delta(s, \cdot\,) \in \widehat{\mathcal{P}}_1(2n), \quad \forall 0 \le s \le 1. \tag{6.3}$$

Let us define

$$B_s(t) = -J\dot{\delta}_s(t)\delta_s^{-1}(t), \quad \forall t \in [0,1]. \tag{6.4}$$

Consider the linear Hamiltonian system (1.1) with $B = B_s$. By choosing the truncated space $Z$ in the saddle point reduction so that $2d = \dim Z$ is large enough for all $s \in [0,1]$, using Theorem 4.1 we obtain

$$m_s^+ = d - i_1(\delta_s) - \nu_1(\delta_s), \quad m_s^0 = \nu_1(\delta_s) = \nu_1(\gamma_0), \quad m_s^- = d + i_1(\delta_s), \tag{6.5}$$

for all $s \in [0,1]$. Then by Lemma 4.2, these Morse indices are constants for all $s \in [0,1]$. Thus we obtain $i_1(\xi_0) = i_1(\xi_1)$. This proves the first equality of (6.2).□

THEOREM 6.4 (Inverse homotopy invariant). *For any two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$ with $i_\tau(\gamma_0) = i_\tau(\gamma_1)$, suppose that there exists a continuous path $h : [0,1] \to \mathrm{Sp}(2n)$ such that $h(0) = \gamma_0(\tau)$, $h(1) = \gamma_1(\tau)$, and $\dim\ker(h(s) - I) = \nu_\tau(\gamma_0)$ for all $s \in [0,1]$. Then $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $h$.*

PROOF. Note that $\gamma_0$ and the joint path $h * \gamma_0 : [0,\tau] \to \mathrm{Sp}(2n)$ are homotopic on $[0,\tau]$ along $h$. We denote this homotopy by $\alpha : [0,1] \times [0,\tau] \to \mathrm{Sp}(2n)$. By Theorem 6.3, we obtain $i_\tau(h|_{[0,s]} * \gamma_0) = i_\tau(\gamma_0)$ for all $s \in [0,1]$, and then $i_\tau(h * \gamma_0)) = i_\tau(\gamma_1)$. So from $h * \gamma_0(\tau) = \gamma_1(\tau)$ and Corollary 6.2 we obtain that $h*\gamma_0$ and $\gamma_1$ are homotopic on $[0,\tau]$ with fixed end points. Denote this homotopy by $\beta : [0,1] \times [0,\tau] \to \mathrm{Sp}(2n)$.

We define a new map $\delta : [0,1] \times [0,\tau] \to \mathrm{Sp}(2n)$ by

$$\delta(s,t) = \begin{cases} \alpha\left(2\dfrac{\tau s}{\tau + t}, t\right) & \text{if } 0 \le s \le \dfrac{t}{2\tau} + \dfrac{1}{2}, \\ \beta\left(2\dfrac{\tau s - t}{\tau - t} - 1, t\right) & \text{if } \dfrac{t}{2\tau} + \dfrac{1}{2} < s \le 1. \end{cases}$$

Then this $\delta$ gives the homotopy of $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $h$. □

A direct consequence of Theorem 6.3 is that the Maslov-type index is invariant under conjugation in $\mathrm{Sp}(2n)$.

COROLLARY 6.5. *Given a path $\gamma \in \mathcal{P}_\tau(2n)$ and a matrix $M \in \mathrm{Sp}(2n)$, let us define $\beta(t) = M^{-1}\gamma(t)M$ for $0 \le t \le \tau$. Then we obtain*

$$i_\tau(\beta) = i_\tau(\gamma) \quad \textit{and} \quad \nu_\tau(\beta) = \nu_\tau(\gamma). \tag{6.6}$$

PROOF. Pick up a path $\alpha \in \mathcal{P}_\tau(2n)$ such that $\alpha(\tau) = M$. Define

$$\delta(s,t) = \alpha^{-1}(s)\gamma(t)\alpha(s), \quad \forall\ (s,t) \in [0,1] \times [0,\tau].$$

Then we obtain $\gamma \sim \beta$ on $[0,\tau]$ along $\delta(\,\cdot\,,\tau)$ via the homotopy $\delta$. Thus (6.6) follows from Theorem 6.3. □

The following theorem gives a characterization of $i_\tau(\gamma)$ when $\gamma \in \mathcal{P}_\tau^0(2n)$, and yields Theorem 1.3.

Theorem 6.6. *For any $\gamma \in \mathcal{P}_\tau^0(2n)$, we obtain* (1.8), *and for every $\beta \in \mathcal{P}_\tau^*(2n)$ which is sufficiently $C^0$-close to $\gamma$, we obtain*

$$i_\tau(\gamma) = i_\tau(\gamma_{-1}) \le i_\tau(\beta) \le i_\tau(\gamma_1) = i_\tau(\gamma) + \nu_\tau(\gamma), \tag{6.7}$$

*where the rotation perturbation paths $\gamma_{-1}$ and $\gamma_1$ of $\gamma$ are defined by* (3.8).

Proof. Without loss of generality, we suppose $\tau = 1$. Since $\mathcal{P}_1^*(2n)$ forms an open subset in $\mathcal{P}_1(2n)$, by Definition 2.3, it suffices to consider the case $\gamma \in \mathcal{P}_1^0(2n)$. By Definition 5.2 and Theorem 6.3, we can replace $\gamma$ by a nearby path $\widehat{\gamma} \in \widehat{\mathcal{P}}_1(2n)$ such that $\gamma \sim \widehat{\gamma}$ with fixed end points. Similarly we can also replace the path $\beta$ by a path $\widehat{\beta} \in \widehat{\mathcal{P}}_1^*(2n)$ such that $\beta \sim \widehat{\beta}$ with fixed end points, and we can require $\widehat{\beta}$ being $C^1$-sufficiently close to $\widehat{\gamma}$, if $\beta$ is $C^0$-sufficiently close to $\gamma$. Then from Proposition 4.5, using notations of the rotation perturbation paths $\widehat{\gamma}_{\pm 1}$ defined by (3.8) for $\widehat{\gamma}$ we obtain

$$\begin{aligned} i_1(\gamma) &= i_1(\widehat{\gamma}) = i_1(\widehat{\gamma}_{-1}) \le i_1(\widehat{\beta}) = i_1(\beta), \\ i_1(\beta) &= i_1(\widehat{\beta}) \le i_1(\widehat{\gamma}_1) = i_1(\widehat{\gamma}) + \nu_1(\widehat{\gamma}) = i_1(\gamma) + \nu_1(\gamma). \end{aligned}$$

This proves (6.7). Then (1.8) follows. □

Remark 6.7. Note that by this theorem, the Maslov-type index theory of paths in $\mathcal{P}_\tau(2n)$ is well defined and is a topological invariant of such paths.

Theorem 6.8 ($\diamond$-product additivity). *Suppose that $\gamma_0 \in \mathcal{P}_\tau(2n_0)$ and $\gamma_1 \in \mathcal{P}_\tau(2n_1)$. Then $\gamma_0 \diamond \gamma_1 \in \mathcal{P}_\tau(2n_0 + 2n_1)$ and*

$$i_\tau(\gamma_0 \diamond \gamma_1) = i_\tau(\gamma_0) + i_\tau(\gamma_1). \tag{6.8}$$

Proof. Set $\tau = 1$. If $\gamma_j \in \mathcal{P}_1^0(2n_j)$ for $j = 0$ and 1, by Definition 5.2 and Theorem 6.3, we can replace them by $\beta_j \in U^{\#}(\gamma_j)$ and then define the rotation perturbation paths $[\beta_j]_{-1}$ of $\beta_j$ by (3.8) respectively. Thus by Definition 5.2 and Proposition 2.5 we obtain

$$\begin{aligned} i_1(\gamma_0 \diamond \gamma_1) &= i_1(\beta_0 \diamond \beta_1) = i_1([\beta_0]_{-1} \diamond [\beta_1]_{-1}) \\ &= i_1([\beta_0]_{-1}) + i_1([\beta_1]_{-1}) \\ &= i_1(\beta_0) + i_1(\beta_1) = i_1(\gamma_0) + i_1(\gamma_1). \end{aligned}$$

Other cases of $\gamma_0$ and $\gamma_1$ being degenerate or not can be proved similarly. □

Finally, we can give

Proof of Theorem 1.4. Without loss of generality we suppose $\tau = 1$. By our previous discussions, the index function $i_1 : \bigcup_{n\in\mathbb{N}} \mathcal{P}_1(2n) \to \mathbb{Z}$ defined by Definition 5.4 satisfies these five conditions. Suppose we have another index function $\mu : \bigcup_{n\in\mathbb{N}} \mathcal{P}_1(2n) \to \mathbb{Z}$ satifying these five conditions. We carry out the proof in five steps.

(I) Claim 1. *For any* $M \in \mathrm{Sp}(2n)^0$*, there exists an integer* $k \in [1, n]$ *such that* $M$ *can be connected to a matrix*

$$Q = Q_1 \diamond \ldots \diamond Q_k \diamond Q_0 \tag{6.9}$$

*by a path* $h : [0,1] \to \mathrm{Sp}(2n)^0$ *satisfying* $Q_i = N_1(b_i)$ *with some* $b_i = \pm 1$ *or* $0$ *defined by* (1.6) *for* $1 \le i \le k$*,* $Q_0 \in \mathrm{Sp}(2n-2k)^*$*, and*

$$\dim\ker(h(t) - I) = \dim\ker(M - I), \quad \forall t \in [0,1]. \tag{6.10}$$

In fact, by applying Theorem 7.3 to $M$, we can choose a path $p \in \mathcal{P}_1(2n)$ such that $p(1) = P$ for $P$ given by (7.8). Let us define $h_1(t) = p(t)Mp(t)^{-1}$ for $t \in [0,1]$. Then if we replace $h$ by $h_1$, we obtain (6.10) and

$$h_1(1) = H_1 \diamond \ldots \diamond H_p \diamond K_1 \diamond \ldots \diamond K_q \diamond M_0, \tag{6.11}$$

where $H_i$ is a first type and $K_j$ is a second type normal form belonging to the eigenvalue 1 for $1 \le i \le p$ and $1 \le j \le q$, and $M_0$ possesses no eigenvalue 1.

For $H_i = N_1(b) \in \mathrm{Sp}(2)$ with $b = \pm 1$ or 0, we define $f_i(t) = H_i$ for all $t \in [0,1]$.

If $H_i = N_{h_i}(b) \in \mathrm{Sp}(2h_i)$ is of the first type with $h_i \ge 2$. Define

$$\xi_i(t) = H_i(I_2 \diamond D(1+t)^{\diamond(h_i - 1)}), \quad \forall t \in [0,1] \text{ and some small } \varepsilon > 0.$$

By direct computation we find that $1 \in \sigma(\xi_i(t))$ possesses algebraic and geometric multiplicities 2 and 1 respectively for all $t \in (0,1]$. Therefore by Theorem 7.3 again as in (6.11), there is a path $\eta_i : [0,1] \to \mathcal{M}(2h_i)$ (cf. (8.2)) which connects $\xi_i(1)$ to $N_1(b_i) \diamond M_i$ for $b_i = 1$ or $-1$. Then $f_i = \eta_i * \xi_i$ connects $H_i$ to $N_1(b) \diamond M_i$ within $\mathcal{M}(2h_i)$.

If $K_j = N_{k_j}(b) \in \mathrm{Sp}(2k_j)$ is of the second type. Then by definition we obtain $k_j \ge 2$ and $b = (b_1, \ldots, b_{k_j - 1}, 0)$. Let us define

$$\alpha_j(t) = N_{k_j}(b(1-t)), \quad \forall t \in [0,1].$$

The path $\alpha_j$ connects $N_{k_j}(b)$ to $N_{k_j}(0)$. Define

$$\beta_j(t) = N_{k_j}(0)[I_2 \diamond D(2^t)^{\diamond(k_j - 2)} \diamond I_2], \quad \forall t \in [0,1].$$

Using (7.2) and (7.4), we define

$$\gamma_j(t) = \mathrm{diag}(A_{k_j}(2, 2(1-t), 1-t, t, 1), C_{k_j}(1/2, 1-t, 1, t)), \quad \forall t \in [0,1].$$

Then the path $\gamma_j * \beta_j$ connects $N_{k_j}(0)$ to $N_2(0) \diamond D(2)^{\diamond(k_j - 2)}$. Using (7.2) and (7.4) we define

$$\zeta_j(t) = \left[\mathrm{diag}(A_2(1+t, 1-t^2, 1-t^2, 1-t^2, 1+t), C_2\left(\frac{1}{1+t}, 1-t, \frac{1}{1+t}, 0\right)\right)\right] \diamond D(2)^{\diamond(k_j - 2)}$$

for all $t \in [0,1]$. Then the path $g_j = \zeta_j * \gamma_j * \eta_j * \xi_j$ connects $K_j$ to $I_2 \diamond D(2)^{\diamond(k_j-1)}$ and satisfies $\dim\ker(g_j(t) - I) = 2$ for all $t \in [0,1]$. Define

$$h_2(t) = (f_1 \diamond \ldots \diamond f_p \diamond g_1 \diamond \ldots \diamond g_q \diamond M_0)(t), \quad \forall t \in [0,1].$$

Then $h_2$ connects $h_1(1)$ to a $\diamond$-product of $N_1(b)$ with $b = \pm 1$ or $0$ and matrices possessing no eigenvalue 1, and satisfies (6.10). By applying Theorem 7.3 to $h_2(1)$, similar to the argument in the first paragraph of this proof, we obtain a path $h_3 : [0,1] \to \mathrm{Sp}(2n)^0$ which connects $h_2(1)$ to $N$ in the Claim 1, and satisfies (6.10).

Now the path $h = h_3 * h_2 * h_1 : [0,1] \to \mathrm{Sp}(2n)^0$ yields the Claim 1.

(II) Claim 2. *The restriction of $\mu$ on $\mathcal{P}_1^0(2n)$ is completely determined by the restriction of $\mu$ on those paths in $\mathcal{P}_1^0(2k)$ whose end points are normal form matrices $N_1(b)$ defined by* (1.6) *or matrices in $\mathrm{Sp}(2k)^*$ for $1 \le k \le n$.*

In fact, for any $\gamma \in \mathcal{P}_1^0(2n)$, the above Claim 1 holds for $M = \gamma(1)$. Then by $1^\circ$ of $\mu$ and (6.10) we obtain

$$\mu(\gamma) = \mu(h * \gamma). \tag{6.12}$$

Since $\mathrm{Sp}(2)^0$ is path connected by [11], for each $Q_i$ in (6.9) with $1 \le i \le k$ there is a path $f_i : [0,1] \to \mathrm{Sp}(2)^0$ which connects $I_2$ to $Q_i$. Since $\mathrm{Sp}(2n-2k)$ is path-connected, there is a path $f_0 \in \mathcal{P}_1(2n-2k)$ which connects the identity matrix to $M_0$. Define

$$f(t) = ([f_1 * \phi_{1,2m\pi}] \diamond f_2 \diamond \ldots \diamond f_k \diamond f_0)(t), \quad \forall t \in [0,1]. \tag{6.13}$$

Because $f(1) = Q = h*\gamma(1)$ where the path $h$ and the matrix $Q$ are given by the Claim 1, we can choose $m \in \mathbb{Z}$ so that $i_1(f) = i_1(h * \gamma)$. Thus by Theorem 6.4, we obtain $f \sim h * \gamma$ with fixed end points. Therefore, by $1^\circ$ of $\mu$, we obtain

$$\mu(h * \gamma) = \mu(f). \tag{6.14}$$

By $2^\circ$ of $\mu$, we obtain

$$\mu(f) = \mu(f_1 * \phi_{1,2m\pi}) + \sum_{i=2}^{k} \mu(f_i) + \mu(f_0). \tag{6.15}$$

Now (6.12), (6.14), and (6.15) prove the Claim 2.

(III) By the above two claims and $1^\circ$–$3^\circ$ of $\mu$ we obtain that the value of $\mu$ on $\mathcal{P}_1^0(2n)$ is completely determined by the value of $\mu$ on $\bigcup_{1\le k\le n} \mathcal{P}_1^*(2k)$.

(IV) Note that $\mathcal{P}_1^*(2n)$ has a homotopy classification by $\mathcal{P}_{1,k}^*(2n)$ for all $k \in \mathbb{Z}$, and we obtain $\widehat{\alpha}_{n,k,1} \in \mathcal{P}_{1,k}^*(2n)$ for all $k \in \mathbb{Z}$. Thus by $1^\circ$ of $\mu$, the value of $\mu$ is completely determined by its value on these $\widehat{\alpha}_{n,k,1}$. Since each $\widehat{\alpha}_{n,k,1}$ is a $\diamond$-product of standard zigzag paths in $\mathcal{P}_1^*(2)$. By 2° of $\mu$, the value of $\mu$ is completely determined by $\widehat{\alpha}_{1,k,1}$.

(V) We consider the paths $\widehat{\alpha}_{1,k,1}$. By 5° of $\mu$, we suppose that for some integer $k \geq 0$ we obtain

$$\mu(\widehat{\alpha}_{1,k,1}) = k. \tag{6.16}$$

Define $h(t) = D(2)R(k\pi)\phi_{1,\pi}(t)$ for $t \in [0,1]$. Then by definition $\widehat{\alpha}_{1,k+1,1}$ is a reparametrization of the path $h * \widehat{\alpha}_{1,k,1}$. The path $h$ intersects $\mathrm{Sp}(2)^0$ only at $M = D(2)R(k\pi + \theta)$ with some $\theta \in (0, \pi)$. Now by Theorem 7.3 we can connect $M$ to $N_1(b)$ for $b = 1$ or $-1$ by a path $p$ satisfying $\dim\ker(p(t) - I) = 1$ for all $t \in [0,1]$. Choose $\theta > 0$ so small that $\psi(t) \equiv N_1(b)R((2t-1)\theta) \notin \mathrm{Sp}(2)^0$ for all $t \in [0,1]\backslash\{1/2\}$. By the structure of $\mathrm{Sp}(2)^*$, there are paths $g^{\pm} : [0,1] \to \mathrm{Sp}(2)^*$ such that $g^-$ connects $\widehat{\alpha}_{1,k,1}(1)$ to $N_1(b)R(-\theta)$ and $g^+$ connects $N_1(b)R(\theta)$ to $\widehat{\alpha}_{1,k+1,1}(1)$ within $\mathrm{Sp}(2)^*$ respectively, and the paths $g^+ * \psi * g^-$ and $\widehat{\alpha}_{1,k,1}(1)\phi_{1,\pi}$ are homotopic with fixed end points. Therefore we obtain

$$\begin{aligned} \widehat{\alpha}_{1,k,1} &\sim g^- * \widehat{\alpha}_{1,k,1}, \\ \widehat{\alpha}_{1,k+1,1} &\sim g^+ * \psi * g^- * \widehat{\alpha}_{1,k+1,1} \sim \psi * g^- * \widehat{\alpha}_{1,k,1}. \end{aligned}$$

Now, together with 1°, 4°, and 3° of $\mu$, we obtain

$$\mu(\widehat{\alpha}_{1,k+1,1}) = \mu(\psi * g^- * \widehat{\alpha}_{1,k,1}) = \mu(g^- * \widehat{\alpha}_{1,k,1}) + 1 = k + 1.$$

Thus by induction and a similar argument for the case of $k < 0$ we obtain (6.16) for all $k \in \mathbb{Z}$.

Thus by the above steps we have proved $\mu(\gamma) = i_1(\gamma)$ for all $\gamma \in \mathcal{P}_1(2n)$. This completes the proof. □

REMARK 6.9. It is easy to construct examples to show that the five axioms in the Theorem 1.4 are independent from each other.

Note that it is possible to define the Maslov-type indices differently by changing the value of $i_\tau(\widehat{\alpha}_{1,0,\tau})$ in the normality condition 5° to some different number, or by changing the conditions 3° and 4° on the behavior of the index map when the symplectic paths cross the singular hypersurface $\mathrm{Sp}(2n)^0$ in $\mathrm{Sp}(2n)$. Note that every second order linear Hamiltonian system can be converted into a linear Hamiltonian system naturally. It is proved in [2] and [18] by different methods that the Maslov-type index theory characterized by Theorem 1.4 for such Hamiltonian systems coincide precisely with the classical Morse index theory for these second order sysems. In this sense our Definition 5.4 of the Maslov-type index theory is natural.

## 7. Basic normal forms of symplectic matrices.

In this section we briefly recall results proved on the basic normal forms of symplectic matrices in [14].

We define the normal form matrix $N_k(b)$ of symplectic matrices belonging to the eigenvalue 1 as follows.

The case of $k=1$: $N_1(b)$ with $b=\pm 1$ or 0 is defined by (1.6).

The case of $k\geq 2$: the matrix $N_k(b)$ is of the following form

$$N_k(b)=\begin{pmatrix} A_k(1,1,1,0,1) & B_k(b) \\ 0 & C_k(1,1,1,0) \end{pmatrix}, \tag{7.1}$$

where for $a$, $c$, $d$, $e\in\mathbb{R}$, and $b=(b_1,\dots,b_k)\in\mathbb{R}^k$, the $k\times k$ real matrices $A_k(a,c,d,e)$, $B_k(t,b)$, and $C_k(a,c,d)$ are defined by

$$A_k(a,c,d,e,f)=\begin{pmatrix} 1 & c & 0 & \dots & 0 & 0 & e \\ 0 & a & c & \dots & 0 & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & c & 0 \\ 0 & 0 & 0 & \dots & 0 & a & d \\ 0 & 0 & 0 & \dots & 0 & 0 & f \end{pmatrix}, \tag{7.2}$$

$$B_k(b)=\begin{pmatrix} b_1 & 0 & \dots & 0 & 0 \\ b_2 & -b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{k-1} & -b_{k-1} & \dots & (-1)^{k-2}b_{k-1} & 0 \\ b_k & -b_k & \dots & (-1)^{k-2}b_k & (-1)^{k-1}b_k \end{pmatrix}, \tag{7.3}$$

(7.4)

$$C_k(a,c,d,e)=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -c & a & 0 & \dots & 0 & 0 & 0 \\ c^2 & -ca & a & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-c)^{k-3} & (-c)^{k-4}a & (-c)^{k-5}a & \cdots & a & 0 & 0 \\ (-c)^{k-2} & (-c)^{k-3}a & (-c)^{k-4}a & \cdots & -ca & a & 0 \\ (-c)^{k-1}-e & (-c)^{k-2}a & (-c)^{k-3}a & \cdots & c^2a & -ca & d \end{pmatrix}.$$

Moreover, when $k\geq 2$, we obtain

$$\dim\ker(N_k(b)-I)=1 \quad \text{if and only if } b_k\neq 0, \tag{7.5}$$

$$\dim\ker(N_k(b)-I)=2 \quad \text{if and only if } b_k=0. \tag{7.6}$$

DEFINITION 7.1. The normal form $N_k(b) \in \mathrm{Sp}(2k)$ belonging to the eigenvalue 1 is of the *first type*, if $k = 1$ or $k \geq 2$ and $\dim\ker(N_k(b) - I) = 1$. Otherwise it is of the *second type*.

The following results on the normal forms can be verified directly. So the proofs are left to the readers.

PROPOSITION 7.2. *For any normal form matrix $N_k(b)$ defined above, there exists $\theta_0 > 0$ such that for $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ with $|\theta| \leq \theta_0$ we obtain*

$$\begin{aligned} \det(N_k(b)[R(\theta_1) \diamond \dots \diamond R(\theta_k)] - I) \\ = \begin{cases} (\sin\theta_1)(\sin\theta_k)[-1 + o(|\theta|)] & \text{if } b_k = 0, \\ (\sin\theta_1) b_k [1 + o(|\theta|)] & \text{if } b_k \neq 0, \end{cases} \end{aligned} \tag{7.7}$$

*where $B(\theta) = o(|\theta|)$ if $\lim_{\theta\to 0} B(\theta) = 0$.*

THEOREM 7.3. *Fix $M \in \mathrm{Sp}(2n)^0$.*

1° *There exist $P \in \mathrm{Sp}(2n)$, two nonnegative integeres $p$ and $q$, an integer $r = 0$ or $1$ satisfying $1 \leq p + q \leq n$, first type normal forms $H_i \in \mathrm{Sp}(2h_i)$ and second type normal forms $K_j \in \mathrm{Sp}(2k_i)$ belonging to the eigenvalue 1 for $1 \leq i \leq p$ and $1 \leq j \leq q$ defined by (1.6) and (7.1), and $M_0 \in \mathrm{Sp}(2n - 2p - 2q)^*$, such that we obtain*

$$PMP^{-1} = H_1 \diamond \dots \diamond H_p \diamond K_1 \diamond \dots \diamond K_q \diamond M_0 \equiv N. \tag{7.8}$$

2° *In the case of 1°, set $k_0 = h_0 = 0$. Define*

$$\begin{cases} m_i = \sum_{s=0}^{i-1} h_s + 1 & \text{for } 1 \leq i \leq p, \\ m_{p+2j-1} = \sum_{s=0}^{p} h_s + \sum_{s=0}^{j-1} k_s + 1 & \text{for } 1 \leq j \leq q, \\ m_{p+2j} = \sum_{s=0}^{p} h_s + \sum_{s=0}^{j-1} k_s + k_j & \text{for } 1 \leq j \leq q, \end{cases} \tag{7.9}$$

*There exists a real number $\theta_0 \in (0, \pi/(8n))$ such that for $0 < |\theta| \leq \theta_0$ and $1 \leq i \leq p + 2q$ holds*

$$1 \leq c(m_i, \theta) \equiv \dim\ker(M - I) - \dim\ker(MP^{-1}R_{m_i}(\theta)P - I) \leq 2, \tag{7.10}$$

*where $R_k(\theta) = I_{2k-2} \diamond R(\theta) \diamond I_{2n-2k}$.*

3° *In the case of 2°, the integer $c(m_i, \theta)$ is independent of $\theta \in [-\theta_0, \theta_0] \backslash \{0\}$, and we denote this constant by $c(m_i)$.*

4° *In the previous notations, for* $1 \le k \le p+2q$, *we define*

$$\begin{aligned}(7.11)\quad d(m_k,\theta_1,\dots,\theta_q) \equiv{}& \dim\ker(MP^{-1}R_{m_1}(\theta_1)\dots R_{m_{k-1}}(\theta_{k-1}) \\ & \cdot R_{m_{k+1}}(\theta_{k+1})\dots R_{m_{p+2q}}(\theta_{p+2q})P-I) \\ & -\dim\ker(MP^{-1}R_{m_1}(\theta_1)\cdots R_{m_{p+2q}}(\theta_{p+2q})P-I),\end{aligned}$$

*for all* $\theta_i \in [-\theta_0,\theta_0]$, $i\in[1,p+2q]\backslash\{k\}$, $\theta_k\in[-\theta_0,\theta_0]\backslash\{0\}$. *Then there exists a sufficiently small* $\theta_0\in(0,\pi/(8n))$ *depending on* $M$ *such that* $d(m_k,\theta_1,\dots,\theta_q) = d(m_k)$ *is a constant independent of all these choices of* $\theta_1,\dots,\theta_{p+2q}$, *and* $d(m_k)$ *takes only the value* 1 *or* 2.

## 8. The topological structure of the regular part of $\mathrm{Sp}(2n)^0$

In this section we prove new results on the topological structures of the regular part $\mathcal{M}(2n)$ of $\mathrm{Sp}(2n)^0$ which is needed in Section 3, where we define

$$(8.1)\qquad \mathcal{M}(2n) = \{M\in\mathrm{Sp}(2n) \mid \dim\ker(M-I)=1\}.$$

THEOREM 8.1. *For any* $M\in\mathcal{M}(2n)$, *let* $P\in\mathrm{Sp}(2n)$ *so that*

$$(8.2)\qquad PMP^{-1} = M_1\diamond N_1\diamond M_2 \equiv N,$$

*where* $N_1\in\mathcal{M}(2h)$ *is a normal form of the eigenevalue* 1 *with* $h\in[1,n]$ *defined in Section* 7 *and* $M_i\in\mathrm{Sp}(2h_i)^*$ *for* $i=1$ *and* 2. *Here* $h+h_1+h_2=n$. *Define* $m_1=h_1+1$. *Then there exists a normal form matrix* $N_1(b)$ *defined in* (1.6) *with* $b=1$ *or* $-1$, *a matrix* $M_0\in\mathrm{Sp}(2n-2)^*$, *a small real number* $\theta_0>0$, *three paths* $\sigma:[0,1]\to\mathcal{M}(2n)$ *and* $\sigma^\pm:[0,1]\to\mathrm{Sp}(2n)^*$ *such that there hold*

$$(8.3)\qquad \sigma(0)=M, \qquad\qquad \sigma(1)=N_1(b)\diamond M_0,$$

$$(8.4)\qquad \sigma^+(0)=MP^{-1}R_{m_1}(\theta_0)P, \qquad \sigma^+(1)=[N_1(b)R(\theta_0)]\diamond M_0,$$

$$(8.5)\qquad \sigma^-(0)=MP^{-1}R_{m_1}(-\theta_0)P, \qquad \sigma^-(1)=[N_1(b)R(-\theta_0)]\diamond M_0,$$

*and* $[N_1(b)R(t\theta_0)]\diamond N_0\in\mathrm{Sp}(2n)^*$ *for all* $t\in[-1,1]\backslash\{0\}$, *where* $R_k(\theta)$ *is defined in* (7.10). *Furthermore, the distance between the paths* $\sigma^\pm$ *and* $\sigma$ *can be chosen to be not greater than the maximum of the distance between* $\sigma^\pm(1)$ *and* $\sigma(1)$.

To continue our study we need the following basic lemma.

LEMMA 8.2. *Let* $q:[0,1]\to\mathcal{M}(2n)$ *be a continuous curve. Fix small* $\theta_0>0$ *sufficiently close to* 0 *so that for all* $s\in[0,1]$ *and* $|\theta|\le\theta_0$ *the matrix*

$$(8.6)\qquad Q_\theta(s)\equiv q(s)R_{m(s)}(\theta),$$

*satisfies* $Q_\theta(s)\in\mathrm{Sp}(2n)^*$ *for* $0<|\theta|\le\theta_0$ *and* $s\in[0,1]$, *and* $m(s)\in[1,n]$ *is the least positive integer which has this property. The existence of* $m(s)$ *is given by*

*Theorem* 1 *of* [11] *with a slight modification of the proof there. Then there exist two path connected sets* $\Omega^+$ *and* $\Omega^-$ *such that we obtain*

$$\{Q_\theta(s) \mid 0 \le s \le 1,\ 0 < \pm\theta \le \theta_0\} \subset \Omega^\pm \subset \mathrm{Sp}(2n)^*. \tag{8.7}$$

*Furthermore, sets* $\Omega^+$ *and* $\Omega^-$ *belong to different path connected components of* $\mathrm{Sp}(2n)^*$*, and we can choose two continuous curves in* $f^\pm : [0,1] \to \Omega^\pm$ *so that* $f^\pm(s)$ *is as close to* $q(s)$ *as we want.*

PROOF. We first construct $\Omega^+$ in several substeps.

(A) Note that for fixed $s \in [0,1]$, the set

$$G(s) \equiv \{Q_\theta(s) \mid 0 < \theta \le \theta_0\}$$

defines a continuous curve in $\mathrm{Sp}(2n)^*$.

Next we fix a $\theta \in (0, \theta_0]$ and define an auxiliary set $\Omega(\theta)$ in the following steps (B) to (E).

(B) In general the map $Q_\theta : [0,1] \to \mathrm{Sp}(2n)^*$ may not be continuous. Suppose there are $k$ integers $A(\theta) \equiv \{m_1, \ldots, m_k\}$ appeared in (8.6) as $m(s)$ for all $s \in [0,1]$ satisfying $1 \le m_1 < \ldots < m_k \le n$ for some $k \in [1,n]$.

If $k = 1$, the map $Q_\theta$ is continuous and we define $\Omega(\theta) = Q_\theta([0,1])$, which is path connected.

Suppose $k > 1$. Then the map $Q_\theta$ is not continuous. We construct $\Omega(\theta)$ as follows.

(C) Suppose $\dim\ker(q(s_0)R_{m_i}(\alpha) - I) = 0$ for some $s_0 \in [0,1]$ and any $0 < \alpha \le \theta$. Without loss of generality we assume $m_i = 1$ here. For fixed $\alpha \in (0,\theta]$, by the effect of the rotation matrix $R_1(\alpha)$, there exists an open subinterval neighborhood $N(\alpha)$ of $s_0$ in $[0,1]$ depending on $\alpha$ such that

$$\det(q(s)R_1(\alpha) - I) \ne 0, \quad \forall s \in N(\alpha).$$

A slight modification of the proof in [11] on the effect of $R_1(\alpha)$ yields

$$\det(q(s)R_1(\alpha) - I) = (-1)^n \sin\alpha[(1 + a_{n+1,1}^2(s))b(s) + o(1)], \tag{8.8}$$

where by the continuity of $q$, both $a_{n+1,1}(s)$ and $b(s)$ are real continuous functions in $s$, independent of $\alpha$, and we obtain $b(0) \ne 0$. The term $o(1)$ is defined as $\lim_{\alpha\to 0} o(1) = 0$. Thus we can further require $\theta \in (0,\theta_0]$ to be sufficiently small so that there exists an open subinterval neighborhood $N$ of $s_0$ in $[0,1]$ satisfying $N \subset \cap\{N(\alpha) \mid 0 < \alpha \le \theta\}$. Thus we have proved the existence of an open subinterval neighborhood $N$ of $s_0$ in $[0,1]$ such that

$$\dim\ker(q(s)R_{m_i}(\theta) - I) = 0, \quad \forall s \in N,\ 0 < \alpha \le \theta. \tag{8.9}$$

Thus for each $m_i \in A(\theta)$ the set $\{s \in [0,1] \mid Q_\theta(s) = q(s)R_{m_i}(\theta)\}$ is a union of subintervals $\{N_{i,j}\}_{j \in E_i}$ in $[0,1]$ with some subindex set $E_i$. Denote by $N_{i,j}^\circ$ the interior of $N_{i,j}$ in $[0,1]$. Note that some $N_{i,j}$ may contain only one point. The unit interval $[0,1]$ is a disjoint union of all these subintervals $N_{i,j}$ for $j \in E_i$ and $i = 1, \ldots, k$.

(D) For any discontinuity point $s_0$ of $Q_\theta$, suppose $Q_\theta(s_0) = q(s_0)R_{m_i}(\theta)$ for some $m_i \in A(\theta)$. By the continuous effect of $R_{m_i}(\alpha)$ for $0 < \alpha \le \theta$ on $q(s)$ with $s$ near $s_0$ given by (8.9) and the openness of $\mathrm{Sp}(2n)^*$ in $\mathrm{Sp}(2n)$, we obtain a small open interval neighborhood $B(s_0)$ of $s_0$ in $[0,1]$ such that we obtain

$$\dim\ker(q(s)R_{m_i}(\alpha) - I) = 0, \quad \forall s \in B(s_0),\ 0 < \alpha \le \theta.$$

For any $s \in B(s_0)$, by definition $Q_\theta(s) = q(s)R_{m(s)}(\theta)$ for some $m(s) \in A(\theta)$. If $m(s) \ne m_i$, similar to (8.8) we obtain

$$\begin{aligned}\det(q(s)R_{m(s)}(\alpha) - I)\det(q(s)R_{m_i}(\alpha) - I)\\ = \sin^2\alpha[(a_{1,1}^2(s) + a_{n+1,1}^2(s))(a_{2,1}^2(s) + a_{n+2,1}^2(s))b(s)^2 + o(1)],\end{aligned}$$

for all $s \in B(s_0)$ and $|\alpha| > 0$ small.

This showes that $q(s)R_{m(s)}(\alpha)$ and $q(s)R_{m_i}(\alpha)$ locate in the same path connected components of $\mathrm{Sp}(2n)^*$ when $|\alpha| > 0$ is small. Thus there exists a continuous curve $\varphi_{\theta,s} : [0,1] \to \mathrm{Sp}(2n)^*$ such that $\varphi_{\theta,s}(0) = q(s)R_{m_i}(\theta)$ and $\varphi_{\theta,s}(1) = q(s)R_{m(s)}(\theta)$. If $m(s) = m_i$, define $\varphi_{\theta,s}([0,1]) = Q_\theta(s)$. Let us define

$$F_\theta(B(s_0)) = \bigcup\{\varphi_{\theta,s}([0,1]) \mid s \in B(s_0)\}.$$

Then $F_\theta(B(s_0))$ is a path connected subset of $\mathrm{Sp}(2n)^*$ containing the non-path connected set $Q_\theta(B(s_0))$ and the path connected set $\{q(s)R_{m_i}(\theta) \mid s \in B(s_0)\}$.

The set $F_\theta(B(s_0))$ is shown in the Figure 1 by thick black curves with $m(s) < m_i = m(s_0)$ and $s \in B(s_0)$.

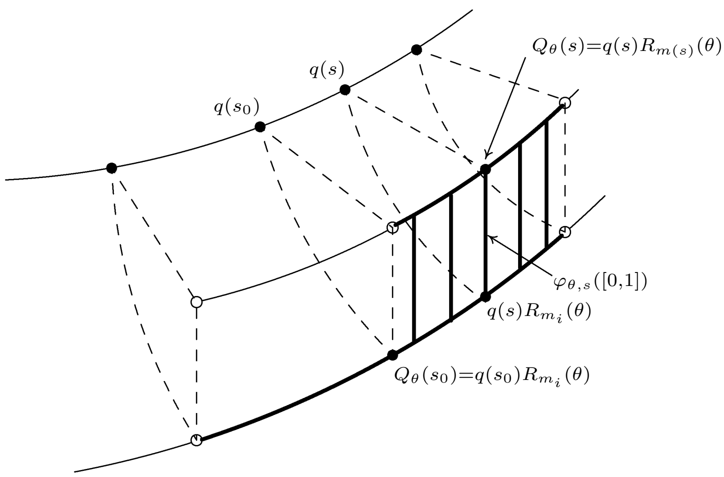


Figure 1. The set $F_\theta(B(s_0))$

(E) Define $\mathcal{F}$ to be the family of open subintervals of $[0,1]$ which consists of $B(s)$ for all discontinuity point $s$ of $Q_\theta$ and $N_{i,j}^\circ$ for all $j \in E_i$ and $i = 1, \dots, k$. Then $\mathcal{F}$ is an open covering of $[0,1]$. Thus $\mathcal{F}$ possesses a finite subcovering $\mathcal{F}_1$ of $[0,1]$. Let us define

$$\begin{aligned}\Omega(\theta) = &\left( \bigcup \{F_\theta(B(s)) \,\middle|\, B(s) \in \mathcal{F}_1,\ s \in [0,1]\} \right) \\ &\cup \left( \bigcup \{Q_\theta(N_{i,j}) \,\middle|\, N_{i,j}^\circ \in \mathcal{F}_1,\ j \in E_i,\ 1 \le i \le k\} \right).\end{aligned}$$

Then by our construction in (D), the set $\Omega(\theta)$ is path connected.

(F) Fix a $\theta \in (0, \theta_0]$ such that (8.9) holds. Using the set $\Omega(\theta)$ constructed in the step (B) or (C)–(E), and the set $G(s)$ defined in (A), we define

$$\Omega^+ = \left( \bigcup \{G(s) \,\middle|\, s \in [0,1]\} \right) \cup \Omega(\theta).$$

Then $\Omega^+$ is path connected and satisfies (8.7).

By a similar proof we obtain the path connected set $\Omega^-$ satisfying (8.7).

That $\Omega^+$ and $\Omega^-$ belong to different path connected components follows from Theorem 2 of [11]. The last conclusion of Lemma 8.2 follows from our construction of $\Omega^\pm$. □

With the aid of Lemma 8.2, we now give

PROOF OF THEOREM 8.1. By the $\mathbb{R}^3$-cylindrical coordinate representation of $\mathrm{Sp}(2)$ introduced in [11], it suffices to study the case for $n \ge 2$. Fix $M \in \mathcal{M}(2n)$ with (8.2) holds. We carry out the proof in three steps.

*Step 1.* Connecting $N$ to $N_1(b) \diamond M_3$ for $N_1(b) \in \mathrm{Sp}(2)^0$ defined in the Section 7 with $b = 1$ or $-1$ and a matrix $M_0 \in \mathrm{Sp}(2n-2)^*$.

By Theorem 7.3, there exists $p_1 \in \mathcal{P}_1(2n)$ such that

$$p_1(1)^{-1} N p_1(1) = H_1 \diamond M_1 \diamond M_2. \tag{8.10}$$

Define

$$f_1(t) = p_1(t)^{-1} N p_1(t), \quad \forall t \in [0,1]. \tag{8.11}$$

Then $f_1 : [0,1] \to \mathcal{M}(2n)$ satisfies $f(0) = N$ and $f_1(1) = H_1 \diamond M_1 \diamond M_2$.

If $h_1 = 1$, the step 1 is done.

If $h_1 \ge 2$, for $d > 1$ define a path in $\mathrm{Sp}(2n)$ by

$$f_2(t) = [H_1(I_2 \diamond D(d^t) \diamond \dots \diamond D(d^t))] \diamond M_1 \diamond M_2. \tag{8.12}$$

Then $f_2(0) = H_1 \diamond M_1 \diamond M_2$, and for $d > 1$ sufficiently close to 1, we have $\sigma(f(t)) = \{1, d^t, d^{-t}\} \cup \sigma(M_1) \cup M_2$, where 1 is a double eigenvalue with geometric multiplicity 1, $d^t$ and $d^{-t}$ are eigenvalues with algebraic multiplicity $(h_1 - 1)$. Thus $f_2$ is a continuous path in $\mathcal{M}(2n)$.

By Theorem 7.3, there exists $p_2 \in \mathcal{P}_1(2n)$ such that

$$p_2(1)^{-1} f_2(1) p_2(1) = N_1(b) \diamond M_0, \tag{8.13}$$

for some $N_1(b) \in \mathrm{Sp}(2)^0$ defined in (1.6) with $b = 1$ or $-1$ and a matrix $M_0 \in \mathrm{Sp}(2n-2)^*$. Define

$$f_3(t) = p_2(t)^{-1} f_2(1) p_2(t), \quad \forall t \in [0,1]. \tag{8.14}$$

Then $f \equiv f_3 * f_2 * f_1 : [0,1] \to \mathcal{M}(2n)$ satisfies $f(0) = N$ and $f(1) = N_1(b) \diamond M_0$.

*Step 2.* Perturbation paths of $f$.

By Lemma 8.2, we obtain two paths $f^{\pm} : [0,1] \to \mathrm{Sp}(2n)^*$ such that

$$\begin{cases} f^{\pm}(0) = NR_{m_1}(\pm\theta), \\ f^{\pm}(1) = [N_1(b) \diamond M_0] R_1(\pm\theta) = [N_1(b) R(\pm\theta)] \diamond M_0, \end{cases} \tag{8.15}$$

and the distance between $f^{\pm}$ and $f$ are not greater than twice the distance between $N$ and $NR_1(\theta)$. Here the distance on $\mathrm{Sp}(2n)$ is the induced metric from $\mathbb{R}^{4n^2}$.

*Step 3.* Construction of the three required paths.

For $P$ defined in (8.2), choose $p \in C([0,1], \mathrm{Sp}(2n))$ such that $p(0) = P$ and $p(1) = I$. Define

$$\sigma(t) = p(t)^{-1} f(t) p(t), \qquad \forall t \in [0,1], \tag{8.16}$$

$$\sigma^{\pm}(t) = p(t)^{-1} f^{\pm}(t) p(t), \quad \forall t \in [0,1]. \tag{8.17}$$

Then these three paths satisfy the requirements of the theorem.

The proof is complete. □

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