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Rui, Hongxing
School of Mathematics and System Science, Shandong University

Tabata, Masahisa
Faculty of Mathematics, Kyushu University



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**by**

**Hongxing RUI** and **Masahisa TABATA**

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

# A Mass-Conservative Characteristic Finite Element Scheme for Convection-Diffusion Problems

Hongxing Rui* and Masahisa Tabata†

* School of Mathematics and System Science
Shandong University , Jinan, 250100 China
E-mail:hxrui@sdu.edu.cn

† Faculty of Mathematics, Kyushu University,
Hakozaki, Higashi-ku, Fukuoka, 812-8581, Japan
E-email:tabata@math.kyushu-u.ac.jp

**Abstract**

We develop a mass-conservative characteristic finite element scheme for convection diffusion problem. This scheme preserves the mass balance identity. It is proved that the scheme is unconditionally stable and convergent with first order in time increment and $k$-th order in element size when the $P_k$ element is employed. Some numerical examples are presented to show the efficiency of the present scheme.

## 1 Introduction

Convection-diffusion problems are solved in various fields of sciences and technologies, e.g., transport problems of heat and solutes in moving fluids. In many applications the Peclet number is high, so the problems become convection dominant. In such circumstances the Galerkin finite element scheme produces easily oscillation solutions. Hence, elaborate numerical schemes based on new ideas such as upwind method, Petrov-Galerkin methods and characteristic(-curve) methods have been developed to perform stable computation. Among them the procedure of the characteristic method is natural from the physical point of view since it approximates particle movements, and it is attractive from the mathematical point of view since it symmetrizes the problem. Many authors have contributed to develop, analyse and apply characteristic finite element schemes; see [1], [4], [6], [7],[8], [13], [14], [15], [16], [21] and references therein.

An important property that the convection-diffusion problems possess is the mass balance; the mass should be preserved if there is no source. In the framework of characteristic methods it it not trivial to maintain this property. Some schemes have been proposed and studied from this point [1], [6], [8], [21].

In this paper we present a new characteristic finite element scheme which preserves the mass balance. Our approach is different from those taken in the previous study. Usually the characteristic method is used to approximate the material derivative term, i.e., the time derivative term plus the convection term of non-divergence form. We do not assume the velocity is incompressible. We use the divergence form and we approximate directly the time derivative term plus the divergence term (Lemma 1). Thus, it is proved that the mass balance is satisfied completely. When the equation is of divergence form like the density equation in the compressible flow field, the mass balance remains true whether the velocity is incompressible or not. Our result corresponds to it. We prove the stability and convergence with first order in time increment and $k$-th order in element size when the $P_k$ element is employed.

We use the Sobolev spaces $L^2(\Omega)$ and $H^m(\Omega)$, $m \geq 1$, with norms denoted by $\|\cdot\|$ and $\|\cdot\|_m$, respectively. We also use the Sobolev space $W^{m,\infty}(\Omega)$. We use the function spaces $H^m(X) = H^m((0,T);X)$ and $C^m(X) = C^m([0,T];X)$ for positive number $T$ and Banach space $X$, whose norms are denoted by $\|\cdot\|_{H^m(X)}$ and $\|\cdot\|_{C^m(X)}$, respectively. We often omit $(0,T)$ and $\Omega$ if there is no confusion, e.g., we write $C^j(H^m)$ in place of $C^j([0,T];H^m(\Omega))$. The symbol $(\cdot,\cdot)$ is used for the inner products in both $L^2(\Omega)$ and $L^2(\Omega)^d$, $d = 2,3$. We use $c$ (with or without subscript) to denote a generic constant independent of discretization parameters $h$, $\Delta t$, and solutions, which can take different values at each occurrence.

The remainder of this paper is organized as follows. In Section 2 we present the mass-conservative characteristic finite element scheme and show the mass balance identity. In Section 3 we analyze the stability and prove the convergence. In Section 4 we give two numerical examples. After stating the conclusion, in Appendix we review two upwind finite element schemes referred in Section 2.

## 2 A mass-conservative characteristic finite element scheme

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ $(d = 2,\ 3)$ with piecewise smooth boundary $\Gamma$, and $T$ be a positive constant. We consider the convection-diffusion operator

$$\mathscr{L}\phi \equiv \frac{\partial \phi}{\partial t} + \nabla\cdot(u\phi - \nu\nabla\phi) \equiv \frac{\partial \phi}{\partial t} + u\cdot\nabla\phi + (\nabla\cdot u)\phi - \nu\Delta\phi, \tag{1}$$

where $\nu(>0)$ is a diffusion coefficient and $u:\ \Omega\times(0,T)\to\mathbb{R}^d$ is a given velocity. We do not assume that the velocity $u$ is incompressible. Let $L$ be a representative length of the domain. When the Peclet number $\mathrm{Pe} \equiv |u|L/\nu$ is high, the conventional Galerkin finite element does not work. For the remedy we focus on schemes based on the method of characteristics.

An important property of the operator $\mathscr{L}$ is the mass-conservation. To describe it we consider the following initial boundary value problem; find $\phi : \Omega\times(0,T)\to\mathbb{R}$ such that

$$\mathscr{L}\phi = f \quad \text{in } \Omega\times(0,T), \tag{2a}$$

$$\nu\frac{d\phi}{dn} - \phi u\cdot n = g \quad \text{on } \Gamma\times(0,T), \tag{2b}$$

$$\phi(\cdot,0) = \phi^0 \quad \text{in } \Omega, \tag{2c}$$

where $f:\Omega\times(0,T)\to\mathbb{R}$, $g:\Gamma\times(0,T)\to\mathbb{R}$, and $\phi^0:\Omega\to\mathbb{R}$ are given functions. A corresponding weak formulation to (2) is to find $\phi:[0,T]\to V$ such that

$$\left(\frac{\partial\phi}{\partial t}(t),\psi\right)+a_1(\phi(t),\psi;\ u(t))+a_0(\phi(t),\psi)=(f(t),\psi)+[g(t),\psi]\quad(\forall\psi\in V), \tag{3a}$$

$$\phi(0)=\phi^0, \tag{3b}$$

where

$$V=H^1(\Omega),\quad a_1(\phi,\psi;\ u)=-(\phi,u\cdot\nabla\psi),\quad a_0(\phi,\psi)=\nu(\nabla\phi,\nabla\psi), \tag{4}$$
$$[\phi,\psi]=\int_\Gamma\phi\psi ds.$$

Substituting $\psi=1$ in (3a), we can easily derive the mass balance identity for $t\in(0,T]$

$$\int_\Omega\phi(x,t)\ dx=\int_\Omega\phi^0(x)\ dx+\int_0^t dt\int_\Omega f(x,t)dx+\int_0^t dt\int_\Gamma g(x,t)dx. \tag{5}$$

This property is preserved for the conventional Galerkin finite element method as follows. Let $V_h$ be a finite dimensional subspace of $V$ and $\Delta t$ be a time increment. We set $N_T=\lfloor T/\Delta t\rfloor$. Let $\phi_h$ be the solution of the problem; find $\{\phi_h^n\}_{n=1}^{N_T}\subset V_h$ such that for $n=1,\cdots,N_T$,

$$\left(\frac{\phi_h^n-\phi_h^{n-1}}{\Delta t},\psi_h\right)+a_1(\phi_h^n,\psi_h;\ u^n)+a_0(\phi_h^n,\psi_h)=(f^n,\psi_h)+[g^n,\psi_h],\quad\forall\psi_h\in V_h, \tag{6}$$

where $\phi_h^0\in V_h$ is an approximation to $\phi^0$ and the super-script $n$ of $u$, $f$ and $g$ means that the functions are evaluated at $t=n\Delta t$. Substituting $\psi_h=1$ in (6) and summing it up from $n=1$ until $m$, we get for $m=1,\cdots,N_T$

$$\int_\Omega\phi_h^m dx=\int_\Omega\phi_h^0 dx+\Delta t\sum_{n=1}^m\left(\int_\Omega f^n dx+\int_\Gamma g^n ds\right). \tag{7}$$

This property, however, does not hold for upwind approximations which modify the convection term $u\cdot\nabla\phi$, e.g., the upwind element choice approximation (32), which corresponds to the upwind finite differencing on the triangular mesh. To realize (7) in the framework of upwind approximations has been done by a mass-conservative upwind finite element method [2], where upwind modification is done for the term $-(\phi,u\cdot\nabla\psi)$. In Appendix we review the ideas of these two approximations.

On the other hand, numerical schemes based on Petrov-Galerkin approximation satisfy (7). In the streamline upwind Petrov-Galerkin method we substitute $\psi_h+\tau u(t)\cdot\nabla\psi_h$ into $\psi_h$ elementwise in (6) or a similar equation to it, where $\tau$ is a positive parameter of element size order [3], [11]. Therefore, by setting $\psi_h=1$, the identity (7) is obtained.

Characteristic methods are usually derived from the approximation of material derivative

$$\mathscr{L}_0\phi\equiv\frac{\partial\phi}{\partial t}+u\cdot\nabla\phi. \tag{8}$$

They are applicable to convection-dominated, or even purely hyperbolic problems and have an advantage that derived matrices are always symmetric. To the best of our knowledge, however, there are no characteristic schemes which satisfy (7), a discrete version of the mass balance identity (5). Our idea is to apply the characteristic approximation to the term

$$\mathscr{L}_1\phi \equiv \frac{\partial \phi}{\partial t} + u\cdot\nabla\phi + (\nabla\cdot u)\phi, \tag{9}$$

but not to the term (8). The present scheme we show below is proved to satisfy (7).

We assume that the velocity $u$ has the following regularity and vanishes on the boundary. The latter assumption leads to Proposition 1 below, which makes the argument simple and clear.

**Hypothesis 1** The velocity $u$ satisfies

$$u \in C^0\left([0,T];W^{1,\infty}(\Omega)\right), \quad u = 0 \text{ on } \partial\Omega.$$

Let $X\colon (0,T) \to \mathbb{R}^d$ be a solution of the ordinary differential equation,

$$\frac{dX}{dt} = u(X,t).$$

Then, we can write (8) as

$$(\mathscr{L}_0\phi)(X(t),t) = \frac{d}{dt}\phi(X(t),t).$$

We set $t^n = n\Delta t$ for $n \in \mathbb{Z}$. Subject to an initial condition $X(t^n) = x$ we get an approximate value of $X$ at $t^{n-1}$ by the Euler method,

$$X_1^n(x) \;=\; x - u^n(x)\Delta t. \tag{10}$$

The following result has been proved in [15].

**Proposition 1** Under Hypothesis 1 and $\Delta t < 1/\|u\|_{C^0(W^{1,\infty})}$, it holds

$$X_1^n(\Omega) = \Omega.$$

Let $\mathscr{T}_h \equiv \{K\}$ be a partition of $\bar{\Omega}$ by elements $K$, $h$ be the maximum diameter, and $V_h \subset H^1(\Omega)$ be a finite element space. In the sequel, we assume that the domain $\Omega$ is polygonal, which leads to $\bigcup\{K;\ K \in \mathscr{T}_h\} = \bar{\Omega}$.

The mass-conservative characteristic (MCC) finite element scheme we propose reads as follows; find $\{\phi_h^n\}_{n=1}^{N_T} \subset V_h$ such that for $n = 1,\cdots,N_T$,

$$\left(\frac{\phi_h^n - \phi_h^{n-1}\circ X_1^n\gamma^n}{\Delta t}, \psi_h\right) + \nu(\nabla\phi_h^n, \nabla\psi_h) = (f^n, \psi_h) + [g^n, \psi_h], \quad \forall\psi_h \in V_h, \tag{11}$$

where $\phi_h^0 \in V_h$ is an approximation to $\phi^0$, $\phi_h^{n-1} \circ X_1^n$ is a composition defined by

$$\left(\phi_h^{n-1} \circ X_1^n\right)(x) = \phi^{n-1}\left(X_1^n(x)\right),$$

and $\gamma^n$ is the Jacobian of the transformation $X_1^n$,

$$\gamma^n = \det\left(\frac{\partial X_1^n}{\partial x}\right) = \det\left(\delta_{ij} - \Delta t \frac{\partial u_i^n}{\partial x_j}\right). \tag{12}$$

**Theorem 1 (mass balance)** Let $\{\phi_h^n\}_{n=1}^{N_T}$ be the solution of (11). Under Hypothesis 1 and $\Delta t < 1/\|u\|_{C^0(W^{1,\infty})}$ it holds that for $m = 1, \cdots, N_T$

$$\int_\Omega \phi_h^m dx = \int_\Omega \phi_h^0 dx + \Delta t \sum_{n=1}^{m} \left(\int_\Omega f^n dx + \int_\Gamma g^n ds\right). \tag{13}$$

*Proof.* Substituting $1 \in V_h$ into $\psi_h$ in (11) and multiplying $\Delta t$, we get

$$(\phi_h^n, 1) - (\phi_h^{n-1} \circ X_1^n \gamma^n, 1) = \Delta t \left\{(f^n, 1) + [g^n, 1]\right\}.$$

By the inverse transformation of $X_1^n$ and Proposition 1, we have

$$\int_\Omega \phi_h^{n-1} \circ X_1^n \gamma^n dx = \int_\Omega \phi_h^{n-1} dx, \tag{14}$$

which implies

$$(\phi_h^n, 1) - (\phi_h^{n-1}, 1) = \Delta t \left\{(f^n, 1) + [g^n, 1]\right\}.$$

Summing up the equations above form $n = 1$ until $m$, we get (13). □

**Remark 1** In the proof of Theorem 1 the numerical computation is assumed to be performed exactly. Let $\phi_{hi}$ be the base function at node $P_i$. In the real computation the integration of composite term $\phi_h^{n-1} \circ X_1^n \gamma^n \phi_{hi}$ may cause numerical errors because the integrand is not smooth on element $K$ which the support of the integrand intersects with. If we do not use the identity (14), the term

$$E_1 \equiv \sum_{n=0}^{m-1} (\phi_h^n - \phi_h^n \circ X_1^n \gamma^n, 1)$$

should be added to the right-hand side of (13). More precisely the integration of composite terms is carried out, smaller $E_1$ becomes.

**Remark 2** The conventional characteristics/Galerkin finite element method which approximates the material derivative term (8) leads to the scheme

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X_1^n}{\Delta t}, \psi_h\right) + \nu(\nabla \phi_h^n, \nabla \psi_h) + ((\nabla \cdot u^n)\phi_h^n, \psi_h) = (f^n, \psi_h) + [g^n, \psi_h], \quad \forall \psi_h \in V_h. \tag{15}$$

This scheme does not satisfy the mass balance even when div $u^n = 0$. Comparing to (15), the present MCC scheme (11) is simpler and of mass-conservative.

**Remark 3** Neglecting the term of second order in $\Delta t$ in (12), we can replace $\gamma^n$ by

$$\gamma_0^n \equiv 1 - \Delta t \ \mathrm{div}\ u^n.$$

When the fluid is incompressible, $\gamma_0^n$ becomes identical. Although this replacement does not affect the convergence rate shown later in Theorem 3, the quantity $E_1$ in Remark 1 may increase because the identity (14) holds no more.

# 3 Stability and convergence

In this section we present two main theorems. The former shows the stability of scheme (11), and the latter gives error estimates.

For a set of function $\{\phi^n\}_{n=0}^{N_T}$ we define the following norms,

$$\|\phi\|_{l^\infty(L^2)} \equiv \max_{0\le n\le N_T} \|\phi^n\|,\ \|\phi\|_{l^2(L^2)} \equiv \left(\sum_{n=0}^{N_T} \Delta t \|\phi^n\|^2\right)^{1/2},\ |\phi|_{l^2(L^2)} \equiv \left(\sum_{n=0}^{N_T} \Delta t [\phi^n, \phi^n]\right)^{1/2}. \tag{16}$$

**Theorem 2 (stability)** Let $\{\phi_h^n\}_{n=0}^{N_T}$ be the solution of (11). Suppose Hypothesis 1 holds. Then, there exists a positive constant $c_1 = c_1(\|u\|_{C^0(W^{1,\infty})}, \nu)$ independent of $h$ and $\Delta t$ such that

$$\|\phi_h\|_{l^\infty(L^2)} + \sqrt{\Delta t}\|\mathscr{L}_{1h}\phi_h\|_{l^2(L^2)} + \sqrt{\nu}\|\nabla\phi_h\|_{l^2(L^2)} \le c_1(\|\phi_h^0\|_{L^2} + \|f\|_{l^2(L^2)} + |g|_{l^2(L^2)}), \tag{17}$$

where

$$\mathscr{L}_{1h}\phi_h^n \equiv \frac{\phi_h^n - \phi_h^{n-1}\circ X_1^n \gamma^n}{\Delta t}.$$

When $g = 0$, $c_1$ is independent of $\nu$.

*Proof.* Substituting $\psi_h = \phi_h^n$ in (11), we have

$$\frac{1}{2\Delta t}\left(\|\phi_h^n\|^2 - \|\phi_h^{n-1}\circ X_1^n\gamma^n\|^2\right) + \frac{\Delta t}{2}\|\mathscr{L}_{1h}\phi_h^n\|^2 + \nu\|\nabla\phi_h^n\|^2 = (f^n, \phi_h^n) + [g^n, \phi_h^n]. \tag{18}$$

Since $\gamma^n \le 1 + c\Delta t$, the inverse transformation of $X_1^n(x)$ leads to

$$\begin{aligned}\|\phi_h^{n-1}\circ X_1^n\gamma^n\|^2 &\le (1+c\Delta t)\int_\Omega (\phi_h^{n-1}\circ X_1^n)^2\gamma^n dx \\ &= (1+c\Delta t)\|\phi_h^{n-1}\|^2.\end{aligned} \tag{19}$$

The right-hand side of (18) is estimated as

$$(f^n, \phi_h^n) \le \frac{1}{2}\|f^n\|^2 + \frac{1}{2}\|\phi_h^n\|^2,$$

and

$$[g^n, \phi_h^n] \le \|g^n\|_{L^2(\Gamma)} \|\phi_h^n\|_{L^2(\Gamma)} \le c\|g^n\|_{L^2(\Gamma)} \|\phi_h^n\|_{H^1(\Omega)}$$
$$\le \frac{c^2}{2\nu}\|g^n\|^2_{L^2(\Gamma)} + \frac{\nu}{2}\|\phi_h^n\|^2_{H^1(\Omega)}.$$

Combining these estimates with (18), we obtain

$$\frac{1}{2\Delta t}\left(\|\phi_h^n\|^2 - \|\phi_h^{n-1}\|^2\right) + \frac{\Delta t}{2}\|\mathscr{L}_{1h}\phi_h^n\|^2 + \frac{\nu}{2}\|\nabla\phi_h^n\|^2$$
$$\le c\|\phi_h^{n-1}\|^2 + \frac{1}{2}\|\phi_h^n\|^2 + \frac{1}{2}\|f^n\|^2 + c_1(\nu)\|g^n\|^2_{L^2(\Gamma)}, \tag{20}$$

which completes the proof by virtue of Gronwall's inequality. □

In order to state error estimates we prepare the following hypotheses. Let $\Pi_h$ be the Lagrange interpolation operator from $C^0(\overline{\Omega})$ into $V_h$ [5].

**Hypothesis 2** There exists a positive integer $k$ such that for $\phi \in H^{k+1} \cap C^0(\overline{\Omega})$

$$\|\Pi_h\phi - \phi\|_s \le ch^{k+1-s}\|\phi\|_{k+1} \quad (s = 0, 1).$$

**Hypothesis 3** $\phi$ has the regularity,

$$\phi \in C^0(H^{k+1}) \cap C^1(H^k) \cap C^2(L^2).$$

**Lemma 1 (consistency)** Suppose functions $u$ and $\phi$ satisfy Hypotheses 1 and 3 for $k = 1$, respectively. Then, it holds that for $n = 1, \cdots, N_T$

$$\|\frac{\partial\phi^n}{\partial t} + \nabla\cdot(u^n\phi^n) - \frac{\phi^n - \phi^{n-1}\circ X_1^n\gamma^n}{\Delta t}\| \le c\Delta t\|\phi\|_{C^2(L^2)\cap C^1(H^1)\cap C^0(H^2)}.$$

*Proof.* The left-hand side is written as $\|I_1 + I_2\|$, where

$$I_1^n = \left(\frac{\partial\phi^n}{\partial t} + u^n\cdot\nabla\phi^n\right) - \frac{\phi^n - \phi^{n-1}\circ X_1^n}{\Delta t}, \tag{21}$$
$$I_2^n = (\nabla\cdot u^n)\phi^n - \phi^{n-1}\circ X_1^n(1-\gamma^n)/\Delta t. \tag{22}$$

We can evaluate $I_1^n$ like [15] and get

$$\|I_1^n\| \le c\Delta t\|\phi\|_{C^2(L^2)\cap C^1(H^1)\cap C^0(H^2)}. \tag{23}$$

From (12) we have

$$\frac{1-\gamma^n}{\Delta t} = \nabla\cdot u^n + O(\Delta t),$$

which leads to

$$\begin{aligned} \|I_2^n\| &= \|\nabla\cdot u^n(\phi^n-\phi^{n-1}\circ X_1^n)+O(\Delta t)\phi^{n-1}\circ X_1^n\|_{L^2} \\ &\le c\Delta t\|\phi\|_{C^0(H^1)\cap C^1(L^2)}. \end{aligned} \tag{24}$$

Combining (23) with (24), we get the result. □

Now we show the error estimate.

**Theorem 3 (error estimate)** Let $\phi$ be the solutions of (2). Suppose Hypotheses 1, 2 and 3 hold for a positive integer $k$. Let $\phi_h$ be the solutions of scheme (11) subject to the initial condition

$$\phi_h^0 = \Pi_h\phi^0.$$

Then there exists a positive constant $c_2 = c_2(\|u\|_{C^0(W^{1,\infty})}, \nu)$ independent of $h$, $\Delta t$ and $\phi$ such that

$$\begin{aligned} &\|\phi_h-\phi\|_{l^\infty(L^2)}+\sqrt{\Delta t}\|\mathscr{L}_{1h}(\phi_h-\phi)\|_{l^2(L^2)}+\sqrt{\nu}\|\nabla(\phi_h-\phi)\|_{l^2(L^2)} \\ \le\ & c_2\big(h^k\|\phi\|_{C^0(H^{k+1})\cap C^1(H^k)}+\Delta t\|\phi\|_{C^2(L^2)\cap C^1(H^1)\cap C^0(H^2)}\big). \end{aligned} \tag{25}$$

When $g=0$, $c_2$ is independent of $\nu$.

*Proof.* (2a) is equivalent to

$$\frac{\partial\phi^n}{\partial t}+u^n\cdot\nabla\phi^n+(\nabla\cdot u^n)\phi^n-\nu\Delta u^n = f^n. \tag{26}$$

From (26) it holds that for any $\psi\in H^1(\Omega)$

$$\left(\frac{\phi^n-\phi^{n-1}\circ X_1^n\gamma^n}{\Delta t},\psi\right)+\nu(\nabla\phi^n,\nabla\psi) = (f^n,\psi)+[g^n,\psi]-(I_1^n+I_2^n,\psi), \tag{27}$$

where $I_1^n$ and $I_2^n$ are defined in (21) and (22). Set $e_h^n=\phi_h^n-\Pi_h\phi^n$ and $\eta^n=\phi^n-\Pi_h\phi^n$. Substituting $\psi_h=e_h^n$ in (11) and $\psi=e_h^n$ (27), and subtracting (27) from (11) , we have

$$\begin{aligned} &\frac{1}{2\Delta t}\left(\|e_h^n\|^2-\|e_h^{n-1}\circ X_1^n\gamma^n\|^2\right)+\frac{\Delta t}{2}\|\mathscr{L}_{1h}e_h^n\|^2+\nu\|\nabla e_h^n\|^2 \\ =\ & \left(\frac{e_h^n-e_h^{n-1}\circ X_1^n\gamma^n}{\Delta t},e_h^n\right)+\nu(\nabla e_h^n,\nabla e_h^n) \\ =\ & (I_1^n+I_2^n,e_h^n)+\left(\frac{\eta^n-\eta^{n-1}\circ X_1^n\gamma^n}{\Delta t},e_h^n\right)+\nu(\nabla\eta^n,\nabla e_h^n). \end{aligned} \tag{28}$$

From Lemma 1 the first term of the right-hand side is estimated as

$$(I_1^n+I_2^n,e_h^n)\le c\Delta t^2\|\phi\|^2_{C^2(L^2)\cap C^1(H^1)\cap C^0(H^2)}+\frac{1}{2}\|e_h^n\|^2.$$

Similarly to [15] we estimate the second term of the right-hand side to obtain

$$
\begin{aligned}
\left(\frac{\eta^n-\eta^{n-1}\circ X_1^n\gamma^n}{\Delta t},e_h^n\right) &= \left(\frac{\eta^n-\eta^{n-1}\circ X_1^n}{\Delta t},e_h^n\right)+\left(\frac{\eta^{n-1}\circ X_1^n(1-\gamma^n)}{\Delta t},e_h^n\right)\\
&\le \left(\|\frac{\eta^n-\eta^{n-1}\circ X_1^n}{\Delta t}\|+c\|\eta^{n-1}\circ X_1^n\|\right)\|e_h^n\|\\
&\le \left(\|\frac{\eta^n-\eta^{n-1}}{\Delta t}\|+c\|\eta^{n-1}\|_1+c\|\eta^{n-1}\|\right)\|e_h^n\|\\
&\le c\left(h^{2k}\|\phi\|^2_{C^0(H^{k+1})\cap C^1(H^k)}\right)+\frac{1}{2}\|e_h^n\|^2.
\end{aligned}
$$

The third term is easily evaluated as

$$
\nu(\nabla\eta^n,\nabla e_h^n)\le c\nu h^k\|\phi\|_{C^0(H^{k+1})}\|\nabla e_h^n\|\le\frac{\nu}{2}\|\nabla e_h^n\|^2+c\nu h^{2k}\|\phi\|^2_{C^0(H^{k+1})}.
$$

Combining these estimates with (28), we get

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|e_h^n\|^2-\|e_h^{n-1}\circ X_1^n\gamma^n\|^2\right)+\frac{\Delta t}{2}\|\mathscr{L}_{1h}e_h^n\|^2+\frac{\nu}{2}\|\nabla e_h^n\|^2\\
&\le\|e_h^n\|^2+c\left(\Delta t^2\|\phi\|^2_{C^2(L^2)\cap C^1(H^1)\cap C^0(H^2)}+h^{2k}\|\phi\|^2_{C^0(H^{k+1})}\right)
\end{aligned}
$$

A similar estimate to (19), the discrete Gronwall inequality and Hypothesis 3 lead to

$$
\begin{aligned}
&\|e_h\|_{l^\infty(L^2)}+\sqrt{\Delta t}\|\mathscr{L}_{1h}e_h\|_{l^2(L^2)}+\sqrt{\nu}\|\nabla e_h\|_{l^2(L^2)}\\
\le\ \ & c\left(h^k\|\phi\|_{C^0(H^{k+1})\cap C^1(H^k)}+\Delta t\|\phi\|_{C^2(L^2)\cap C^1(H^1)\cap C^0(H^2)}\right). \qquad (29)
\end{aligned}
$$

Combining (29) with the estimate of $\eta^n$, we complete the proof. □

# 4 Numerical examples

In this section we show some numerical results to observe the efficiency of the present MCC finite element scheme. We compare numerical results of the MCC scheme (11) with those of the mass-conservative upwind FEM (35) and the conventional characteristics/Galerkin FEM (15).

In schemes (11) and (15) composite functions are integrated on elements. We approximate the integral $\int_K \phi_h^{n-1}\circ X_1^n\psi_h\gamma^n dx$ by a numerical integration formula. We use the same numerical integration method as the one in [15]. We divide the triangle $K$ into 16 congruent small triangles. Approximating $\phi_h^{n-1}\circ X_1^n\psi_h\gamma^n$ by the linear interpolation on each small triangle and integrating the interpolated function, we get an approximate value of the integral. In the following examples we use the $P_1$ element. Hence, Hypothesis 2 is satisfied for $k=1$ [5], which implies the convergence order is $O(\Delta t+h)$ by Theorem 3. As we take $\Delta t=O(h)$ in the examples, the final convergence order becomes $O(h)$.

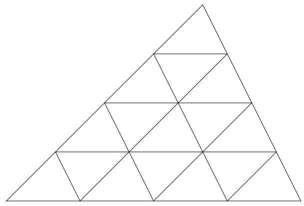

Figure 1: Figure 1: A triangle $K$ divided into 16 congruent triangles.

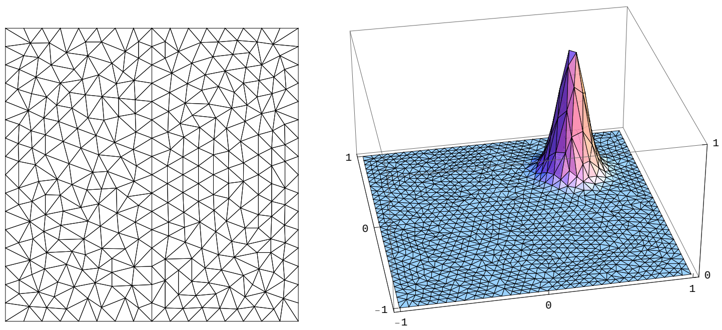

Figure 2: A mesh(N=16) and the exact solution $\phi$ of Example 1 at $t = 0.5$.

**Example 1** The data and $\phi$ are as follows,

$$\begin{aligned}
\Omega &= (-1,1) \times (-1,1),\ T = 0.5, \\
u(x,t) &= (1+\sin(t-x_1), 1+\sin(t-x_2))^T,\ \nu = 0.01, \\
\phi(x,t) &= \exp\left(-\frac{1-\cos(t-x_1)}{\nu}\right)\exp\left(-\frac{1-\cos(t-x_2)}{\nu}\right).
\end{aligned} \tag{30}$$

Then, the right-hand side $f$ in (2a) is identically equal to 0. The velocity is not incompressible, $\nabla \cdot u \neq 0$. Although Hypothesis 1 is not satisfied, i.e., $u \neq 0$ on $\Gamma$, the value of $\phi$ on $\Gamma$ is almost equal to zero, less $5.0 \times 10^{-6}$, we can neglect the effect of the flux $\phi u \cdot n$ on the boundary. Dividing each side of the square into $N$ segments,

$$N = 16,\ 32,\ 64,\ 128,\ 256,$$

we make partitions $\{\mathscr{T}_h\}$ consisting of non-uniform triangular elements; see Fig. 2. We set $h = 2/N$ and $\Delta t = h$. Fig. 3 shows relative errors in $\ell^\infty(L^2)$-norm, $\ell^2(H_0^1)$-seminorm, and of mass at $T$,

$$\frac{\|\phi_h - \Pi_h\phi\|_X}{\|\Pi_h\phi\|_X} \quad \left(X = \ell^\infty(L^2),\ \ell^2(H_0^1)\right), \qquad \frac{\left|\int_\Omega \phi_h^{N_T}\,dx - \int_\Omega \Pi_h\phi^{N_T}\,dx\right|}{\left|\int_\Omega \Pi_h\phi^{N_T}\,dx\right|}$$

by the MCC FEM (•), the conventional characteristics/Galerkin FEM (○) and the mass-conservative upwind FEM (⋄). The errors of mass by the mass-conservative upwind FEM (35) are too small to be plotted in the graph. We can see good convergence results of the

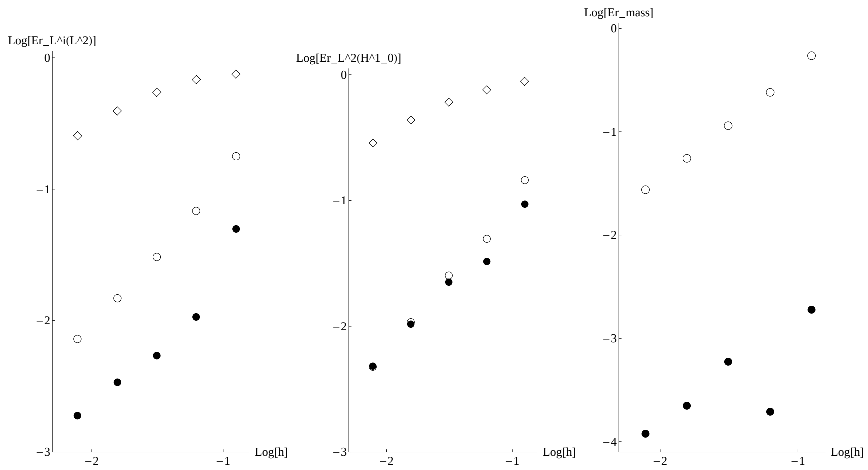


Figure 3: Errors vs. $h$ (Example 1). $\ell^\infty(L^2)$-norm (left), $\ell^2(H_0^1)$-seminorm (center), and mass(right).

Table 1: The slopes of the graphs by the results on the finest two meshes (Example 1).

| Scheme | symbol | $\ell^\infty(L^2)$ | $\ell^2(H_0^1)$ | mass |
|---|---|---|---|---|
| (35) | $\diamond$ | 0.63 | 0.61 | - |
| (15) | $\circ$ | 1.02 | 1.19 | 1.02 |
| MCC (11) | $\bullet$ | 0.84 | 1.11 | 0.90 |

MCC FEM in both norm and seminorm. Caused by numerical integration error, the mass balance by MCC is not preserved exactly, but is much better than (15). The slopes of the graphs obtained from the finest two meshes, $N = 128, 256$, are listed in Table 1. The theoretical convergence results $O(h)$ are reflected.

**Example 2** The data and $\phi$ are as follows,

$$\Omega = (0,1)\times(0,1),\ T = 1,$$
$$u(x) = (-x_2 \sin\pi x_1 \sin\pi x_2, x_1 \sin\pi x_1 \sin\pi x_2)^T, \quad \nu = 0.1,$$
$$\phi(x,t) = x_1 x_2 (1-x_2)\cos(t+x_1+x_2).$$

The velocity is not incompressible, $\nabla\cdot u \neq 0$. By substituting the above $\phi$, $u$ and $\nu$ in (2a) and (2b), respectively, $f$ and $g$ are obtained; they are not equal to zero. Dividing each side of the square into $N$ segments,

$$N = 8,\ 16,\ 32,\ 64,$$

we make partitions $\{\mathscr{T}_h\}$ consisting of non-uniform triangular elements. We set $h = 1/N$ and $\Delta t = 0.8h$. Fig. 5 shows relative errors in $\ell^\infty(L^2)$-norm, $\ell^2(H_0^1)$-seminorm, and of mass at $T$ by the MCC FEM (•), the conventional characteristics/Galerkin FEM (○) and the mass-conservative upwind FEM (◇). The errors of mass by the mass-conservative upwind FEM (35) are too small to be plotted in the graph. In this example the results by (15) are slightly better than MCC in both norm and seminorm, but the mass balance by MCC is much better than (15). The slopes of the graphs obtained from the finest two meshes, $N = 32, 64$, are listed in Table 2. The theoretical convergence results $O(h)$ are reflected.

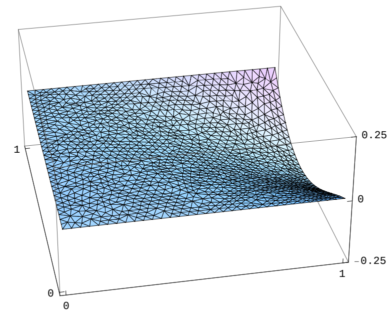


Figure 4: The exact solution $\phi$ of Example 2 at $t = 1$.

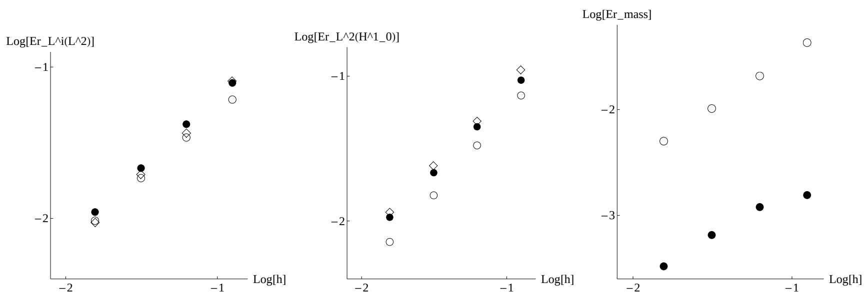


Figure 5: Errors vs. $h$ (Example 2). $\ell^\infty(L^2)$-norm (left), $\ell^2(H_0^1)$-seminorm (center), and mass(right).

Table 2: The slopes of the graphs by the results on the finest two meshes (Example 2).

| Scheme | symbol | $\ell^\infty(L^2)$ | $\ell^2(H_0^1)$ | mass |
|---|---|---|---|---|
| (35) | ◇ | 1.05 | 1.07 | - |
| (15) | ○ | 0.94 | 1.06 | 1.01 |
| MCC (11) | • | 0.96 | 1.02 | 0.97 |

# 5 Conclusions

We have presented a new mass-conservative characteristic finite element scheme of first order in time increment. The modification from the conventional characteristics/Galerkin method is very small, i.e., only the multiplication of the Jacobian to the composite term and the elimination of the term $(\nabla \cdot u)\phi$. The scheme is unconditionally stable. We have proved the stability and convergence of order $\Delta t + h^k$ , which has been recognized by numerical results for $k = 1$. In the forthcoming paper we will present a corresponding scheme of second order in time increment $\Delta t$.

# Appendix

Here we review two kinds of upwind finite element approximations developed in the early days. The one has monotone property and the other does mass-conservation property. The ideas used for these approximations are simple and natural. From them many improved upwind finite element/volume schemes have been developed. Let $\mathscr{T}_h = \{K\}$ be a partition of $\Omega$ by simplices, i.e., triangles ($d = 2$) or tetrahedron ($d = 3$).

## A.1 The upwind element choice approximation [17]

Let $V = H^1_0(\Omega)$ and $a_1^{(\infty)}$ be the bilinear form on $V$ defined by

$$a_1^{(\infty)}(\phi, \psi;\ u) = (u \cdot \nabla\phi, \psi), \qquad \phi, \psi \in V. \tag{31}$$

Let $V_h \subset V$ be the P1 finite element space. The upwind element choice approximation $a_{1h}^{(\infty)}$ to $a_1^{(\infty)}$is defined by

$$a_{1h}^{(\infty)}(\phi_h, \psi_h;\ u) = \sum_P u(P) \cdot \nabla\phi_{h|K_P^u}\ \psi_h(P) \mathrm{meas} D_P, \qquad \phi_h, \psi_h \in V_h, \tag{32}$$

where $P$ runs over all the nodes in $\Omega$, $K_P^u$ is the upwind element at $P$ with respect to $u(P)$, and $D_P$ is the barycentric domain at $P$. The definitions of $K_P^u$ and $D_P$ are as follows. Upwind element $K_P^u$ is an element $K \in \mathscr{T}_h$ such that

(i) $P$ is a vertex of $K$,
(ii) The vector $u(P)(\neq 0)$ with endpoint $P$ intersects $K \backslash \{P\}$.

In the case when $u(P)$ is parallel to an edge (or face) including $P$, $K_P^u$ is not uniquely defined, but even in this case the definition (32) is well-defined. Barycentric domain $D_P$ is defined by

$$D_P = \bigcup_K \{D_P^K;\ P \text{ is a vertex of } K \in \mathscr{T}_h\},$$
$$D_P^K = \bigcap_Q \{x \in K;\ Q(\neq P) \text{ is a vertex of } K, \lambda_P(x) \geq \lambda_Q(x)\},$$

where $\lambda_R$, $R = P, Q$, is the barycentric coordinate associated with vertex $R$ of $K$.

(32) was presented in 1977 and it is one of the upwind finite element approximations developed in the earliest stage. It has a similar property to the first-order upwind finite difference approximation. When the mesh $\mathscr{T}_h$ is of weakly acute type, i.e., all angles of triangles are less than or equal to $\pi/2$ in $d = 2$, we can derive monotone finite element schemes. The solution satisfies a discrete maximum principle when the original problem has the maximum principle [10], [18]. Approximation (32) is not mass-conservative even if the definition is extended appropriately to $V_h \subset H^1(\Omega)$. (32) is extended to second- and third-order upwind approximations for high-Reynolds number flow problems [9], [19].

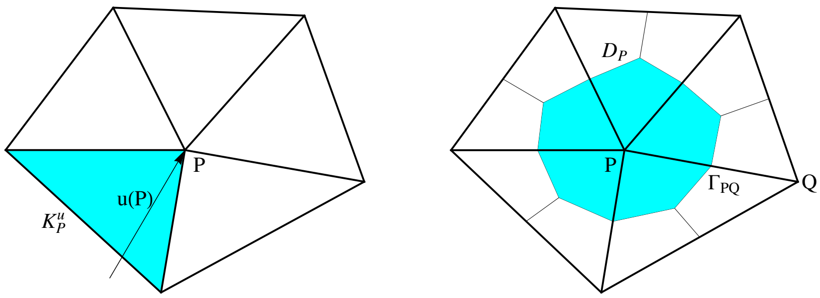


Figure 6: Upwind element $K_P^u$ at $P$ (left) and the barycentric domain $D_P$ (right).

## A.2 The mass-conservative upwind approximation [2]

Let $V = H^1(\Omega)$ and $a_1$ be the bilinear form on $V$ defined by (4),

$$a_1(\phi, \psi;\ u) = -(\phi, u \cdot \nabla \psi), \qquad \phi, \psi \in V.$$

Let $V_h \subset V$ be the P1 finite element space. The mass-conservative upwind approximation $a_{1h}$ to $a_1$is defined by

$$a_{1h}(\phi_h, \psi_h;\ u) = \sum_P \psi_h(P) \sum_{Q \in \Lambda_P} \left(\beta_{PQ}(u)^+ \phi_h(P) - \beta_{PQ}(u)^- \phi_h(Q)\right), \qquad \phi_h, \psi_h \in V_h, \tag{33}$$

where $P$ runs over all the nodes in $\bar{\Omega}$,

$$\Lambda_P = \{Q \in \bar{\Omega};\ \text{node Q is adjacent to P}\}, \quad \beta_{PQ}(u) = \int_{\Gamma_{PQ}} u \cdot n\, ds, \tag{34}$$

$$\Gamma_{PQ} = D_P \cap D_Q, \quad \beta^+ = \max(\beta, 0), \quad \beta^- = \max(-\beta, 0),$$

and $n$ is the outer normal to $\Gamma_{PQ}$ from the barycentric domain $D_P$. (33) is derived as follows,

$$\begin{aligned} a_1(\phi,\psi;\, u) &= (\nabla\cdot(u\phi),\psi) - [\phi u\cdot n,\psi] \\ &= \sum_P \int_{D_P} \nabla\cdot(u\phi)\psi\, dx - [\phi u\cdot n,\psi] \\ &= \sum_P \int_{\partial D_P} n\cdot u\phi\psi\, ds - [\phi u\cdot n,\psi] \\ &\approx \sum_P \psi(P) \int_{\partial D_P} n\cdot u\phi\, ds - [\phi u\cdot n,\psi] \\ &\approx \sum_P \psi(P) \int_{\Gamma_{PQ}} n\cdot u\phi\, ds \\ &\approx a_{1h}(\phi,\psi;\, u), \end{aligned}$$

where the relation $\beta = \beta^+ - \beta^-$ is used at the last line. A mass-conservative finite element scheme for (2) is to find $\{\phi_h^n\}_{n=1}^{N_T} \subset V_h$ such that for $n = 1,\cdots,N_T$,

$$\left(\frac{\bar{\phi}_h^n - \bar{\phi}_h^{n-1}}{\Delta t}, \bar{\psi}_h\right) + a_{1h}(\phi_h^n,\psi_h;\, u) + \nu(\nabla\phi_h^n,\nabla\psi_h) = (f^n,\psi_h) + [g^n,\psi_h], \quad \forall\psi_h \in V_h. \tag{35}$$

where $\phi_h^0 \in V_h$ is an approximation to $\phi^0$ and $- : V_h \to L^2(\Omega)$ is a lumping operator defined by

$$\bar{\phi}_h = \sum_P \phi_h(P)\chi_P, \quad \chi_P(x) = \begin{cases} 1, & (x \in D_p), \\ 0, & (x \notin D_p). \end{cases}$$

Since $a_{1h}$ can be written as

$$a_{1h}(\phi_h,\psi_h;\, u) = \sum_{\Gamma_{PQ}} (\psi_h(P) - \psi_h(Q))\left(\beta_{PQ}(u)^+\phi_h(P) - \beta_{PQ}(u)^-\phi_h(Q)\right), \qquad \phi_h,\psi_h \in V_h,$$

it holds that $a_{1h}(\phi_h, 1; u) = 0$. Hence, substituting $\phi_h = 1$ in (35), we obtain for $m = 1,\cdots,N_T$

$$\int_\Omega \bar{\phi}_h^m\, dx = \int_\Omega \bar{\phi}_h^0\, dx + \Delta t \sum_{n=1}^m \left(\int_\Omega f^n\, dx + \int_\Gamma g^n\, ds\right). \tag{36}$$

Using the fact

$$\int_\Omega \bar{\phi}_h\, dx = \int_\Omega \phi_h\, dx, \quad \phi_h \in V_h,$$

(7) is derived from (36).

Nowadays, $a_{1h}$ can be regarded as a vertex-centered finite volume approximation with $D_P$ as control volume. It is extended to various schemes for the computation of hyperbolic type equations as Euler equations on unstructured meshes [9], [14], [20]. It is not necessary to take $\beta_{PQ}$ exactly in (34), but is sufficient to satisfy

$$|\beta_{PQ} - \int_{\Gamma_{PQ}} u\cdot n\, ds| \le c|PQ|^d.$$

When $\nabla \cdot u = 0$ in $\Omega$ and $u \cdot n = 0$ on $\Gamma$, the solution of (2) satisfies the maximum principle. This property is maintained by (35) with the choice (34) of $\beta_{PQ}$ [2]. For the other choice of the control volume in place of $D_P$ we refer to [10], [12].

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