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A mathematical analysis of a system of Caputo–Fabrizio fractional differential equations for the anthrax disease model in animals

Shahram Rezapour[1,2,3], Sina Etemad[4] and Hakimeh Mohammadi[5*]

*Correspondence:
hakimeh.mohammadi@yahoo.com
[5]Department of Mathematics,
Miandoab Branch, Islamic Azad
University, Miandoab, Iran
Full list of author information is
available at the end of the article

Abstract

We study a fractional-order model for the anthrax disease between animals based on the Caputo–Fabrizio derivative. First, we derive an existence criterion of solutions for the proposed fractional $\mathcal{CF}$-system of the anthrax disease model by utilizing the Picard–Lindelof technique. By obtaining the basic reproduction number $\mathcal{R}_0$ of the fractional $\mathcal{CF}$-system we compute two disease-free and endemic equilibrium points and check the asymptotic stability property. Moreover, by applying an iterative approach based on the Sumudu transform we investigate the stability of the fractional $\mathcal{CF}$-system. We obtain approximate series solutions of this system by means of the homotopy analysis transform method, in which we invoke the linear Laplace transform. Finally, after the convergence analysis of the numerical method HATM, we present a numerical simulation of the $\mathcal{CF}$-fractional anthrax disease model and review the dynamical behavior of the solutions of this $\mathcal{CF}$-system during a time interval.

MSC: Primary 34A08; secondary 34A12

Keywords: Anthrax disease; Homotopy analysis method; Mathematical modeling; Numerical simulation; The Caputo–Fabrizio derivative

1 Introduction

Analysis and investigation of various mathematical models of different natural processes is an applied branch of mathematics in which the researchers study dynamics of desired systems by means of some logical and computational tools. In this way, new fractional operators play an important role in modeling such natural phenomena and processes. Recently, the Caputo–Fabrizio fractional operator is utilized by many authors to analyze the existing systems (see, e.g., [1–8]). Also, there are many works on applications of fractional calculus (see, e.g., [9–26]). We apply this new fractional operator in the present research paper and recall its properties in the sequel.

Anthrax is considered as an infectious disease caused by the Bacillus Anthracis bacterium. Anthrax disease is categorized under zoonotic diseases and affects both animal

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and human population [27]. Naturally, the anthrax disease can be found in soil and mostly has influence on herbivores as compared to carnivores [28]. This disease is one of the most dangerous infectious diseases in the world causing a vast and uncontrolled mortality in some animal populations such as pigs, sheep, horses, goats, cattle [29, 30]. According to Gutting et al. [31], this group of animals gets infected with Bacillus Anthracis bacterium through several ways including the consumption of infected water or grass, the inhalation of its spores, or contact with infected animals. Note that carcasses of infective animals can also pollute the environment. Grass and soil are the most important reservoirs of anthrax spores, which can cause the transmission of this disease between animals, because anthrax spores persist in the soil or grass for a long time under very extreme weather conditions. Also, the clinical symptoms of anthrax disease in infective animals take time to manifest since the incubation period of this disease is about three to eight days before these animals succumb to death.

The first simple model for dynamics of transmission of anthrax disease is formulated by Mushayabasa [32] in 2015. In this model the author regards three compartments entitled Susceptible, Contamination, and Pathogens. Mushayabasa does not discuss the role of infective animals in his model as a key factor in the transmission of anthrax infectious disease. One year later, Zerihun et al. [33] extended the Mushayabasa model and designed a new model of anthrax disease supplemented with four compartments entitled Susceptible, Contamination, Infective, and Pathogens. The compartment "Infective animals" has a key importance in this model, in which the clinical symptoms of anthrax transmit to susceptible animals [33]. After aforementioned works, some authors also studied various models of anthrax disease furnished with different compartments (see [34–36]).

For the proposed model of anthrax disease, in the present research, we are motivates by a research paper of Kimathi et al. [37], in which the usefulness of vaccination policy on SIR model is regarded in the context of a novel fractional modeling. In fact, the novelty of this work is that the compartment "Vaccinated animals" is added to the existing SIR model, and we generalize the classical system to a new fractional system based on a new fractional operator without singular kernel named the Caputo–Fabrizio derivative for the first time. We observe that the obtained approximate solutions of the fractional $\mathcal{CF}$-model of anthrax disease approach those of the classical integer-order system by passing the time.

More precisely, the contents of the paper is as follows. In the first step, we derive an existence criterion of solutions for the proposed fractional $\mathcal{CF}$-system of the anthrax disease model by utilizing the Picard–Lindelof technique. Then by obtaining the basic reproduction number $\mathcal{R}_0$ of the fractional $\mathcal{CF}$-system we compute two disease-free and endemic equilibrium points and check the asymptotic stability property. Moreover, by applying an iterative approach based on the Sumudu transform we investigate the stability of the fractional $\mathcal{CF}$-system. We obtain the approximate series solutions of this system by means of the homotopy analysis transform method, in which we invoke the linear Laplace transform [38–40]. Finally, after the convergence analysis of the numerical method HATM, we present a numerical simulation of the $\mathcal{CF}$-fractional anthrax disease model and review the dynamical behavior of the solutions of this $\mathcal{CF}$-system during a time interval.

2 Preliminaries

In this part, we review some auxiliary and primitive concepts on the fractional operators. Assume that $\varrho \in (n-1, n)$ so that $n = [\varrho] + 1$. For a function $\breve{w} \in \mathcal{AC}_{\mathbb{R}}^{(n)}([0, +\infty))$, the frac-

tional derivative of Caputo type is given by

$$
{}^{C}\mathcal{D}_0^{\varrho}\breve{w}(t) = \int_0^t \frac{(t-z)^{n-\varrho-1}}{\Gamma(n-\varrho)} \breve{w}^{(n)}(z)\, \mathrm{d}z,
$$

provided that the integral is finite-valued [41, 42]. After that, a new fractional operator with no singular kernel is introduced by two Italian mathematicians Caputo and Fabrizio [43]. They assume that $a < b$ and $\breve{w} \in \mathcal{H}^1(a,b)$. Then the Caputo–Fabrizio or $(\mathcal{FC})$-derivative of order $\varrho \in (0,1]$ for a function $\breve{w}$ is given by

$$
{}^{C\mathcal{F}}\mathcal{D}_a^{\varrho}\breve{w}(t) = \frac{(2-\varrho)M(\varrho)}{2(1-\varrho)} \int_a^t \exp\left(\frac{-\varrho}{1-\varrho}(t-z)\right)\breve{w}'(z)\, \mathrm{d}z \quad (t \geq 0),
$$

where $M(\varrho)$ is a normalization function depending on the order ϱ with $M(0) = M(1) = 1$ [43]. Further, for $n \geq 1$ and $\varrho \in (0,1]$, we have ${}^{C\mathcal{F}}\mathcal{D}_a^{\varrho+n}\breve{w}(t) = {}^{C\mathcal{F}}\mathcal{D}_a^{\varrho}(\mathcal{D}^n\breve{w}(t))$ [3]. In 2015, Losada and Nieto [44] obtained a new explicit formula for the function $M(\varrho) = \frac{2}{2-\varrho}$ for $\varrho \in (0,1]$. In this case the fractional $\mathcal{CF}$-derivative for $\breve{w}$ is represented by

$$
{}^{C\mathcal{F}}\mathcal{D}_0^{\varrho}\breve{w}(t) = \frac{1}{1-\varrho} \int_0^t \exp\left(\frac{-\varrho}{1-\varrho}(t-z)\right)\breve{w}'(z)\, \mathrm{d}z \quad (t \geq 0).
$$

It is clear that for each $\varrho \in (0,1]$, the equality ${}^{C\mathcal{F}}\mathcal{D}_0^{\varrho}\breve{w}(t) = 0$ is equivalent to $\breve{w}(t) = c^*$, where c^* is an arbitrary constant. Also, Losada and Nieto defined the fractional $\mathcal{CF}$-integral of order $\varrho \in (0,1]$ for $\breve{w}$ as follows:

$$
{}^{C\mathcal{F}}\mathcal{I}_0^{\varrho}\breve{w}(t) = \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)}\breve{w}(t) + \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \breve{w}(z)\, \mathrm{d}z
$$

for $t > 0$ [44]. In this direction the authors prove that the unique solution of the fractional-order differential equation ${}^{C\mathcal{F}}\mathcal{D}_0^{\varrho}\breve{w}(t) = \breve{h}(t)$ is obtained by

$$
\breve{w}(t) = \breve{w}(0) + \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)}\left(\breve{h}(t) - \breve{h}(0)\right) + \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \breve{h}(z)\, \mathrm{d}z \tag{1}
$$

for $t \geq 0$ ([44]). For $\varrho \in (0,1]$, the Laplace transform of the fractional $\mathcal{CF}$-derivative is defined by

$$
\mathbb{L}\left[{}^{C\mathcal{F}}\mathcal{D}_0^{\varrho+n}\breve{w}(t)\right](s) = \frac{s^{n+1}\mathbb{L}[\breve{w}(t)] - s^n\breve{w}(0) - s^{n-1}\breve{w}'(0) - \cdots - \breve{w}^{(n)}(0)}{s+\varrho(1-s)},
$$

where $n \geq 1$ and $M(\varrho) = 1$ [44]. In particular, for $n = 1$ and $n = 0$, we have

$$
\begin{cases}
\mathbb{L}[{}^{C\mathcal{F}}\mathcal{D}_0^{\varrho+1}\breve{w}(t)](s) = \frac{s^2\mathbb{L}[\breve{w}(t)] - s\breve{w}(0) - \breve{w}'(0)}{s+\varrho(1-s)}, \\
\mathbb{L}[{}^{C\mathcal{F}}\mathcal{D}_0^{\varrho}\breve{w}(t)](s) = \frac{s\mathbb{L}[\breve{w}(t)]\breve{w}(0)}{s+\varrho(1-s)}.
\end{cases}
$$

In the light of the classical definition of the Fourier integral, the Sumudu transform can be derived [45–47]. For this aim, construct the following set

$$
\mathbb{A} = \left\{ \breve{w} : \exists \beta, c_1, c_2 \geq 0 \text{ such that } |\breve{w}(t)| < \beta \exp\left(\frac{t}{c_j}\right), t \in (-1)^j \times [0, \infty) \right\}.
$$

Then the Sumudu transform of a function $\check{w}(t) \in \mathbb{A}$ is represented by $\mathbb{ST}[\check{w}(t)](s) = \check{W}(s)$ and is defined as

$$\check{W}(s) = \mathbb{ST}\big[\check{w}(t)\big](s) = \frac{1}{s} \int_0^\infty \exp\left(\frac{-t}{s}\right) \check{w}(t)\,\mathrm{d}t \quad \big(s \in (-c_1, c_2)\big)$$

for $t \geq 0$, and the inverse Sumudu transform of $\check{W}(s)$ is denoted by $\check{w}(t) = \mathbb{ST}^{-1}[\check{W}(s)]$ [47]. Moreover, the Sumudu transform of the fractional derivative of the Caputo type is given by

$$\mathbb{ST}\big[{}^{C}\mathcal{D}_0^\varrho \check{w}(t)\big](s) = \frac{1}{s^\varrho}\left[\check{W}(s) - \sum_{j=0}^n s^{\varrho-j}\big[{}^{C}\mathcal{D}_0^{\varrho-j}\check{w}(t)\big]_{t=0}\right],$$

where $n - 1 < \varrho \leq n$ [46]. Now assume that $\check{w}$ is a function such that its $\mathcal{CF}$-derivative of fractional order exists. The Sumudu transform of the fractional $\mathcal{CF}$-derivative for $\check{w}$ is defined by

$$\mathbb{ST}\big[{}^{C\mathcal{F}}\mathcal{D}_0^\varrho \check{w}(t)\big](s) = \frac{M(\varrho)}{1 - \varrho + \varrho s}\big(\mathbb{ST}\big[\check{w}(t)\big](s) - \check{w}(0)\big)$$

for $t \geq 0$ [48]. In the following, we review some notions about the stability. Let $(\mathbb{W}, d)$ be a metric space. We say that a self-map $\Psi : \mathbb{W} \to \mathbb{W}$ is the Picard operator if there is $p^* \in \mathbb{W}$ such that $\mathcal{FIX}(\Psi) = \{p^*\}$ and, consequently, the convergent sequence $\{\Psi^n(p)\}_{n\in\mathbb{N}}$ tends to p^* for all $p \in \mathbb{W}$ [49].

In this position, let us assume that $(\mathbb{W}, \|\cdot\|)$ is a Banach space and $\Psi : \mathbb{W} \to \mathbb{W}$ is a self-map on $\mathbb{W}$. Suppose that $\mathcal{FIX}(\Psi) = \{p \in \mathbb{W} : \Psi(p) = p\} \neq \emptyset$ is the collection of all fixed points of Ψ. Moreover, let $\{P_n\}_{n\geq 0} \subset \mathbb{W}$ be a sequence generated by the Picard iteration as follows: $P_{n+1} = \varphi(\Psi, P_n)$ $(n = 0, 1, 2, \dots)$, where $P_0 \in \mathbb{W}$ is the initial approximation, φ is some function, and also $\lim_{n\to\infty} P_n = p \in \mathcal{FIX}(\Psi)$. Suppose that $\{\check{f}_n\}_{n\geq 0} \subset \mathbb{W}$ and put

$$\varepsilon_n = \big\|\check{f}_{n+1} - \varphi(\Psi, \check{f}_n)\big\| \quad (n = 0, 1, 2, \dots).$$

Then the recursive algorithm $P_{n+1} = \varphi(\Psi, P_n)$ is said to be Picard Ψ-stable with respect to Ψ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} \check{f}_n = p$ [49].

Remark 2.1 ([49]) Note that if the sequence $\{\check{f}_n\}$ has an upper bound, then $P_{n+1} = \Psi P_n$ is Picard Ψ-Stable whenever the Picard iteration $P_{n+1} = \Psi P_n$ satisfies all above assumptions.

The following theorem is utilized to prove the stability of the proposed fractional anthrax disease model.

Theorem 2.2 ([49]) *Suppose that $(\mathbb{W}, \|\cdot\|)$ is a Banach space and Ψ is a self-map on $\mathbb{W}$ satisfying the inequality*

$$\|\Psi_p - \Psi_{p'}\| \leq K\|p - \Psi_p\| + k\|p - p'\| \tag{2}$$

for all $p, p' \in \mathbb{W}$, where $K \geq 0$ and $0 \leq k < 1$. Then Ψ is Picard Ψ-Stable.

We further derive an important criterion to confirm the asymptotic stability of a fractional linear system of the Caputo–Fabrizio type at free equilibrium point.

Proposition 2.3 ([50]) *Let $\breve{w}(t) \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$. Then the characteristic equation related to the linear system*

$$^{\mathcal{CF}}\mathcal{D}_0^\varrho \breve{w}(t) = M\breve{w}(t) \tag{3}$$

supplemented with the Caputo–Fabrizio derivative of order $\varrho \in (0, 1)$ is given by

$$\det\big[s\big(I_{n \times n} - (1 - \varrho)M\big) - \varrho M\big] = 0. \tag{4}$$

Theorem 2.4 ([50]) *Suppose that the matrix $(I_{n \times n} - (1 - \varrho)M)$ is invertible. Then the fractional linear $\mathcal{CF}$-system (3) has the asymptotic stability property at a free equilibrium point if and only if all roots of the characteristic equation (4) for $\mathcal{CF}$-system (3) have negative real parts.*

3 Fractional mathematical model of the anthrax disease

In this section, we introduce a new fractional model of the anthrax disease in animals by applying a novel fractional operator with no singular kernel. In view of the implemented study by Kimathi and Wainaina [37], the classical first-order SIRV model of the anthrax disease in animals is formulated by the following four nonlinear differential equations:

$$\begin{cases} \frac{dS}{dt} = \omega - \delta S(t)I(t) - (\rho + \upsilon)S(t) + \zeta R(t) + \varpi V(t), \\ \frac{dI}{dt} = \delta S(t)I(t) - (\rho + \tau + \kappa)I(t), \\ \frac{dR}{dt} = \kappa I(t) - (\rho + \zeta)R(t), \\ \frac{dV}{dt} = \upsilon S(t) - (\rho + \varpi)V(t), \end{cases} \tag{5}$$

supplemented with initial conditions $S(0) = \breve{S}_0$, $I(0) = \breve{I}_0$, $R(0) = \breve{R}_0$, and $V(0) = \breve{V}_0$ [37]. Although human contribution in the transmission of the anthrax disease between animals is negligible, it becomes a very important subject of discussing the transmission of this disease in the animal population only. The fractional-order system (FDE) is related to systems with memory, history, or nonlocal effects, which exist in many biological systems that show the realistic biphasic decline behavior of infection or diseases but at a slower rate. In this model, since the internal memory effects of the biological system of the anthrax infection are not included, it is better that we extend the proposed ordinary model to a new fractional model. This shows that the fractional model of this animal disease yields the approximate results similar to the classical integer-order model. To modify the existing model, we convert the first-order ordinary derivative into the $\mathcal{CF}$-derivative of fractional order $\varrho \in (0, 1]$ as follows:

$$\begin{cases} ^{\mathcal{CF}}\mathcal{D}_0^\varrho S(t) = \omega - \delta S(t)I(t) - (\rho + \upsilon)S(t) + \zeta R(t) + \varpi V(t), \\ ^{\mathcal{CF}}\mathcal{D}_0^\varrho I(t) = \delta S(t)I(t) - (\rho + \tau + \kappa)I(t), \\ ^{\mathcal{CF}}\mathcal{D}_0^\varrho R(t) = \kappa I(t) - (\rho + \zeta)R(t), \\ ^{\mathcal{CF}}\mathcal{D}_0^\varrho V(t) = \upsilon S(t) - (\rho + \varpi)V(t), \end{cases} \tag{6}$$

furnished with initial conditions $S(0) = \check{S}_0, I(0) = \check{I}_0, R(0) = \check{R}_0$, and $V(0) = \check{V}_0$. In this mathematical framework, $S(t)$ represents the number of animals at risk of the anthrax infection at time t (Susceptible), $I(t)$ indicates the number of animals with symptoms of this disease at time t (Infected), $R(t)$ stands for the number of recovered animals from the anthrax infection and acquired temporal immunity at time t (Recovered), and $V(t)$ denotes the number of vaccinated animals against attacks of mentioned anthrax disease at time t (Vaccinated). In this case, it is obvious that the total number of animals included in these four classes at time t equals $N(t) = S(t) + I(t) + R(t) + V(t)$.

Moreover, this new fractional model includes eight nonnegative parameters. The parameter ω denotes the recruitment rate, δ shows the contact rate, ρ indicates the natural death rate, υ represents the vaccinated rate, ζ is the waning recovery rate, ϖ stands for the waning immunity rate of vaccinated animals, τ indicates the disease-induced death rate, and the parameter κ represents the recovery rate of animals. Besides, we need to notice that in the first-order ordinary system (5) of the disease model, the right-hand sides of four equations have dimensions $(time)^{-1}$, but when we convert an integer order of these equations into the fractional order ϱ, the dimensions of the left-hand sides of four equations equal $(time)^{-\varrho}$. To match the dimensions of both sides of these differential equations, we have to change the dimensions of all nonnegative parameters $\omega, \delta, \rho, \upsilon, \zeta, \varpi, \tau$, and κ. In this position the modified version of the fractional system of the anthrax disease model formulated by (6) is as follows:

$$
\begin{cases}
{}^{CF}\mathcal{D}_0^\varrho S(t) = \omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t), \\
{}^{CF}\mathcal{D}_0^\varrho I(t) = \delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t), \\
{}^{CF}\mathcal{D}_0^\varrho R(t) = \kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t), \\
{}^{CF}\mathcal{D}_0^\varrho V(t) = \upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t).
\end{cases}
\tag{7}
$$

Numerical solutions of the modified fractional model (7) are obtained by utilizing the homotopy analysis transform method (HATM). To do this, the fractional differential equations of the above model are converted into algebraic equations by means of the Laplace transform. In the next section, we first derive an existence criterion of solutions for the fractional system (7).

4 The existence criterion by Picard–Lindelof technique

Hereafter, we consider the following fractional model of the anthrax disease by employing the Caputo–Fabrizio derivative:

$$
\begin{cases}
{}^{CF}\mathcal{D}_0^\varrho S(t) = \omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t), \\
{}^{CF}\mathcal{D}_0^\varrho I(t) = \delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t), \\
{}^{CF}\mathcal{D}_0^\varrho R(t) = \kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t), \\
{}^{CF}\mathcal{D}_0^\varrho V(t) = \upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t),
\end{cases}
\tag{8}
$$

furnished with initial conditions $S(0) = \check{S}_0, I(0) = \check{I}_0, R(0) = \check{R}_0$, and $V(0) = \check{V}_0$. To check the existence of solutions for the modified fractional system (8) of the anthrax disease model, we utilize the Picard–Lindelof technique. To do this, we first need to convert the anthrax

disease model (8) into a fractional integral equation. In other words, we apply the fractional $\mathcal{CF}$-integral operator defined by Losada and Nieto [44] to both sides of differential equations (8). Then taking into account $(S(0), I(0), R(0), V(0)) = (\check{S}_0, \check{I}_0, \check{R}_0, \check{V}_0)$, we have

$$
\begin{aligned}
S(t) = \check{S}_0 \\
+ \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \big(\omega^\varrho - \delta^\varrho S(t)I(t) - \big(\rho^\varrho + \upsilon^\varrho\big)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \big[\omega^\varrho - \delta^\varrho S(z)I(z) - \big(\rho^\varrho + \upsilon^\varrho\big)S(z) + \zeta^\varrho R(z) + \varpi^\varrho V(z) \big] \, dz,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
I(t) = \check{I}_0 + \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \big(\delta^\varrho S(t)I(t) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)I(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \big[\delta^\varrho S(z)I(z) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)I(z) \big] \, dz,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
R(t) = \check{R}_0 + \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \big(\kappa^\varrho I(t) - \big(\rho^\varrho + \zeta^\varrho\big)R(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \big[\kappa^\varrho I(z) - \big(\rho^\varrho + \zeta^\varrho\big)R(z) \big] \, dz,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
V(t) = \check{V}_0 + \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \big(\upsilon^\varrho S(t) - \big(\rho^\varrho + \varpi^\varrho\big)V(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \big[\upsilon^\varrho S(z) - \big(\rho^\varrho + \varpi^\varrho\big)V(z) \big] \, dz.
\end{aligned} \tag{12}
$$

Now, due to (9)–(12), we define the Picard iterative algorithm as follows $(n = 0, 1, 2, \dots)$:

$$
S_0(t) = \check{S}_0, \qquad I_0(t) = \check{I}_0, \qquad R_0(t) = \check{R}_0, \qquad V_0(t) = \check{V}_0, \tag{13}
$$

and

$$
\begin{aligned}
S_{n+1}(t) = \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \\
\times \big(\omega^\varrho - \delta^\varrho S_n(t)I_n(t) - \big(\rho^\varrho + \upsilon^\varrho\big)S_n(t) + \zeta^\varrho R_n(t) + \varpi^\varrho V_n(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \\
\times \int_0^t \big[\omega^\varrho - \delta^\varrho S_n(z)I_n(z) - \big(\rho^\varrho + \upsilon^\varrho\big)S_n(z) + \zeta^\varrho R_n(z) + \varpi^\varrho V_n(z) \big] \, dz,
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
I_{n+1}(t) = \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \big(\delta^\varrho S_n(t)I_n(t) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)I_n(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \big[\delta^\varrho S_n(z)I_n(z) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)I_n(z) \big] \, dz,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
R_{n+1}(t) = \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \big(\kappa^\varrho I_n(t) - \big(\rho^\varrho + \zeta^\varrho\big)R_n(t) \big) \\
+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \big[\kappa^\varrho I_n(z) - \big(\rho^\varrho + \zeta^\varrho\big)R_n(z) \big] \, dz,
\end{aligned} \tag{16}
$$

$$V_{n+1}(t) = \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \left(\upsilon^\varrho S_n(t) - \left(\rho^\varrho + \varpi^\varrho \right) V_n(t) \right)$$

$$+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \left[\upsilon^\varrho S_n(z) - \left(\rho^\varrho + \varpi^\varrho \right) V_n(z) \right] dz. \tag{17}$$

Now we assume that we can obtain the exact solutions of the fractional system (8) by taking the limits of both sides of (14)–(17) as n tends to infinity. In other words, the solutions are obtained as follows:

$$\begin{cases} \lim_{n\to\infty} S_n(t) = S(t), \\ \lim_{n\to\infty} I_n(t) = I(t), \\ \lim_{n\to\infty} R_n(t) = R(t), \\ \lim_{n\to\infty} V_n(t) = V(t). \end{cases} \tag{18}$$

Here we are ready to derive the existence criterion and the uniqueness of the solutions based on the Picard–Lindelof approach. To reach this goal, define the following operators:

$$\begin{cases} \Upsilon_1(t,S) := \omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t), \\ \Upsilon_2(t,I) := \delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t), \\ \Upsilon_3(t,R) := \kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t), \\ \Upsilon_4(t,V) := \upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t), \end{cases} \tag{19}$$

where $\Upsilon_1(t,S)$, $\Upsilon_2(t,I)$, $\Upsilon_3(t,R)$, and $\Upsilon_4(t,V)$ are contractions with respect to S, I, R, and V for the first, second, third, and fourth functions, respectively. Furthermore, we consider the following product spaces:

$$\begin{cases} \mathbb{I}_{a,b_1} := [t-a, t+a] \times [S-b_1, S+b_1] = A \times B_1, \\ \mathbb{I}_{a,b_2} := [t-a, t+a] \times [I-b_2, I+b_2] = A \times B_2, \\ \mathbb{I}_{a,b_3} := [t-a, t+a] \times [R-b_3, R+b_3] = A \times B_3, \\ \mathbb{I}_{a,b_4} := [t-a, t+a] \times [V-b_4, V+b_4] = A \times B_4. \end{cases} \tag{20}$$

Take

$$\Upsilon_1^* = \sup_{(t,S) \in \mathbb{I}_{a,b_1}} \left\| \Upsilon_1(t, S(t)) \right\|, \qquad \Upsilon_2^* = \sup_{(t,I) \in \mathbb{I}_{a,b_2}} \left\| \Upsilon_2(t, I(t)) \right\|,$$

$$\Upsilon_3^* = \sup_{(t,R) \in \mathbb{I}_{a,b_3}} \left\| \Upsilon_3(t, R(t)) \right\|,$$

and

$$\Upsilon_4^* = \sup_{(t,V) \in \mathbb{I}_{a,b_4}} \left\| \Upsilon_4(t, V(t)) \right\|.$$

In this position, we define the Picard operator

$$\mathcal{O} : \mathcal{C}(A, B_1, B_2, B_3, B_4) \to \mathcal{C}(A, B_1, B_2, B_3, B_4)$$

as follows:

$$\mathcal{O}\big(\mathbb{W}(t)\big) = \mathbb{W}_0(t) + \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)}\mathcal{G}\big(t,\mathbb{W}(t)\big) + \frac{2\varrho}{(2-\varrho)M(\varrho)}\int_0^t \mathcal{G}\big(z,\mathbb{W}(z)\big)\,\mathrm{d}z, \qquad (21)$$

so that $\mathbb{W}(t) = \{S(t), I(t), R(t), V(t)\}$, $\mathbb{W}_0(t) = \{\check{S}_0, \check{I}_0, \check{R}_0, \check{V}_0\}$, and

$$\mathcal{G}\big(t,\mathbb{W}(t)\big) = \big\{\Upsilon_1\big(t,S(t)\big), \Upsilon_2\big(t,I(t)\big), \Upsilon_3\big(t,R(t)\big), \Upsilon_4\big(t,V(t)\big)\big\}. \qquad (22)$$

To apply the Picard theorem, we define the uniform norm on the space

$$\mathcal{C}(A, B_1, B_2, B_3, B_4)$$

as $\|\mathbb{W}\|_\infty = \sup_{t\in[t-a,t+a]=A} |\mathbb{W}(t)|$. In the following, we assume that all solution functions are bounded during a time interval, that is,

$$\|\mathbb{W}\|_\infty \le \max\{b_1, b_2, b_3, b_4\} = b. \qquad (23)$$

Moreover, let us assume that $\Upsilon^* = \max\{\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*, \Upsilon_4^*\}$ and that there is t_0 with $t \le t_0$. Then we have

$$\big\|\mathcal{O}\mathbb{W}(t) - \mathbb{W}_0(t)\big\|$$

$$= \left\|\frac{2(1-\varrho)}{(2-\varrho)M(\varrho)}\mathcal{G}\big(t,\mathbb{W}(t)\big) + \frac{2\varrho}{(2-\varrho)M(\varrho)}\int_0^t \mathcal{G}\big(z,\mathbb{W}(z)\big)\,\mathrm{d}z\right\|$$

$$\le \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)}\big\|\mathcal{G}\big(t,\mathbb{W}(t)\big)\big\| + \frac{2\varrho}{(2-\varrho)M(\varrho)}\int_0^t \big\|\mathcal{G}\big(z,\mathbb{W}(z)\big)\big\|\,\mathrm{d}z$$

$$\le \left[\frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} + \frac{2\varrho t_0}{(2-\varrho)M(\varrho)}\right]\Upsilon^* = \mu^*\Upsilon^* \le b,$$

where we assume that $\mu^* < \frac{b}{\Upsilon^*}$ and also $\mu^* = \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} + \frac{2\varrho t_0}{(2-\varrho)M(\varrho)}$. Finally, we intend to show that the Picard operator $\mathcal{O}$ is a contraction. Since the functions Υ_1, Υ_2, Υ_3, and Υ_4 are contractions, for all $\mathbb{W}_1, \mathbb{W}_2 \in \mathcal{C}(A, B_1, B_2, B_3, B_4)$, we can write

$$\big\|\mathcal{G}\big(t,\mathbb{W}_1(t)\big) - \mathcal{G}\big(t,\mathbb{W}_2(t)\big)\big\| \le \lambda^*\big\|\mathbb{W}_1(t) - \mathbb{W}_2(t)\big\|, \qquad (24)$$

where $\lambda^* < 1$ is the contraction constant. At this moment, using the definition of the Picard operator $\mathcal{O}$ given in (21), inequality (24), and the equality

$$\|\mathcal{O}\mathbb{W}_1 - \mathcal{O}\mathbb{W}_1\| = \sup_{t\in A}\big|\mathbb{W}_1(t) - \mathbb{W}_2(t)\big|,$$

we get

$$\big\|\mathcal{O}\big(\mathbb{W}_1(t)\big) - \mathcal{O}\big(\mathbb{W}_2(t)\big)\big\| = \left\|\frac{2(1-\varrho)}{(2-\varrho)M(\varrho)}\big[\mathcal{G}\big(t,\mathbb{W}_1(t)\big) - \mathcal{G}\big(t,\mathbb{W}_2(t)\big)\big]\right.$$

$$\left. + \frac{2\varrho}{(2-\varrho)M(\varrho)}\int_0^t \big[\mathcal{G}\big(z,\mathbb{W}_1(z)\big) - \mathcal{G}\big(z,\mathbb{W}_2(z)\big)\big]\,\mathrm{d}z\right\|$$

$$\leq \frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} \left\| \mathcal{G}\big(t, \mathbb{W}_1(t)\big) - \mathcal{G}\big(t, \mathbb{W}_2(t)\big) \right\|$$

$$+ \frac{2\varrho}{(2-\varrho)M(\varrho)} \int_0^t \left\| \mathcal{G}\big(z, \mathbb{W}_1(z)\big) - \mathcal{G}\big(z, \mathbb{W}_2(z)\big) \right\| dz$$

$$\leq \frac{2(1-\varrho)\lambda^*}{(2-\varrho)M(\varrho)} \left\| \mathbb{W}_1(t) - \mathbb{W}_2(t) \right\|$$

$$+ \frac{2\varrho\lambda^*}{(2-\varrho)M(\varrho)} \int_0^t \left\| \mathbb{W}_1(z) - \mathbb{W}_2(z) \right\| dz$$

$$\leq \left[\frac{2(1-\varrho)}{(2-\varrho)M(\varrho)} + \frac{2\varrho t_0}{(2-\varrho)M(\varrho)} \right] \lambda^* \left\| \mathbb{W}_1(t) - \mathbb{W}_2(t) \right\|$$

$$= \mu^* \lambda^* \left\| \mathbb{W}_1(t) - \mathbb{W}_2(t) \right\|.$$

Thus we obtain

$$\| \mathcal{O}\mathbb{W}_1 - \mathcal{O}\mathbb{W}_2 \|_\infty \leq \mu^* \lambda^* \| \mathbb{W}_1 - \mathbb{W}_2 \|_\infty,$$

which indicates that the operator $\mathcal{O}$ is a contraction with constant $\mu^* \lambda^* < 1$ since $\lambda^* < 1$. Hence the Banach fixed point theorem implies that the fractional system (8) of the anthrax disease model has a unique solution.

5 Equilibrium points of the fractional $\mathcal{CF}$-model (8)

In this section, we intend to obtain the equilibrium points of the fractional anthrax disease $\mathcal{CF}$-model (8). For this aim, we first solve the following homogeneous equations:

$$^{\mathcal{CF}}\mathcal{D}_0^\varrho S(t) = {}^{\mathcal{CF}}\mathcal{D}_0^\varrho I(t) = {}^{\mathcal{CF}}\mathcal{D}_0^\varrho R(t) = {}^{\mathcal{CF}}\mathcal{D}_0^\varrho V(t) = 0. \tag{25}$$

Consequently, a disease-free equilibrium point of the fractional $\mathcal{CF}$-system (8) is given by $\mathcal{E}^0 = (S^0, I^0, R^0, V^0)$, where

$$S^0 = \frac{(\rho^\varrho + \varpi^\varrho)\omega^\varrho}{\rho^\varrho(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)}, \qquad I^0 = 0, \qquad R^0 = 0, \qquad V^0 = \frac{\upsilon^\varrho \omega^\varrho}{\rho^\varrho(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)}. \tag{26}$$

To find the endemic equilibrium point for the fractional $\mathcal{CF}$-system (8), we need to determine a basic reproduction number $\mathcal{R}_0$. This quantity appears by applying the next-generation matrix process introduced by Van den Driessche [51]. To obtain the basic reproduction number $\mathcal{R}_0$, set

$$A = \begin{bmatrix} \delta^\varrho SI \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I \\ -\omega^\varrho + \delta^\varrho SI + (\rho^\varrho + \upsilon^\varrho)S - \zeta^\varrho R - \varpi^\varrho V \\ -\kappa^\varrho I + (\rho^\varrho + \zeta^\varrho)R \\ -\upsilon^\varrho S + (\rho^\varrho + \varpi^\varrho)V \end{bmatrix}.$$

Then the Jacobian matrices of both matrices A and B at disease-free equilibrium point $\mathcal{E}^0$ given in (26) are defined as follows:

$$
\left[\mathfrak{J}(A)\right]\left(\mathcal{E}^0\right) = \begin{bmatrix} 0 & \frac{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}{\rho^\varrho (\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}
$$

and

$$
\left[\mathfrak{J}(B)\right]\left(\mathcal{E}^0\right) = \begin{bmatrix} 0 & \rho^\varrho + \tau^\varrho + \kappa^\varrho & 0 & 0 \\ 0 & \frac{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}{\rho^\varrho (\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} & -\zeta^\varrho & -\varpi^\varrho \\ 0 & -\kappa^\varrho & \rho^\varrho + \zeta^\varrho & 0 \\ -\upsilon^\varrho & 0 & 0 & \rho^\varrho + \varpi^\varrho \end{bmatrix}. \tag{28}
$$

In view of (27) and (28), by some routine computations we obtain

$$
\left[\mathfrak{J}(A)\right]\left(\mathcal{E}^0\right) \cdot \left[\mathfrak{J}(B)\right]^{-1}\left(\mathcal{E}^0\right) = \begin{bmatrix} \frac{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}{\rho^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho)(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

In the final step, we find the eigenvalue of the characteristic equation

$$
\det\left[I_{4\times4} - \lambda\left(\left[\mathfrak{J}(A)\right]\left(\mathcal{E}^0\right) \cdot \left[\mathfrak{J}(B)\right]^{-1}\left(\mathcal{E}^0\right)\right)\right] = 0,
$$

and so the basic reproduction number $\mathcal{R}_0$ is obtained as follows:

$$
\mathcal{R}_0 = \lambda = \frac{\rho^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho)(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)}{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}. \tag{29}
$$

The basic reproduction number $\mathcal{R}_0$ is a metric to measure the transmission potential of a infectious disease over the time. When the value of $\mathcal{R}_0$ is greater than one, the fractional $\mathcal{CF}$-system (8) has an endemic equilibrium point $\mathcal{E}^* = (S^*, I^*, R^*, V^*)$. More precisely, to obtain an endemic equilibrium point $\mathcal{E}^*$, we have to solve equations (25) assuming that all variables $S(t)$, $I(t)$, $R(t)$, and $V(t)$ are nonzero. Equations (25) can be rewritten as follows:

$$
\omega^\varrho - \delta^\varrho S(t)I(t) - \left(\rho^\varrho + \upsilon^\varrho\right)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t) = 0, \tag{30}
$$

$$
\delta^\varrho S(t)I(t) - \left(\rho^\varrho + \tau^\varrho + \kappa^\varrho\right)I(t) = 0, \tag{31}
$$

$$
\kappa^\varrho I(t) - \left(\rho^\varrho + \zeta^\varrho\right)R(t) = 0, \tag{32}
$$

$$
\upsilon^\varrho S(t) - \left(\rho^\varrho + \varpi^\varrho\right)V(t) = 0. \tag{33}
$$

From equation (31) we have $I(t)[\delta^\varrho S(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)] = 0$. Since $I(t) \neq 0$, we can obtain $S^*(t) = \frac{\rho^\varrho + \tau^\varrho + \kappa^\varrho}{\delta^\varrho}$. Moreover, from equation (33) we have $V^*(t) = \frac{\upsilon^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho)}{\delta^\varrho (\rho^\varrho + \varpi^\varrho)}$. Finally, if we combine equations (30) and (31), then by solving the obtained system we get

$$
R^*(t) = \frac{\kappa^\varrho \left[\omega^\varrho \delta^\varrho (\rho^\varrho + \varpi^\varrho) - \rho^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho)(\rho^\varrho + \upsilon^\varrho + \varpi^\varrho)\right]}{\delta^\varrho (\rho^\varrho + \varpi^\varrho)\left[\rho^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho + \zeta^\varrho) + \zeta^\varrho \tau^\varrho\right]}
$$

and

$$I^*(t) = \frac{(\rho^\varrho + \zeta^\varrho)[\omega^\varrho \delta^\varrho (\rho^\varrho + \varpi^\varrho) - \rho^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho)(\rho^\varrho + \upsilon^\varrho + \varpi^\varrho)]}{\delta^\varrho (\rho^\varrho + \varpi^\varrho)[\rho^\varrho (\rho^\varrho + \tau^\varrho + \kappa^\varrho + \zeta^\varrho) + \zeta^\varrho \tau^\varrho]}.$$

Hence the components of an endemic equilibrium point $\mathcal{E}^* = (S^*, I^*, R^*, V^*)$ for the fractional $\mathcal{CF}$-system (8) are obtained as before.

In this position, we want to check the asymptotic stability property of the disease-free equilibrium point $\mathcal{E}^0$ obtained in (26) for the fractional $\mathcal{CF}$-system (8) of the anthrax disease model. By some simple computations we get that the Jacobian matrix of the fractional $\mathcal{CF}$-system (8) at disease-free equilibrium point $\mathcal{E}^0$ is defined by

$$\mathfrak{J}(\mathcal{E}^0) = \begin{bmatrix} -(\rho^\varrho + \upsilon^\varrho) & -\frac{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}{\rho^\varrho (\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} & \zeta^\varrho & \varpi^\varrho \\ 0 & \frac{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}{\rho^\varrho (\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} - (\rho^\varrho + \tau^\varrho + \kappa^\varrho) & 0 & 0 \\ 0 & \kappa^\varrho & -(\rho^\varrho + \zeta^\varrho) & 0 \\ \upsilon^\varrho & 0 & 0 & -(\rho^\varrho + \varpi^\varrho) \end{bmatrix}.$$

Hence the characteristic equation of the mentioned $\mathcal{CF}$-system (8) is given by

$$\det\big[s\big(I_{4\times4} - (1-\varrho)\mathcal{J}\big(\mathcal{E}^0\big)\big) - \varrho\mathcal{J}\big(\mathcal{E}^0\big)\big] = 0. \tag{34}$$

Then we can state the following theorem and confirm that the disease-free equilibrium point $\mathcal{E}^0$ of $\mathcal{CF}$-system (8) is asymptotically stable.

Theorem 5.1 *The disease-free equilibrium point $\mathcal{E}^0$ of the fractional $\mathcal{CF}$-system of the anthrax disease model* (8) *has the asymptotic stability property whenever real parts of all roots of the characteristic equation* (34) *are negative.*

Proof In view of the Jacobian matrix $\mathcal{J}(\mathcal{E}^0)$, applying the matrix equation (34), we obtain the characteristic equation of the fractional $\mathcal{CF}$-system (8)

$$\begin{aligned} &\big[s\big(1 - (1-\varrho)P^*\big) - \varrho P^*\big]\big[s\big(1 + (1-\varrho)\big(\rho^\varrho + \zeta^\varrho\big)\big) + \varrho\big(\rho^\varrho + \zeta^\varrho\big)\big] \\ &\quad \times \big(\big[s\big(1 + (1-\varrho)\big(\rho^\varrho + \upsilon^\varrho\big)\big) + \varrho\big(\rho^\varrho + \upsilon^\varrho\big)\big] \\ &\quad \times \big[s\big(1 + (1-\varrho)\big(\rho^\varrho + \varpi^\varrho\big)\big) + \varrho\big(\rho^\varrho + \varpi^\varrho\big)\big] \\ &\quad - \big[s(1-\varrho)\upsilon^\varrho - \varrho\upsilon^\varrho\big]\big[s(1-\varrho)\varpi^\varrho - \varrho\varpi^\varrho\big]\big) \\ &= 0, \end{aligned} \tag{35}$$

where $P^* = \frac{\delta^\varrho \omega^\varrho (\rho^\varrho + \varpi^\varrho)}{\rho^\varrho (\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)$. The eigenvalues of this characteristic equation are

$$s_1 = \frac{\varrho P^*}{1 - (1-\varrho)P^*}, \qquad s_2 = \frac{-\varrho(\rho^\varrho + \zeta^\varrho)}{1 + (1-\varrho)(\rho^\varrho + \zeta^\varrho)},$$

and the roots of the equation $s^2 + B^* s + C^* = 0$ where

$$B^* = \frac{\varrho(2\rho^\varrho + \varpi^\varrho + \upsilon^\varrho) + 2\varrho(1-\varrho)[\rho^{2\varrho} + \rho^\varrho \varpi^\varrho + \rho^\varrho \upsilon^\varrho + 2\upsilon^\varrho \varpi^\varrho]}{1 + (1-\varrho)[(\rho^\varrho + \varpi^\varrho)(\rho^\varrho + \upsilon^\varrho)] + (1-\varrho)^2[\rho^\varrho (\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)]}$$

and

$$C^* = \frac{\varrho^2[\rho^\varrho(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)]}{1 + (1 - \varrho)[(\rho^\varrho + \varpi^\varrho)(\rho^\varrho + \upsilon^\varrho)] + (1 - \varrho)^2[\rho^\varrho(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)]}.$$

If $(1 - \varrho)P^* > 1$, then since $\varrho \in (0, 1]$, $P^* > 0$, and so $\frac{\delta^\varrho \omega^\varrho(\rho^\varrho + \varpi^\varrho)}{\rho^\varrho(\rho^\varrho + \varpi^\varrho + \upsilon^\varrho)} > (\rho^\varrho + \tau^\varrho + \kappa^\varrho)$. This means that s_1 is a root with negative sign. Also, as we said before, all parameters are positive, so it is clear that s_2 is negative. Moreover, the roots of equation $s^2 + B^*s + C^* = 0$ must also be negative. To reach this goal, since $\varrho \in (0, 1]$, $B^* > 0$ and $C^* > 0$, and thus by the Routh–Hurwitz criterion we find that all roots of the characteristic equation (35) are negative. Hence if $(1 - \varrho)P^* > 1$, then the disease-free equilibrium point $\mathcal{E}^0$ of the fractional $\mathcal{CF}$-system of the anthrax disease model (8) has the asymptotic stability property, and the proof is completed.   □

6 Stability analysis via iterative approach

To analyze the stability of the fractional anthrax disease model (8), we provide an iterative formula by means of the Sumudu transform. For this aim, we get

$$\begin{cases} \mathbb{ST}[^{\mathcal{CF}}\mathcal{D}_0^\varrho S(t)](s) = \mathbb{ST}[\omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t)](s), \\ \mathbb{ST}[^{\mathcal{CF}}\mathcal{D}_0^\varrho I(t)](s) = \mathbb{ST}[\delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t)](s), \\ \mathbb{ST}[^{\mathcal{CF}}\mathcal{D}_0^\varrho R(t)](s) = \mathbb{ST}[\kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t)](s), \\ \mathbb{ST}[^{\mathcal{CF}}\mathcal{D}_0^\varrho V(t)](s) = \mathbb{ST}[\upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t)](s). \end{cases} \tag{36}$$

By the definition of the Sumudu transform for the fractional $\mathcal{CF}$-derivative we obtain

$$\begin{cases} \frac{M(\varrho)}{1 - \varrho + \varrho s}(\mathbb{ST}[S(t)](s) - S(0)) \\ \quad = \mathbb{ST}[\omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t)](s), \\ \frac{M(\varrho)}{1 - \varrho + \varrho s}(\mathbb{ST}[I(t)](s) - I(0)) = \mathbb{ST}[\delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t)](s), \\ \frac{M(\varrho)}{1 - \varrho + \varrho s}(\mathbb{ST}[R(t)](s) - R(0)) = \mathbb{ST}[\kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t)](s), \\ \frac{M(\varrho)}{1 - \varrho + \varrho s}(\mathbb{ST}[V(t)](s) - V(0)) = \mathbb{ST}[\upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t)](s). \end{cases} \tag{37}$$

By rewriting the formulas we obtain the following equalities:

$$\begin{cases} \mathbb{ST}[S(t)](s) \\ \quad = S(0) + \frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}[\omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t)](s), \\ \mathbb{ST}[I(t)](s) = I(0) + \frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}[\delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t)](s), \\ \mathbb{ST}[R(t)](s) = R(0) + \frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}[\kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t)](s), \\ \mathbb{ST}[V(t)](s) = V(0) + \frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}[\upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t)](s). \end{cases} \tag{38}$$

Now, after taking the inverse Sumudu transform on both sides of system (38), the we obtain the following recursive equations for the fractional $\mathcal{CF}$-model (8):

$$
\begin{cases}
S_{n+1}(t) \\
\quad = S_n(0) + \mathbb{ST}^{-1}\big[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}[\omega^\varrho - \delta^\varrho S_n(t)I_n(t) - (\rho^\varrho + \upsilon^\varrho)S_n(t) + \zeta^\varrho R_n(t) + \varpi^\varrho V_n(t)](s)\big], \\
I_{n+1}(t) = I_n(0) + \mathbb{ST}^{-1}\big[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}[\delta^\varrho S_n(t)I_n(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I_n(t)](s)\big], \\
R_{n+1}(t) = R_n(0) + \mathbb{ST}^{-1}\big[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}[\kappa^\varrho I_n(t) - (\rho^\varrho + \zeta^\varrho)R_n(t)](s)\big], \\
V_{n+1}(t) = V_n(0) + \mathbb{ST}^{-1}\big[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}[\upsilon^\varrho S_n(t) - (\rho^\varrho + \varpi^\varrho)V_n(t)](s)\big].
\end{cases}
\tag{39}
$$

On the other hand, we obtain the approximate solutions of this $\mathcal{CF}$-system by

$$
\begin{aligned}
S(t) = \lim_{n\to\infty} S_n(t), \qquad I(t) = \lim_{n\to\infty} I_n(t), \\
R(t) = \lim_{n\to\infty} R_n(t), \qquad V(t) = \lim_{n\to\infty} V_n(t).
\end{aligned}
\tag{40}
$$

Now we can check the stability of the fractional $\mathcal{CF}$-system by considering the above notions and relations.

Theorem 6.1 *Suppose that Ψ is a self-map defined as follows*:

$$
\begin{aligned}
\Psi\big(S_n(t)\big) &= S_{n+1}(t) = S_n(t) \\
&\quad + \mathbb{ST}^{-1}\bigg[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}\big[\omega^\varrho - \delta^\varrho S_n(t)I_n(t) \\
&\quad - \big(\rho^\varrho + \upsilon^\varrho\big)S_n(t) + \zeta^\varrho R_n(t) + \varpi^\varrho V_n(t)\big](s)\bigg], \\[4pt]
\Psi\big(I_n(t)\big) &= I_{n+1}(t) = I_n(t) \\
&\quad + \mathbb{ST}^{-1}\bigg[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}\big[\delta^\varrho S_n(t)I_n(t) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)I_n(t)\big](s)\bigg], \\[4pt]
\Psi\big(R_n(t)\big) &= R_{n+1}(t) = R_n(t) \\
&\quad + \mathbb{ST}^{-1}\bigg[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}\big[\kappa^\varrho I_n(t) - \big(\rho^\varrho + \zeta^\varrho\big)R_n(t)\big](s)\bigg], \\[4pt]
\Psi\big(V_n(t)\big) &= V_{n+1}(t) = V_n(t) \\
&\quad + \mathbb{ST}^{-1}\bigg[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}\big[\upsilon^\varrho S_n(t) - \big(\rho^\varrho + \varpi^\varrho\big)V_n(t)\big](s)\bigg].
\end{aligned}
\tag{41}
$$

Then the iteration fractional $\mathcal{CF}$-system (41) is Ψ-stable in $\mathcal{L}^1(a,b)$ whenever we have

$$
\begin{cases}
1 - \delta^\varrho K_1^* \Phi_1(t) - \delta^\varrho K_2^* \Phi_2(t) - (\rho^\varrho + \upsilon^\varrho)\Phi_3(t) + \zeta^\varrho \Phi_4(t) + \varpi^\varrho \Phi_5(t) < 1, \\
1 + \delta^\varrho K_1^* \Phi_6(t) + \delta^\varrho K_2^* \Phi_7(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)\Phi_8(t) < 1, \\
1 + \kappa^\varrho \Phi_9(t) - (\rho^\varrho + \zeta^\varrho)\Phi_{10}(t) < 1, \\
1 + \upsilon^\varrho \Phi_{11}(t) - (\rho^\varrho + \varpi^\varrho)\Phi_{12}(t) < 1,
\end{cases}
\tag{42}
$$

where the functions Φ_j, $j = 1, 2, \ldots, 12$, are introduced further.

Proof To begin the proof, we intend to prove that the operator Ψ has a fixed point. For all $n, m \in \mathbb{N}$, we may write

$$
\begin{aligned}
&\left\| \Psi\big(S_n(t)\big) - \Psi\big(S_m(t)\big)\right\| \\
&\quad = \left\| S_{n+1}(t) - S_{m+1}(t)\right\| \\
&\quad = \Bigg\| S_n(t) + \mathbb{ST}^{-1}\bigg[\frac{1 - \varrho + \varrho s}{M(\varrho)} \\
&\qquad \times \mathbb{ST}\big[\omega^\varrho - \delta^\varrho S_n(t)I_n(t) - \big(\rho^\varrho + \upsilon^\varrho\big)S_n(t) + \zeta^\varrho R_n(t) + \varpi^\varrho V_n(t)\big](s)\bigg] \\
&\qquad - S_m(t) - \mathbb{ST}^{-1}\bigg[\frac{1 - \varrho + \varrho s}{M(\varrho)} \\
&\qquad \times \mathbb{ST}\big[\omega^\varrho - \delta^\varrho S_m(t)I_m(t) - \big(\rho^\varrho + \upsilon^\varrho\big)S_m(t) + \zeta^\varrho R_m(t) + \varpi^\varrho V_m(t)\big](s)\bigg]\Bigg\| \\
&\quad \leq \left\| S_n(t) - S_m(t)\right\| + \Bigg\| \mathbb{ST}^{-1}\bigg[\frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}\big[-\delta^\varrho\big(S_n(t)I_n(t) - S_m(t)I_m(t)\big) \\
&\qquad - \big(\rho^\varrho + \upsilon^\varrho\big)\big(S_n(t) - S_m(t)\big) + \zeta^\varrho\big(R_n(t) - R_m(t)\big) + \varpi^\varrho\big(V_n(t) - V_m(t)\big)\big](s)\bigg]\Bigg\| \\
&\quad \leq \left\| S_n(t) - S_m(t)\right\| + \mathbb{ST}^{-1}\bigg[\frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}\big[\left\|-\delta^\varrho S_n(t)\big(I_n(t) - I_m(t)\big)\right\| \\
&\qquad + \left\|-\delta^\varrho I_m(t)\big(S_n(t) - S_m(t)\big)\right\| \\
&\qquad + \left\|-\big(\rho^\varrho + \upsilon^\varrho\big)\big(S_n(t) - S_m(t)\big)\right\| + \left\|\zeta^\varrho\big(R_n(t) - R_m(t)\big)\right\| \\
&\qquad + \left\|\varpi^\varrho\big(V_n(t) - V_m(t)\big)\right\|\big](s)\bigg].
\end{aligned}
\tag{43}
$$

Because of the same role of all four solutions, we will consider

$$
\left\| S_n(t) - S_m(t)\right\| \simeq \left\| I_n(t) - I_m(t)\right\| \simeq \left\| R_n(t) - R_m(t)\right\| \simeq \left\| V_n(t) - V_m(t)\right\|.
\tag{44}
$$

Then from (43) and (44) we have

$$
\begin{aligned}
\left\| \Psi\big(S_n(t)\big) - \Psi\big(S_m(t)\big)\right\| &\leq \left\| S_n(t) - S_m(t)\right\| \\
&\quad + \mathbb{ST}^{-1}\bigg[\frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}\big[\left\|-\delta^\varrho S_n(t)\big(S_n(t) - S_m(t)\big)\right\| \\
&\quad + \left\|-\delta^\varrho I_m(t)\big(S_n(t) - S_m(t)\big)\right\| + \left\|-\big(\rho^\varrho + \upsilon^\varrho\big)\big(S_n(t) - S_m(t)\big)\right\| \\
&\quad + \left\|\zeta^\varrho\big(S_n(t) - S_m(t)\big)\right\| + \left\|\varpi^\varrho\big(S_n(t) - S_m(t)\big)\right\|\big](s)\bigg].
\end{aligned}
$$

Since S_n, I_m, R_n, and V_n are convergent sequences, they are bounded. Hence there are constants K_1^*, K_2^*, K_3^*, and K_4^* such that for all t and $m, n \in \mathbb{N}$, we have

$$
\left\| S_n(t)\right\| \leq K_1^*, \qquad \left\| I_m(t)\right\| \leq K_2^*, \qquad \left\| R_n(t)\right\| \leq K_3^*, \qquad \left\| V_n(t)\right\| \leq K_4^*.
$$

Therefore we obtain

$$\left\| \Psi\big(S_n(t)\big) - \Psi\big(S_m(t)\big)\right\|$$
$$\leq \left\| S_n(t) - S_m(t)\right\|$$
$$+ \mathbb{ST}^{-1}\bigg[\frac{1 - \varrho + \varrho s}{M(\varrho)}\mathbb{ST}\big[-\delta^\varrho K_1^* \left\| S_n(t) - S_m(t)\right\|$$
$$- \delta^\varrho K_2^* \left\| S_n(t) - S_m(t)\right\| - \big(\rho^\varrho + \upsilon^\varrho\big)\left\| S_n(t) - S_m(t)\right\|$$
$$+ \zeta^\varrho \left\| S_n(t) - S_m(t)\right\| + \varpi^\varrho \left\| S_n(t) - S_m(t)\right\|\big](s)\bigg]$$
$$= \big(1 - \delta^\varrho K_1^* \Phi_1(t) - \delta^\varrho K_2^* \Phi_2(t) - \big(\rho^\varrho + \upsilon^\varrho\big)\Phi_3(t) + \zeta^\varrho \Phi_4(t) + \varpi^\varrho \Phi_5(t)\big)$$
$$\times \left\| S_n(t) - S_m(t)\right\|,$$

where Φ_j, $j = 1, 2, \ldots, 5$, are functions arising from $\mathbb{ST}^{-1}[\frac{1-\varrho+\varrho s}{M(\varrho)}\mathbb{ST}[\cdot]]$. In the same manner, we get

$$\left\| \Psi\big(I_n(t)\big) - \Psi\big(I_m(t)\big)\right\| \leq \big(1 + \delta^\varrho K_1^* \Phi_6(t) + \delta^\varrho K_2^* \Phi_7(t) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)\Phi_8(t)\big)$$
$$\times \left\| I_n(t) - I_m(t)\right\|,$$
$$\left\| \Psi\big(R_n(t)\big) - \Psi\big(R_m(t)\big)\right\| \leq \big(1 + \kappa^\varrho \Phi_9(t) - \big(\rho^\varrho + \zeta^\varrho\big)\Phi_{10}(t)\big)\left\| R_n(t) - R_m(t)\right\|,$$

and

$$\left\| \Psi\big(V_n(t)\big) - \Psi\big(V_m(t)\big)\right\| \leq \big(1 + \upsilon^\varrho \Phi_{11}(t) - \big(\rho^\varrho + \varpi^\varrho\big)\Phi_{12}(t)\big)\left\| V_n(t) - V_m(t)\right\|.$$

Under hypotheses (42), the self-map Ψ is a contraction, and thus it possesses a fixed point. Now we claim that Ψ satisfies all assumptions of Theorem 2.2. To prove this claim, we can easily assume that $K = (0, 0, 0)$ and

$$k = \begin{cases} 1 - \delta^\varrho K_1^* \Phi_1(t) - \delta^\varrho K_2^* \Phi_2(t) - (\rho^\varrho + \upsilon^\varrho)\Phi_3(t) + \zeta^\varrho \Phi_4(t) + \varpi^\varrho \Phi_5(t), \\ 1 + \delta^\varrho K_1^* \Phi_6(t) + \delta^\varrho K_2^* \Phi_7(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)\Phi_8(t), \\ 1 + \kappa^\varrho \Phi_9(t) - (\rho^\varrho + \zeta^\varrho)\Phi_{10}(t), \\ 1 + \upsilon^\varrho \Phi_{11}(t) - (\rho^\varrho + \varpi^\varrho)\Phi_{12}(t). \end{cases} \tag{45}$$

Then all assumptions of Theorem 2.2 are fulfilled, and so Ψ is Picard Ψ-stable, and the proof is completed. $\qquad\square$

7 Analytical solutions of model (8) by HATM method

In this section, we implement the homotopy analysis transform method (HATM) to solve the fractional anthrax disease model (8). This method is an elegant combination of the standard Laplace transform method [38] and homotopy analysis method [39]. The advantage of this well-developed method is its flexible capability of combining two powerful methods to obtain exact and approximate analytical solutions for the existing fractional nonlinear equations. To solve the $\mathcal{CF}$-fractional anthrax disease model (8) by means of

HATM, we first take the Laplace transform of both sides of fractional differential equations of $\mathcal{CF}$-system (8). Thus we have

$$
\begin{cases}
\mathbb{L}[^{\mathcal{CF}}\mathcal{D}_0^\varrho S(t)](s) = \mathbb{L}[\omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t)](s), \\
\mathbb{L}[^{\mathcal{CF}}\mathcal{D}_0^\varrho I(t)](s) = \mathbb{L}[\delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t)](s), \\
\mathbb{L}[^{\mathcal{CF}}\mathcal{D}_0^\varrho R(t)](s) = \mathbb{L}[\kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t)](s), \\
\mathbb{L}[^{\mathcal{CF}}\mathcal{D}_0^\varrho V(t)](s) = \mathbb{L}[\upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t)](s).
\end{cases}
$$

Now by the definition of the Laplace transform of the fractional $\mathcal{CF}$-derivative we obtain

$$
\begin{cases}
\frac{s\mathbb{L}[S(t)]-S(0)}{s+\varrho(1-s)} = \mathbb{L}[\omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t)](s), \\
\frac{s\mathbb{L}[I(t)]-I(0)}{s+\varrho(1-s)} = \mathbb{L}[\delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t)](s), \\
\frac{s\mathbb{L}[R(t)]-R(0)}{s+\varrho(1-s)} = \mathbb{L}[\kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t)](s), \\
\frac{s\mathbb{L}[V(t)]-V(0)}{s+\varrho(1-s)} = \mathbb{L}[\upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t)](s).
\end{cases}
$$

Rewriting these equalities, we get

$$
\begin{cases}
\mathbb{L}[S(t)] - \frac{\check{S}_0}{s} - \frac{s+\varrho(1-s)}{s}\mathbb{L}[\omega^\varrho - \delta^\varrho S(t)I(t) - (\rho^\varrho + \upsilon^\varrho)S(t) \\
\qquad + \zeta^\varrho R(t) + \varpi^\varrho V(t)](s) = 0, \\
\mathbb{L}[I(t)] - \frac{\check{I}_0}{s} - \frac{s+\varrho(1-s)}{s}\mathbb{L}[\delta^\varrho S(t)I(t) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)I(t)](s) = 0, \\
\mathbb{L}[R(t)] - \frac{\check{R}_0}{s} - \frac{s+\varrho(1-s)}{s}\mathbb{L}[\kappa^\varrho I(t) - (\rho^\varrho + \zeta^\varrho)R(t)](s) = 0, \\
\mathbb{L}[V(t)] - \frac{\check{V}_0}{s} - \frac{s+\varrho(1-s)}{s}\mathbb{L}[\upsilon^\varrho S(t) - (\rho^\varrho + \varpi^\varrho)V(t)](s) = 0.
\end{cases}
\tag{46}
$$

By utilizing the homotopy analysis method we further find series solutions for the $\mathcal{CF}$-fractional anthrax disease model (8). To reach this goal, we consider $q \in [0,1]$ as the embedding parameter. The HAM technique is based on continuous mappings

$$
\mathcal{Q}_1(t;q) \to S(t), \qquad \mathcal{Q}_2(t;q) \to I(t), \qquad \mathcal{Q}_3(t;q) \to R(t), \qquad \mathcal{Q}_4(t;q) \to V(t),
$$

so that as q increases from 0 to 1, $\mathcal{Q}_j(t;q)$ ($j = 1,2,3,4$) vary from the initial approximation to the exact solution. To observe this subject, we define the following nonlinear operators:

$$
\mathcal{K}_1\big(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)\big)
$$

$$
:= \mathbb{L}\big[\mathcal{Q}_1(t;q)\big] - \frac{\check{S}_0}{s} - \frac{s+\varrho(1-s)}{s}
$$

$$
\times \mathbb{L}\big[\omega^\varrho - \delta^\varrho \mathcal{Q}_1(t;q)\mathcal{Q}_2(t;q) - (\rho^\varrho + \upsilon^\varrho)\mathcal{Q}_1(t;q) + \zeta^\varrho \mathcal{Q}_3(t;q) + \varpi^\varrho \mathcal{Q}_4(t;q)\big](s),
$$

$$
\mathcal{K}_2\big(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)\big)
$$

$$
:= \mathbb{L}\big[\mathcal{Q}_2(t;q)\big] - \frac{\check{I}_0}{s} - \frac{s+\varrho(1-s)}{s}
$$

$$
\times \mathbb{L}\big[\delta^\varrho \mathcal{Q}_1(t;q)\mathcal{Q}_2(t;q) - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)\mathcal{Q}_2(t;q)\big](s),
$$

$$
\mathcal{K}_3\big(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)\big)
$$

$$
:= \mathbb{L}\big[\mathcal{Q}_3(t;q)\big] - \frac{\check{R}_0}{s} - \frac{s+\varrho(1-s)}{s}
$$

$$\times \mathbb{L}\big[\kappa^{\varrho} \mathcal{Q}_2(t;q) - \big(\rho^{\varrho} + \zeta^{\varrho}\big)\mathcal{Q}_3(t;q)\big](s),$$

$$\mathcal{K}_4\big(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)\big)$$

$$:= \mathbb{L}\big[\mathcal{Q}_4(t;q)\big] - \frac{\check{V}_0}{s} - \frac{s + \varrho(1-s)}{s}$$

$$\times \mathbb{L}\big[\nu^{\varrho} \mathcal{Q}_1(t;q) - \big(\rho^{\varrho} + \varpi^{\varrho}\big)\mathcal{Q}_4(t;q)\big](s).$$

Then we construct the following collection of zero-order deformation equations [39]:

$$\begin{cases} (1-q)\mathbb{L}[\mathcal{Q}_1(t;q) - \check{S}_0] = qhH(t)\mathcal{K}_1(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)), \\ (1-q)\mathbb{L}[\mathcal{Q}_2(t;q) - \check{I}_0] = qhH(t)\mathcal{K}_1(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)), \\ (1-q)\mathbb{L}[\mathcal{Q}_3(t;q) - \check{R}_0] = qhH(t)\mathcal{K}_1(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)), \\ (1-q)\mathbb{L}[\mathcal{Q}_4(t;q) - \check{V}_0] = qhH(t)\mathcal{K}_1(\mathcal{Q}_1(t;q), \mathcal{Q}_2(t;q), \mathcal{Q}_3(t;q), \mathcal{Q}_4(t;q)), \end{cases} \tag{47}$$

supplemented with initial conditions

$$\mathcal{Q}_1(0;q) = \check{S}_0, \qquad \mathcal{Q}_2(0;q) = \check{I}_0, \qquad \mathcal{Q}_3(0;q) = \check{R}_0, \qquad \mathcal{Q}_4(0;q) = \check{V}_0,$$

where $q \in [0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, H is an auxiliary nonzero function, $\check{S}_0, \check{I}_0, \check{R}_0$, and $\check{V}_0$ are initial guesses of $S(t), I(t), R(t)$, and $V(t)$, $\mathcal{Q}_j(t;q)$ ($j = 1, 2, 3, 4$) are unknown functions, and $\mathbb{L}$ is the Laplace linear operator. It is necessary to have great freedom to choose auxiliary things in HAM. It is obvious that by letting $q = 0$ and $q = 1$ we have

$$\begin{cases} \mathcal{Q}_1(t;0) = \check{S}_0, & \mathcal{Q}_1(t;1) = S(t), \\ \mathcal{Q}_2(t;0) = \check{I}_0, & \mathcal{Q}_2(t;1) = I(t), \\ \mathcal{Q}_3(t;0) = \check{R}_0, & \mathcal{Q}_3(t;1) = R(t), \\ \mathcal{Q}_4(t;0) = \check{V}_0, & \mathcal{Q}_4(t;1) = V(t). \end{cases}$$

Then we can observe that by increasing q from 0 to 1 the solutions $\mathcal{Q}_1(t;q)$, $\mathcal{Q}_2(t;q)$, $\mathcal{Q}_3(t;q)$, and $\mathcal{Q}_4(t;q)$ vary from the initial guesses $\check{S}_0, \check{I}_0, \check{R}_0$, and $\check{V}_0$ to $S(t), I(t), R(t)$, and $V(t)$, respectively. In this step, we expand the functions $\mathcal{Q}_1(t;q)$, $\mathcal{Q}_2(t;q)$, $\mathcal{Q}_3(t;q)$, and $\mathcal{Q}_4(t;q)$ by using Taylor's series with respect to q. Then we get

$$\begin{cases} \mathcal{Q}_1(t;q) = \check{S}_0 + \sum_{r=1}^{\infty} S_r(t)q^r, \\ \mathcal{Q}_2(t;q) = \check{I}_0 + \sum_{r=1}^{\infty} I_r(t)q^r, \\ \mathcal{Q}_3(t;q) = \check{R}_0 + \sum_{r=1}^{\infty} R_r(t)q^r, \\ \mathcal{Q}_4(t;q) = \check{V}_0 + \sum_{r=1}^{\infty} V_r(t)q^r, \end{cases} \tag{48}$$

where $S_r(t) = \frac{1}{r!}\frac{\partial^r \mathcal{Q}_1(t;q)}{\partial q^r}|_{q=0}$, $I_r(t) = \frac{1}{r!}\frac{\partial^r \mathcal{Q}_2(t;q)}{\partial q^r}|_{q=0}$, $R_r(t) = \frac{1}{r!}\frac{\partial^r \mathcal{Q}_3(t;q)}{\partial q^r}|_{q=0}$, and $V_r(t) = \frac{1}{r!} \times \frac{\partial^r \mathcal{Q}_4(t;q)}{\partial q^r}|_{q=0}$ are the constant coefficients of the series (48). If we choose a suitable auxiliary linear operator, suitable initial guesses, a suitable auxiliary parameter h, and a suitable auxiliary function H, then the series (48) is convergent at $q = 1$, as proved by Liao [39] (also

see [52, 53]). Thus we have

$$
\begin{cases}
\mathcal{Q}_1(t;q) = \check{S}_0 + \sum_{r=1}^{\infty} S_r(t), \\
\mathcal{Q}_2(t;q) = \check{I}_0 + \sum_{r=1}^{\infty} I_r(t), \\
\mathcal{Q}_3(t;q) = \check{R}_0 + \sum_{r=1}^{\infty} R_r(t), \\
\mathcal{Q}_4(t;q) = \check{V}_0 + \sum_{r=1}^{\infty} V_r(t),
\end{cases}
\implies
\begin{cases}
S(t) = \check{S}_0 + \sum_{r=1}^{\infty} S_r(t), \\
I(t) = \check{I}_0 + \sum_{r=1}^{\infty} I_r(t), \\
R(t) = \check{R}_0 + \sum_{r=1}^{\infty} R_r(t), \\
V(t) = \check{V}_0 + \sum_{r=1}^{\infty} V_r(t),
\end{cases}
\tag{49}
$$

which must be the solutions of the $\mathcal{CF}$-fractional anthrax disease model (8). Now we produce the following rth-order deformation equations. Define the vectors

$$
\begin{aligned}
\vec{S}_r(t) &= \left\{ S_0(t), S_1(t), \ldots, S_r(t) \right\} \quad (r = 1, 2, 3, \ldots), \\
\vec{I}_r(t) &= \left\{ I_0(t), I_1(t), \ldots, I_r(t) \right\} \quad (r = 1, 2, 3, \ldots), \\
\vec{R}_r(t) &= \left\{ R_0(t), R_1(t), \ldots, R_r(t) \right\} \quad (r = 1, 2, 3, \ldots), \\
\vec{V}_r(t) &= \left\{ V_0(t), V_1(t), \ldots, V_r(t) \right\} \quad (r = 1, 2, 3, \ldots).
\end{aligned}
$$

We differentiate the zero-order deformation equations (47) r times with respect to the embedding parameter q. Next, we take $q = 0$ and finally divide them by $r!$. In this case, we obtain the following rth-order linear deformation equations:

$$
\begin{cases}
\mathbb{L}[S_r(t) - \sigma_r S_{r-1}(t)] = hH\mathfrak{R}_{S,r}(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)) \quad (r = 1, 2, 3, \ldots), \\
\mathbb{L}[I_r(t) - \sigma_r I_{r-1}(t)] = hH\mathfrak{R}_{I,r}(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)) \quad (r = 1, 2, 3, \ldots), \\
\mathbb{L}[R_r(t) - \sigma_r R_{r-1}(t)] = hH\mathfrak{R}_{R,r}(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)) \quad (r = 1, 2, 3, \ldots), \\
\mathbb{L}[V_r(t) - \sigma_r V_{r-1}(t)] = hH\mathfrak{R}_{V,r}(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)) \quad (r = 1, 2, 3, \ldots),
\end{cases}
\tag{50}
$$

furnished with initial values

$$
S_0(t) = \check{S}_0, \qquad I_0(t) = \check{I}_0, \qquad R_0(t) = \check{R}_0, \qquad V_0(t) = \check{V}_0,
\tag{51}
$$

where

$$
\sigma_r = \begin{cases} 0, & r \le 1, \\ 1, & r > 1, \end{cases}
$$

and

$$
\begin{aligned}
& \mathfrak{R}_{S,r}(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)) \\
& \quad = \mathbb{L}[S_{r-1}(t)] - \frac{\check{S}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1-s)}{s} \\
& \qquad \times \mathbb{L}[\omega^\varrho - \delta^\varrho S_{r-1}(t) I_{r-1}(t) - (\rho^\varrho + \upsilon^\varrho) S_{r-1}(t) \\
& \qquad + \zeta^\varrho R_{r-1}(t) + \varpi^\varrho V_{r-1}(t)](s), \\
& \mathfrak{R}_{I,r}(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)) \\
& \quad = \mathbb{L}[I_{r-1}(t)] - \frac{\check{I}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1-s)}{s}
\end{aligned}
$$

$$\times \mathbb{L}\big[\delta^\varrho S_{r-1}(t)I_{r-1}(t) - \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big)I_{r-1}(t)\big](s),$$

$$\mathfrak{R}_{R,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big)$$

$$= \mathbb{L}\big[R_{r-1}(t)\big] - \frac{\check{R}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1-s)}{s}$$

$$\times \mathbb{L}\big[\kappa^\varrho I_{r-1}(t) - \big(\rho^\varrho + \zeta^\varrho\big)R_{r-1}(t)\big](s),$$

and

$$\mathfrak{R}_{V,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big)$$

$$= \mathbb{L}\big[V_{r-1}(t)\big] - \frac{\check{V}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1-s)}{s}$$

$$\times \mathbb{L}\big[\upsilon^\varrho S_{r-1}(t) - \big(\rho^\varrho + \varpi^\varrho\big)V_{r-1}(t)\big](s).$$

Applying the inverse Laplace transform to both sides of (50), we obtain

$$S_r(t) = \sigma_r S_{r-1}(t) + hH\mathbb{L}^{-1}\big[\mathfrak{R}_{S,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big)\big], \tag{52}$$

$$I_r(t) = \sigma_r I_{r-1}(t) + hH\mathbb{L}^{-1}\big[\mathfrak{R}_{I,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big)\big], \tag{53}$$

$$R_r(t) = \sigma_r R_{r-1}(t) + hH\mathbb{L}^{-1}\big[\mathfrak{R}_{R,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big)\big], \tag{54}$$

$$V_r(t) = \sigma_r V_{r-1}(t) + hH\mathbb{L}^{-1}\big[\mathfrak{R}_{V,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big)\big] \tag{55}$$

for $r = 1, 2, 3, \dots$. For convenience, we can consider the nonzero auxiliary function H to be equal to unity. Now, if we solve equation (52) for $r = 1$, then in view of initial conditions (51), we have

$$S_1(t) = \sigma_1 S_0(t) + hH\mathbb{L}^{-1}\big[\mathfrak{R}_{S,1}\big(\vec{S}_0(t), \vec{I}_0(t), \vec{R}_0(t), \vec{V}_0(t)\big)\big]$$

$$= hH\mathbb{L}^{-1}\bigg[\mathbb{L}[\check{S}_0] - \frac{\check{S}_0}{s}(1 - \sigma_1)$$

$$- \frac{s + \varrho(1-s)}{s}\mathbb{L}\big[\omega^\varrho - \delta^\varrho \check{S}_0\check{I}_0 - \big(\rho^\varrho + \upsilon^\varrho\big)\check{S}_0 + \zeta^\varrho\check{R}_0 + \varpi^\varrho\check{V}_0\big]\bigg]$$

$$= -hH\mathbb{L}^{-1}\bigg[\frac{s + \varrho(1-s)}{s^2}\big(\omega^\varrho - \delta^\varrho\check{S}_0\check{I}_0 - \big(\rho^\varrho + \upsilon^\varrho\big)\check{S}_0 + \zeta^\varrho\check{R}_0 + \varpi^\varrho\check{V}_0\big)\bigg]$$

$$= -\hat{\Delta}_1 hH\mathbb{L}^{-1}\bigg[\frac{s + \varrho(1-s)}{s^2}\bigg]$$

$$= -\hat{\Delta}_1 hH\big(1 + \varrho(t-1)\big), \tag{56}$$

where $\hat{\Delta}_1 = \omega^\varrho - \delta^\varrho\check{S}_0\check{I}_0 - \big(\rho^\varrho + \upsilon^\varrho\big)\check{S}_0 + \zeta^\varrho\check{R}_0 + \varpi^\varrho\check{V}_0$. Hence, continuing similar computations on equations (53)–(55), we get

$$\begin{cases} I_1(t) = -\hat{\Delta}_2 hH(1 + \varrho(t-1)), \\ R_1(t) = -\hat{\Delta}_3 hH(1 + \varrho(t-1)), \\ V_1(t) = -\hat{\Delta}_4 hH(1 + \varrho(t-1)), \end{cases} \tag{57}$$

where $\hat{\Delta}_2 = \delta^\varrho \check{S}_0 \check{I}_0 - (\rho^\varrho + \tau^\varrho + \kappa^\varrho)\check{I}_0$, $\hat{\Delta}_3 = \kappa^\varrho \check{I}_0 - (\rho^\varrho + \zeta^\varrho)\check{R}_0$, and $\hat{\Delta}_4 = \upsilon^\varrho \check{S}_0 - (\rho^\varrho + \varpi^\varrho)\check{V}_0$. Again, if we solve equation (52) for $r = 2$, then in view of (56)–(57), we have

$$
\begin{aligned}
S_2(t) = {} & S_1(t) - \hat{\Delta}_1 h^2 H^2 \big(1 + \varrho(t-1)\big) - hH\omega^\varrho \big(1 + \varrho(t-1)\big) \\
& + \hat{\Delta}_1 \hat{\Delta}_2 h^3 H^3 \delta^\varrho \bigg(1 + 3\varrho(t-1) \\
& + \varrho^2 \big(2t^2 - 6t + 3\big) + \varrho^3 \bigg(\frac{t^3}{3} - 2t^2 + 3t - 1 \bigg) \bigg) \\
& - \hat{\Delta}_1 h^2 H^2 \big(\rho^\varrho + \upsilon^\varrho\big) \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg) \\
& + \hat{\Delta}_3 h^2 H^2 \zeta^\varrho \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg) \\
& + \hat{\Delta}_4 h^2 H^2 \varpi^\varrho \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg).
\end{aligned}
\tag{58}
$$

Similarly, by solving the equations (53)–(55) for $r = 2$ we get the following functions with respect to t:

$$
\begin{aligned}
I_2(t) = {} & I_1(t) - \hat{\Delta}_2 h^2 H^2 \big(1 + \varrho(t-1)\big) \\
& - \hat{\Delta}_1 \hat{\Delta}_2 h^3 H^3 \delta^\varrho \bigg(1 + 3\varrho(t-1) \\
& + \varrho^2 \big(2t^2 - 6t + 3\big) + \varrho^3 \bigg(\frac{t^3}{3} - 2t^2 + 3t - 1 \bigg) \bigg) \\
& - \hat{\Delta}_1 h^2 H^2 \big(\rho^\varrho + \tau^\varrho + \kappa^\varrho\big) \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg),
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
R_2(t) = {} & R_1(t) - \hat{\Delta}_3 h^2 H^2 \big(1 + \varrho(t-1)\big) \\
& + \hat{\Delta}_2 h^2 H^2 \kappa^\varrho \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg) \\
& - \hat{\Delta}_3 h^2 H^2 \big(\rho^\varrho + \zeta^\varrho\big) \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg),
\end{aligned}
\tag{60}
$$

and

$$
\begin{aligned}
V_2(t) = {} & V_1(t) - \hat{\Delta}_4 h^2 H^2 \big(1 + \varrho(t-1)\big) \\
& + \hat{\Delta}_1 h^2 H^2 \upsilon^\varrho \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg) \\
& - \hat{\Delta}_4 h^2 H^2 \big(\rho^\varrho + \varpi^\varrho\big) \bigg(1 + 2\varrho(t-1) + \varrho^2 \bigg(\frac{t^2}{2} - 2t + 1 \bigg) \bigg).
\end{aligned}
\tag{61}
$$

According to the series (49), if we continue this precess and solve equations (52)–(55) for $r = 3, 4, \dots$, then by (56)–(61) the series solutions for the $\mathcal{CF}$-fractional anthrax disease

model (8) are given by

$$S(t) = \check{S}_0 + \sum_{r=1}^{\infty} S_r(t) = \check{S}_0 + S_1(t) + S_2(t) + \cdots = \check{S}_0 - \hat{\Delta}_1 hH\big(1 + \varrho(t-1)\big) + \cdots,$$

$$I(t) = \check{I}_0 + \sum_{r=1}^{\infty} I_r(t) = \check{I}_0 + I_1(t) + I_2(t) + \cdots = \check{I}_0 - \hat{\Delta}_2 hH\big(1 + \varrho(t-1)\big) + \cdots,$$

$$R(t) = \check{R}_0 + \sum_{r=1}^{\infty} R_r(t) = \check{R}_0 + R_1(t) + R_2(t) + \cdots = \check{R}_0 - \hat{\Delta}_3 hH\big(1 + \varrho(t-1)\big) + \cdots,$$

$$V(t) = \check{V}_0 + \sum_{r=1}^{\infty} V_r(t) = \check{V}_0 + V_1(t) + V_2(t) + \cdots = \check{V}_0 - \hat{\Delta}_4 hH\big(1 + \varrho(t-1)\big) + \cdots, \quad (62)$$

where the constants $\hat{\Delta}_1$, $\hat{\Delta}_2$, $\hat{\Delta}_3$, and $\hat{\Delta}_4$ were introduced above.

8 Convergence analysis of HATM for the $\mathcal{CF}$-model

In this section, we prove the convergence of HATM method utilized for the fractional $\mathcal{CF}$-system (46) of the anthrax disease model.

Theorem 8.1 *Let $\sum_{r=0}^{\infty} S_r(t)$, $\sum_{r=0}^{\infty} I_r(t)$, $\sum_{r=0}^{\infty} R_r(t)$, and $\sum_{r=0}^{\infty} V_r(t)$ be uniformly convergent series approaching to $S(t)$, $I(t)$, $R(t)$, and $V(t)$, respectively, where $S_r(t)$, $I_r(t)$, $R_r(t)$, and $V_r(t)$ belonging to $\mathcal{L}(\mathbb{R}^+)$ are produced by the rth-order deformation equations (50), and, in addition, $\sum_{r=0}^{\infty} {}^{\mathcal{CF}}\mathcal{D}_0^\varrho S_r(t)$, $\sum_{r=0}^{\infty} {}^{\mathcal{CF}}\mathcal{D}_0^\varrho I_r(t)$, $\sum_{r=0}^{\infty} {}^{\mathcal{CF}}\mathcal{D}_0^\varrho R_r(t)$, and $\sum_{r=0}^{\infty} {}^{\mathcal{CF}}\mathcal{D}_0^\varrho V_r(t)$ also are convergent series. Then the functions $S(t)$, $I(t)$, $R(t)$, and $V(t)$ are exact solutions of the fractional $\mathcal{CF}$-system (46) of the anthrax disease model.*

Proof Suppose that $\sum_{r=0}^{\infty} S_r(t)$ is an uniformly convergent series approaching to $S(t)$. Then, it is evident that $\lim_{r \to \infty} S_r(t) = 0$ for any $t \in \mathbb{R}^+$. Since the Laplace operator is linear, we have

$$\sum_{r=1}^{n} \mathbb{L}\big[S_r(t) - \sigma_r S_{r-1}(t)\big]$$

$$= \sum_{r=1}^{n} \big(\mathbb{L}\big[S_r(t)\big] - \sigma_r \mathbb{L}\big[S_{r-1}(t)\big]\big)$$

$$= \mathbb{L}\big[S_1(t)\big] + \big(\mathbb{L}\big[S_2(t)\big] - \mathbb{L}\big[S_1(t)\big]\big) + \cdots + \big(\mathbb{L}\big[S_n(t)\big] - \mathbb{L}\big[S_{n-1}(t)\big]\big)$$

$$= \mathbb{L}\big[S_n(t)\big].$$

Therefore we get

$$\sum_{r=1}^{\infty} \mathbb{L}\big[S_r(t) - \sigma_r S_{r-1}(t)\big] = \lim_{n \to \infty} \mathbb{L}\big[S_n(t)\big] = \mathbb{L}\Big[\lim_{n \to \infty} S_n(t)\Big] = 0,$$

and so

$$\sum_{r=1}^{\infty} \mathbb{L}\big[S_r(t) - \sigma_r S_{r-1}(t)\big] = hH \sum_{r=1}^{\infty} \mathfrak{R}_{S,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\big) = 0.$$

Since $h \neq 0$ and $H \neq 0$, this yields

$$\sum_{r=1}^{\infty} \mathfrak{R}_{S,r}\left(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\right) = 0. \tag{63}$$

Similarly, we can prove that

$$\sum_{r=1}^{\infty} \mathfrak{R}_{I,r}\left(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\right) = 0, \tag{64}$$

$$\sum_{r=1}^{\infty} \mathfrak{R}_{R,r}\left(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\right) = 0, \tag{65}$$

$$\sum_{r=1}^{\infty} \mathfrak{R}_{V,r}\left(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\right) = 0. \tag{66}$$

Now, from Equation (63) we have

$$
\begin{aligned}
0 &= \sum_{r=1}^{\infty} \mathfrak{R}_{S,r}\left(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\right) \\
&= \sum_{r=1}^{\infty}\left(\mathbb{L}\left[S_{r-1}(t)\right] - \frac{\check{S}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s}\right. \\
&\quad \times \mathbb{L}\left[\omega^\varrho - \delta^\varrho S_{r-1}(t) I_{r-1}(t) - \left(\rho^\varrho + \upsilon^\varrho\right) S_{r-1}(t)\right. \\
&\quad \left.\left. + \zeta^\varrho R_{r-1}(t) + \varpi^\varrho V_{r-1}(t)\right](s)\right) \\
&= \mathbb{L}\left[\sum_{r=1}^{\infty} S_{r-1}(t)\right] - \frac{\check{S}_0}{s}\sum_{r=1}^{\infty}(1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s} \\
&\quad \times \mathbb{L}\left[\sum_{r=1}^{\infty}\left(\omega^\varrho - \delta^\varrho S_{r-1}(t) I_{r-1}(t)\right.\right. \\
&\quad \left.\left. - \left(\rho^\varrho + \upsilon^\varrho\right) S_{r-1}(t) + \zeta^\varrho R_{r-1}(t) + \varpi^\varrho V_{r-1}(t)\right)\right](s) \\
&= \mathbb{L}\left[S(t)\right] - \frac{\check{S}_0}{s} - \frac{s + \varrho(1 - s)}{s}\mathbb{L}\left[\omega^\varrho - \delta^\varrho S(t) I(t)\right. \\
&\quad \left. - \left(\rho^\varrho + \upsilon^\varrho\right) S(t) + \zeta^\varrho R(t) + \varpi^\varrho V(t)\right](s).
\end{aligned}
$$

Similarly, from (64)–(66) we have

$$
\begin{aligned}
0 &= \sum_{r=1}^{\infty} \mathfrak{R}_{I,r}\left(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t)\right) \\
&= \sum_{r=1}^{\infty}\left(\mathbb{L}\left[I_{r-1}(t)\right] - \frac{\check{I}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s}\right. \\
&\quad \left. \times \mathbb{L}\left[\delta^\varrho S_{r-1}(t) I_{r-1}(t) - \left(\rho^\varrho + \tau^\varrho + \kappa^\varrho\right) I_{r-1}(t)\right](s)\right)
\end{aligned}
$$

$$= \mathbb{L}\left[\sum_{r=1}^{\infty} I_{r-1}(t) \right] - \frac{\check{I}_0}{s} \sum_{r=1}^{\infty} (1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s}$$

$$\times \mathbb{L}\left[\sum_{r=1}^{\infty} \left(\delta^{\varrho} S_{r-1}(t) I_{r-1}(t) - \left(\rho^{\varrho} + \tau^{\varrho} + \kappa^{\varrho} \right) I_{r-1}(t) \right) \right](s)$$

$$= \mathbb{L}\big[I(t) \big] - \frac{\check{I}_0}{s} - \frac{s + \varrho(1 - s)}{s} \mathbb{L}\big[\delta^{\varrho} S(t) I(t) - \left(\rho^{\varrho} + \tau^{\varrho} + \kappa^{\varrho} \right) I(t) \big](s),$$

$$0 = \sum_{r=1}^{\infty} \mathfrak{R}_{R,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t) \big)$$

$$= \sum_{r=1}^{\infty} \left(\mathbb{L}\big[R_{r-1}(t) \big] - \frac{\check{R}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s} \right.$$

$$\left. \times \mathbb{L}\big[\kappa^{\varrho} I_{r-1}(t) - \left(\rho^{\varrho} + \zeta^{\varrho} \right) R_{r-1}(t) \big](s) \right)$$

$$= \mathbb{L}\left[\sum_{r=1}^{\infty} R_{r-1}(t) \right] - \frac{\check{R}_0}{s} \sum_{r=1}^{\infty} (1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s}$$

$$\times \mathbb{L}\left[\sum_{r=1}^{\infty} \left(\kappa^{\varrho} I_{r-1}(t) - \left(\rho^{\varrho} + \zeta^{\varrho} \right) R_{r-1}(t) \right) \right](s)$$

$$= \mathbb{L}\big[R(t) \big] - \frac{\check{R}_0}{s} - \frac{s + \varrho(1 - s)}{s} \mathbb{L}\big[\kappa^{\varrho} I(t) - \left(\rho^{\varrho} + \zeta^{\varrho} \right) R(t) \big](s),$$

and

$$0 = \sum_{r=1}^{\infty} \mathfrak{R}_{V,r}\big(\vec{S}_{r-1}(t), \vec{I}_{r-1}(t), \vec{R}_{r-1}(t), \vec{V}_{r-1}(t) \big)$$

$$= \sum_{r=1}^{\infty} \left(\mathbb{L}\big[V_{r-1}(t) \big] - \frac{\check{V}_0}{s}(1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s} \right.$$

$$\left. \times \mathbb{L}\big[\upsilon^{\varrho} S_{r-1}(t) - \left(\rho^{\varrho} + \varpi^{\varrho} \right) V_{r-1}(t) \big](s) \right)$$

$$= \mathbb{L}\left[\sum_{r=1}^{\infty} V_{r-1}(t) \right] - \frac{\check{V}_0}{s} \sum_{r=1}^{\infty} (1 - \sigma_r) - \frac{s + \varrho(1 - s)}{s}$$

$$\times \mathbb{L}\left[\sum_{r=1}^{\infty} \left(\upsilon^{\varrho} S_{r-1}(t) - \left(\rho^{\varrho} + \varpi^{\varrho} \right) V_{r-1}(t) \right) \right](s)$$

$$= \mathbb{L}\big[V(t) \big] - \frac{\check{V}_0}{s} - \frac{s + \varrho(1 - s)}{s} \mathbb{L}\big[\upsilon^{\varrho} S(t) - \left(\rho^{\varrho} + \varpi^{\varrho} \right) V(t) \big](s).$$

Therefore $S(t)$, $I(t)$, $R(t)$, and $V(t)$ are solutions of the fractional $\mathcal{CF}$-system (46) of the anthrax disease model. The proof is completed. □
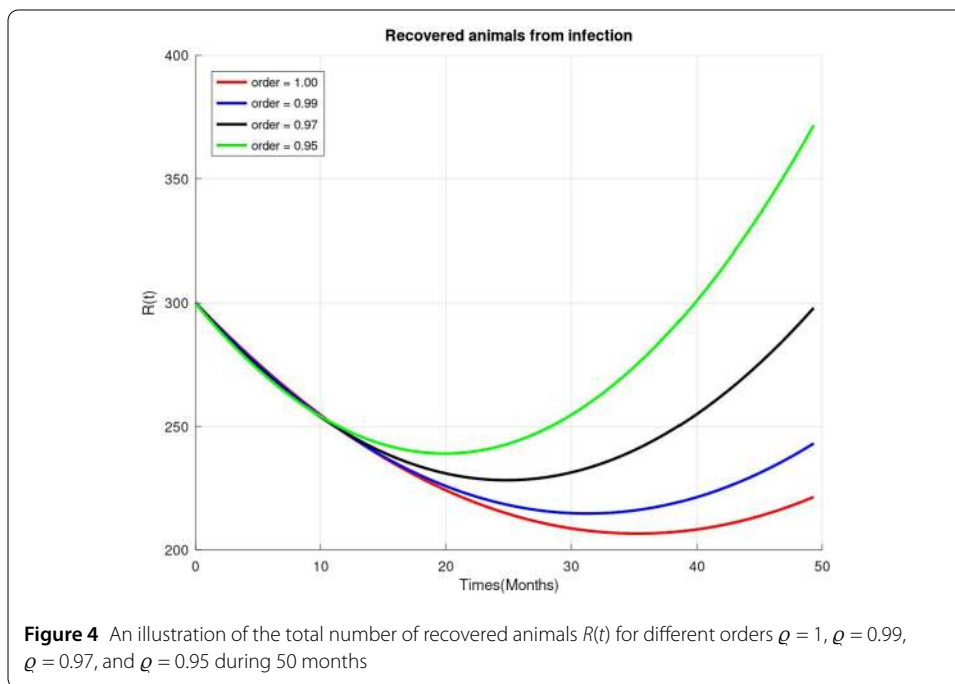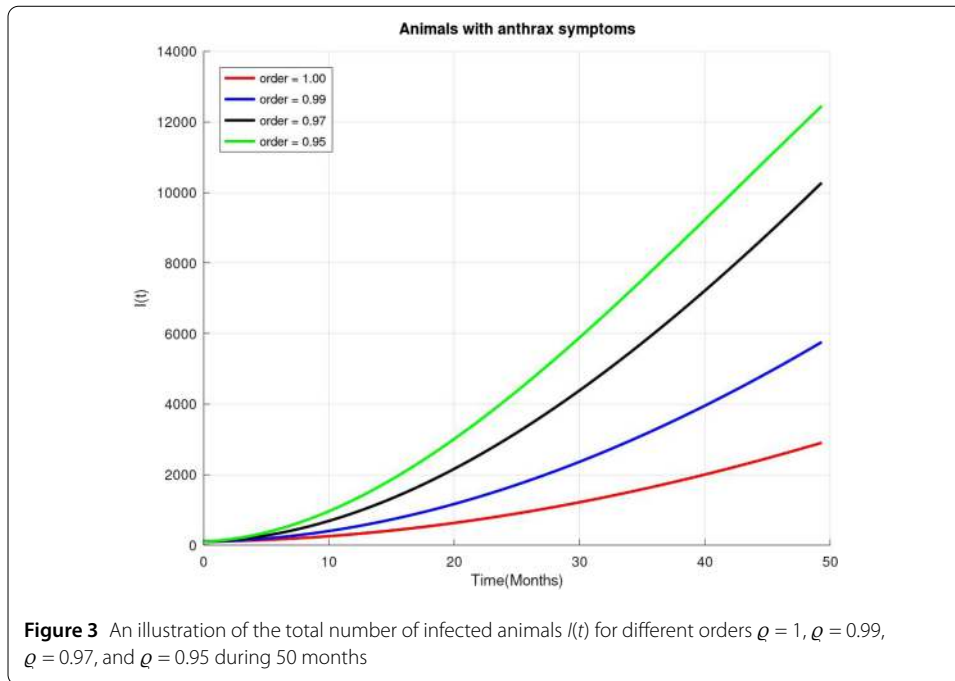
9 Numerical simulations

In this section, we present the results of numerical simulations based on the theoretical findings for the fractional anthrax disease $\mathcal{CF}$-system (8) over a period of $t = 50$ months. In other words, we numerically calculate $S(t)$, $I(t)$, $R(t)$, and $V(t)$ at integer and noninteger orders $\varrho = 1$, $\varrho = 0.99$, $\varrho = 0.97$, and $\varrho = 0.95$. These solution functions for the fractional

Figure 1 An illustration of the total number of each class of animal populations $S(t)$, $I(t)$, $R(t)$, and $V(t)$ for order $\varrho = 0.99$

Figure 2 An illustration of the total number of susceptible animals $S(t)$ for different orders $\varrho = 1$, $\varrho = 0.99$, $\varrho = 0.97$, and $\varrho = 0.95$ during 50 months
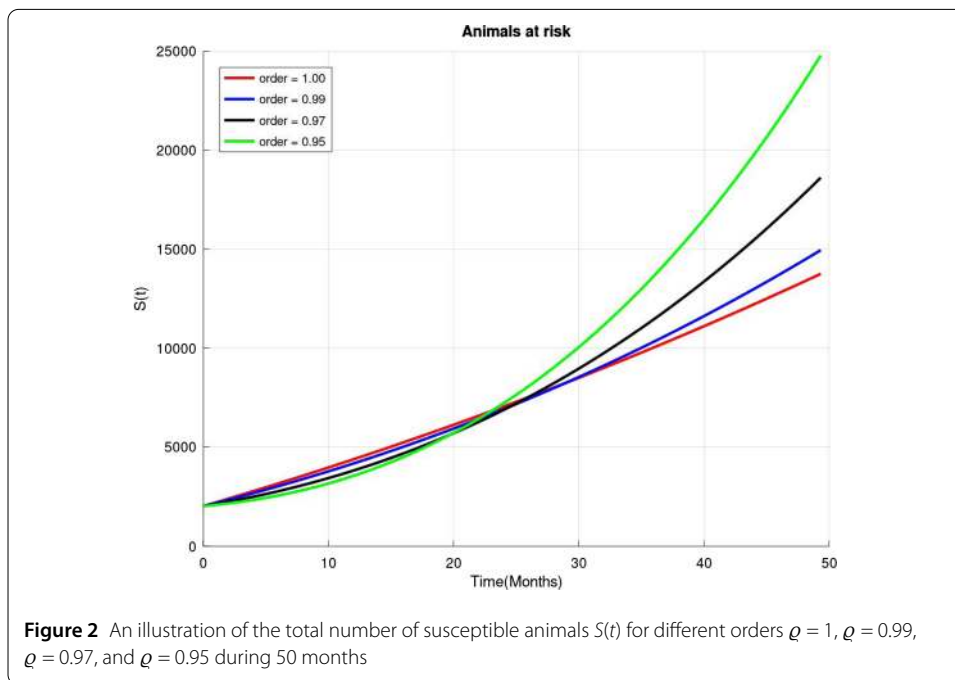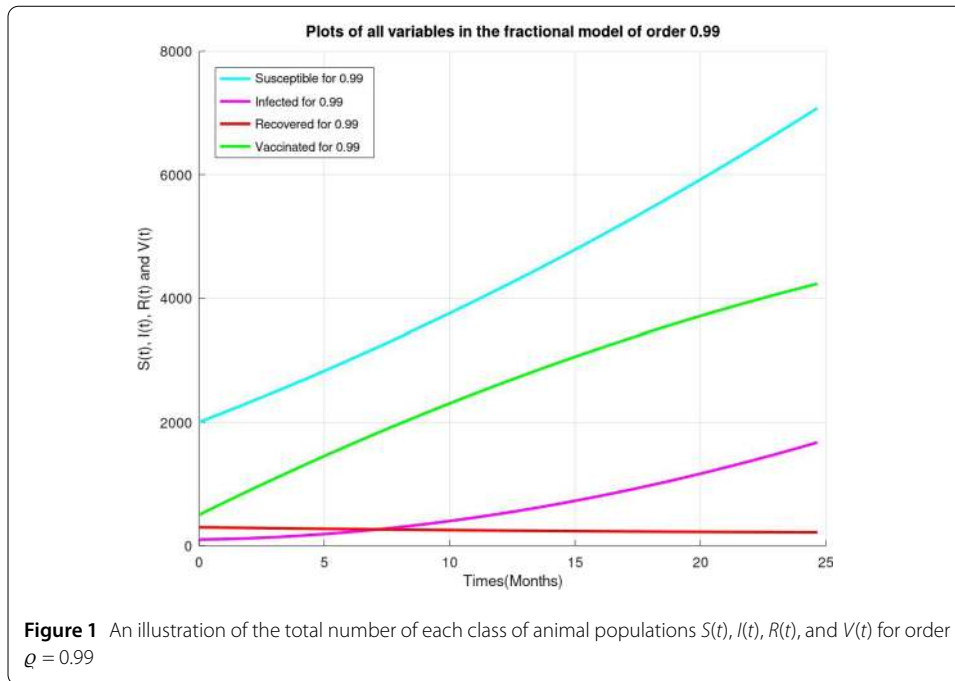
anthrax disease $\mathcal{CF}$-model (8) are obtained by implementing HATM technique. We use distinct values of nonnegative parameters $\omega = 200$, $\delta = 0.0001$, $\rho = 0.001$, $\upsilon = 0.1$, $\zeta = 0.02$, $\varpi = 0.003$, and $\kappa = 0.01$ and select the initial values $\check{S}_0 = 2000$, $\check{I}_0 = 100$, $\check{R}_0 = 300$, and $\check{V}_0 = 500$. Note that these numerical values for parameters are taken from the existing data given in [29, 33, 35, 36].

Figure 1 shows the total number of each class of animal populations $S(t)$, $I(t)$, $R(t)$, and $V(t)$ during a time interval including 25 months for the fractional order $\varrho = 0.99$. Based
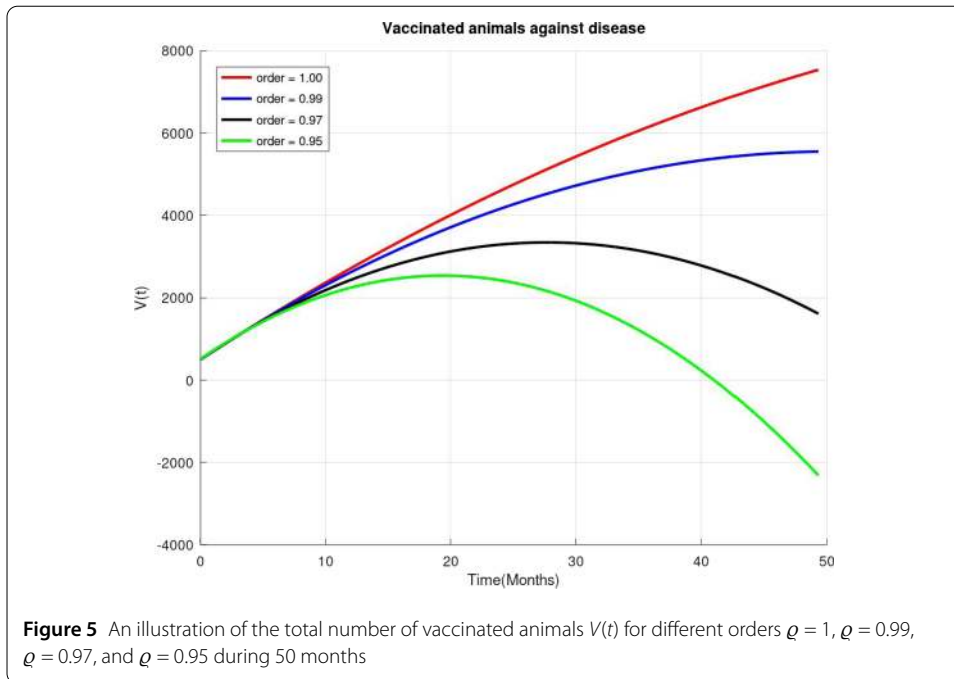
Figure 3 An illustration of the total number of infected animals $I(t)$ for different orders $\varrho = 1$, $\varrho = 0.99$, $\varrho = 0.97$, and $\varrho = 0.95$ during 50 months

Figure 4 An illustration of the total number of recovered animals $R(t)$ for different orders $\varrho = 1$, $\varrho = 0.99$, $\varrho = 0.97$, and $\varrho = 0.95$ during 50 months

on the observations of this figure, we see that the total number of the susceptible animals increases by passing the time, this animal class is highly infected by anthrax disease, and the transmission rate of this disease remains high. In Figs. 2, 3, 4, 5, we plot the three-term approximate solutions of the fractional anthrax disease model by applying the homotopy analysis transform method (HATM) with auxiliary parameter $h = -1$ and auxiliary function $H(t) = 1$ for different values of order ϱ. In each diagram, the solid red line illustrates the solution functions for integer order $\varrho = 1$. These four Figs. 2, 3, 4, 5 indicate that by

Figure 5 An illustration of the total number of vaccinated animals $V(t)$ for different orders $\varrho = 1$, $\varrho = 0.99$, $\varrho = 0.97$, and $\varrho = 0.95$ during 50 months

Table 1 Values of $S(t)$ for four different orders of $^{\mathcal{CF}}\mathcal{D}_0^\varrho$

ϱ	$t = 0$	$t = 3$	$t = 6$	$t = 9$	$t = 12$	$t = 15$	$t = 18$	$t = 21$	$t = 24$
1	2000	2565.3	3148.6	3750	4369.8	5008.5	5666.3	6343.5	7040.4
0.99	2001.6	2489.2	3010.9	3567.7	4160.6	4790.6	5458.7	6165.9	6913
0.97	2003.2	2347.5	2759.6	3243.5	3803	4441.9	5164	5973.2	6873.2
0.95	2002.9	2218.6	2539.1	2974.2	3533.5	4226.8	5063.8	6054.3	7208.1

Table 2 Values of $I(t)$ for four different orders of $^{\mathcal{CF}}\mathcal{D}_0^\varrho$

ϱ	$t = 0$	$t = 3$	$t = 6$	$t = 9$	$t = 12$	$t = 15$	$t = 18$	$t = 21$	$t = 24$
1	100	122.15	165.01	228.27	311.63	414.77	537.4	679.2	839.88
0.99	100.06	139.14	222.51	349.19	518.19	728.54	979.23	1269.3	1597.8
0.97	100.29	174.81	338.35	587.07	917.15	1324.8	1806	2357.2	2974.4
0.95	100.73	213.06	455.07	817.03	1289.2	1861.8	2525.1	3269.4	4084.9

Table 3 Values of $R(t)$ for four different orders of $^{\mathcal{CF}}\mathcal{D}_0^\varrho$

ϱ	$t = 0$	$t = 3$	$t = 6$	$t = 9$	$t = 12$	$t = 15$	$t = 18$	$t = 21$	$t = 24$
1	300	284.78	270.91	258.39	247.22	237.41	228.95	221.84	216.09
0.99	299.94	284.36	270.35	257.91	247.04	237.74	230.02	223.86	219.27
0.97	299.82	283.54	269.36	257.27	247.28	239.38	233.58	229.87	228.26
0.95	299.68	282.74	268.57	257.16	248.51	242.62	239.5	239.13	241.53

letting $\varrho \to 1$ the approximate solutions approach the classic integer solution with $\varrho = 1$. More precisely, we provide Tables 1, 2, 3 and 4 for solutions $S(t)$, $I(t)$, $R(t)$, and $V(t)$, which represent a comparison between the obtained values for the fractional order $\mathcal{CF}$-model (8) with $\varrho = 0.99$, $\varrho = 0.97$, and $\varrho = 0.95$ and the integer-order model (5) with $\varrho = 1$. Based on this data, we find that the impact of the vaccination rate to control the spread of the an-

Table 4 Values of $V(t)$ for four different orders of ${}^{\mathcal{CF}}\mathcal{D}_0^{\varrho}$

ϱ	$t = 0$	$t = 3$	$t = 6$	$t = 9$	$t = 12$	$t = 15$	$t = 18$	$t = 21$	$t = 24$
1	500	1083.9	1647.6	2191.2	2714.6	3217.8	3700.8	4163.6	4606.3
0.99	502.03	1085.4	1633	2144.8	2620.9	3061.1	3465.6	3834.3	4167.3
0.97	506.35	1088.3	1603.5	2052	2433.9	2749.1	2997.6	3179.5	3294.7
0.95	511.06	1090.8	1573.1	1958	2245.4	2435.5	2528.2	2523.4	2421.3

thrax disease between animals is vital, and hence this implies that the vaccination policies should be considered seriously to overcome this animal infection.

10 Conclusions

In this research work, we provide a fractional-order modeling of the anthrax disease between animals based on the Caputo–Fabrizio derivative. In the first step, we derive an existence criterion of solutions for proposed fractional $\mathcal{CF}$-system of the anthrax disease model by utilizing the Picard–Lindelof technique. Then by obtaining the basic reproduction number $\mathcal{R}_0$ of the fractional $\mathcal{CF}$-system we compute two disease-free and endemic equilibrium points and check the asymptotic stability. Moreover, by applying an iterative approach based on the Sumudu transform, we investigate the stability of the fractional $\mathcal{CF}$-system. The approximate series solutions of this system are obtained by means of the homotopy analysis transform method, in which we invoke the linear Laplace transform. Finally, after the convergence analysis of numerical method HATM, we present a numerical simulation of the $\mathcal{CF}$-fractional anthrax disease model and review the dynamical behavior of the solutions of this $\mathcal{CF}$-system during a time interval.

Author details
[1]Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam. [2]Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam. [3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan. [4]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. [5]Department of Mathematics, Miandoab Branch, Islamic Azad University, Miandoab, Iran.

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