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A MATRIX REPRESENTATION OF A PAIR OF PROJECTIONS IN A HILBERT SPACE

BY
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Let H be a complex Hilbert space and let $K=H \oplus H$. Then K can be identified with the set of all column matrices

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \psi_i \in H$$

equipped with componentwise addition and scalar multiplication and the scalar product

$$(\boldsymbol{\psi} \mid \boldsymbol{\phi}) = (\psi_1 \mid \phi_1)+(\psi_2 \mid \phi_2).$$

Using this representation of $K=H \oplus H$ the algebra $L(K)$ of all bounded operators on K may be identified with the algebra $\mathbf{M}_2(L(H))$ of all 2×2 matrices over the ring $L(H)$ with the involution

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^* = \begin{bmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{bmatrix}.$$

In the sequel we mean by the word *projection* an orthogonal projection, i.e. a selfadjoint idempotent. Moreover we use the notation $L(H)^+$ for the positive part of $L(H)$.

LEMMA 1. *Let $a, b \in L(H)^+$ and assume that $a+b, q_1, q_2$ are projections, such that $aq_1=bq_2=0$. Then a and b commute and*

$$\mathbf{e}_\pm = \begin{bmatrix} a+q_1 & \pm(ab)^{1/2} \\ \pm(ab)^{1/2} & b+q_2 \end{bmatrix}$$

are projections belonging to $L(K)$.

Proof. Since $a, b \le a+b$ and $a+b$ is a projection, a commutes with $a+b$. Hence a commutes with $(a+b-a)=b$. Moreover $(a+b)a=a$ and $(a+b)b=b$. Therefore

$$\mathbf{e}_\pm^2 = \mathbf{e}_\pm .$$

Finally $\mathbf{e}_\pm$ are selfadjoint, since their matrices are invariant under the *-operation in $\mathbf{M}_2(L(H))$, Q.E.D.

Let $\mathbf{I}(H, K)$ be the set of all isometries of H into K. Then every element $\rho \in \mathbf{I}(H, K)$ induces a *-isomorphism ρ' of $L(H)$ into $L(K)$ defined by the equations

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$$\rho'(a)\psi = \rho(a\rho^{-1}(\psi)), \quad \psi \in \rho(H),$$
$$\rho'(a)\psi = 0, \qquad \psi \perp \rho(H).$$

Indeed we have

$$\sup_{\|\psi\|\leq 1} \|\rho'(a)\psi\| = \sup_{\substack{\|\psi\|\leq 1\\ \psi\in\rho(H)}} \|\rho'(a)\psi\| = \sup_{\|\rho^{-1}\psi\|\leq 1} \|\rho(a\rho^{-1}(\psi))\|$$
$$= \sup_{\|\rho^{-1}\psi\|\leq 1} \|a\rho^{-1}(\psi)\| = \|a\|.$$

That ρ' is a *-homomorphism is easily verified. The following theorem asserts that for every pair of projections $e, f \in L(H)$ there exists $\rho \in \mathrm{I}(H, K)$ such that $\rho'(e)$ and $\rho'(f)$ have the form (1) or alternatively such that

$$\rho'(e) = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$$

while $\rho'(f)$ has a representation of the form (1). More explicitly:

THEOREM 2. *Let e and f be projections in $L(H)$. Then*

(i) *There is a linear isometry $\rho: H \to K = H \oplus H$ such that, under the corresponding injection $\rho': L(H) \to L(K)$,*

$$\rho'(e) = \begin{bmatrix} e\wedge f + e\wedge f^{\perp} + a & (ab)^{1/2} \\ (ab)^{1/2} & b \end{bmatrix},$$

$$\rho'(f) = \begin{bmatrix} e\wedge f + e^{\perp}\wedge f + a & -(ab)^{1/2} \\ -(ab)^{1/2} & b \end{bmatrix},$$

where $I \geq a \geq b \geq 0$, $a+b$ is a projection orthogonal to $e\wedge f + e^{\perp}\wedge f + e\wedge f^{\perp} + e^{\perp}\wedge f^{\perp}$, and $\frac{1}{2}$ and 1 do not belong to the point spectrum of a or b.

(ii) *There is a linear isometry $\tau: H \to K$ such that, under the corresponding injection $\tau': L(H) \to L(K)$,*

$$\tau'(e) = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tau'(f) = \begin{bmatrix} e\wedge f + c & (cd)^{1/2} \\ (cd)^{1/2} & d + e^{\perp}\wedge f \end{bmatrix},$$

where $I \geq c \geq 0$, $I \geq d \geq 0$, $c+d$ is the projection $e - e\wedge f - e\wedge f^{\perp}$, and 1 does not belong to the point spectrum of c or d.

Here $e^{\perp} = I - e$, $f^{\perp} = I - f$, $e\wedge f$ is the projection corresponding to $eH \cap fH$, and I is the identity in $L(H)$. The virtue of these representations stems from the fact that in each case the elements of $L(H)$ appearing as entries in the matrices all commute. The noncommutativity of e and f is thus embodied entirely in the matrix form of the representations.

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Proof of (i). Let $s=e+f$, $d=e-f$. Since $2I\geq s\geq 0$ the spectrum of s lies in the closed interval [0, 2]. Now we have:

LEMMA 3. *If λ is the spectral measure corresponding to s then*

(a) $\lambda(\{2\})=e\wedge f$,

(b) $\lambda(\{1\})=e^{\perp}\wedge f+e\wedge f^{\perp}$,

(c) $\lambda(\{0\})=e^{\perp}\wedge f^{\perp}$.

Proof of Lemma 3. (a) If $\psi\in(e\wedge f)H$ then $s\psi=2\psi$ so $\psi\in\lambda(\{2\})H$. Conversely, assume $\psi\in\lambda(\{2\})H$ and $\|\psi\|=1$. Then $(\psi, e\psi)+(\psi, f\psi)=2$, so that $e\psi=f\psi=\psi$ whence $\psi\in(e\wedge f)H$.

(c) is proved in the same way.

(b) If $\psi\in(e^{\perp}\wedge f+e\wedge f^{\perp})H$ then we can write $\psi=\psi_1+\psi_2$ with $\psi_1\in(e^{\perp}\wedge f)H$ and $\psi_2\in(e\wedge f^{\perp})H$. But then $(e+f)\psi=\psi_1+\psi_2=\psi$ so $\psi\in\lambda(\{1\})H$. Conversely, if $\psi\in\lambda(\{1\})H$ then $(e+f-I)^2\psi=0$. But $(e+f-I)^2=I-(e-f)^2$ so $(e-f)^2\psi=\psi$. It now follows from the spectral theorem that ψ is a linear combination $\psi=\alpha\psi_1+\beta\psi_2$ with $\|\psi_1\|=\|\psi_2\|=1$ and $(e-f)\psi_1=\psi_1$, $(e-f)\psi_2=-\psi_2$. But then $1=(\psi_1, (e-f)\psi_1)$ $=(\psi_1, e\psi_1)-(\psi_1, f\psi_1)$ whence $(\psi_1, e\psi_1)=1$ and $(\psi_1, f\psi_1)=0$ so that $\psi_1\in(e\wedge f^{\perp})H$. Similarly $\psi_2\in(e^{\perp}\wedge f)H$ so that $\psi\in(e^{\perp}\wedge f+e\wedge f^{\perp})H$.

Let e_2, e_+, e_1, e_-, e_0 denote $\lambda(\{2\})$, $\lambda((1, 2))$, $\lambda(\{1\})$, $\lambda((0, 1))$, $\lambda(\{0\})$ respectively.

We now examine the structure of d. Since $d^2=2s-s^2$ the support of d is e_++e_1 $+e_-$ and since d is Hermitian its polar decomposition (see, for instance, [1, p. 334]) takes the form

$$d = u(2s-s^2)^{1/2},$$

where u is a partial isometry commuting with $(2s-s^2)^{1/2}$ with $u=u^*$ and $u^2=e_+$ $+e_1+e_-$. From the identity $sd+ds-2d=0$ we obtain $su=u(2I-s)$. This implies $f(s)u=uf(2I-s)$ for any polynomial f and hence, by the separate weak continuity of multiplication, for any Borel function f defined on the closed interval [0, 2]. In particular $e_+u=ue_-$ and $e_1u=ue_1$.

We have directly $d(e\wedge f^{\perp})=e\wedge f^{\perp}$ and $d(e^{\perp}\wedge f)=-(e^{\perp}\wedge f)$ which, since $de_1=ue_1$, gives $u(e\wedge f^{\perp})=e\wedge f^{\perp}$ and $u(e^{\perp}\wedge f)=-(e^{\perp}\wedge f)$.

Let $v=u(e_++e_-)$. Then v is a partial isometry, $v=v^*$, and $v^2=e_++e_-$.

Let $\rho: H\to K$ be given by

$$\rho(\psi) = \begin{bmatrix}(I-e_-)\psi\\ ve_-\psi\end{bmatrix}.$$

Then $(\rho(\psi), \rho(\phi))=(\psi, [(I-e_-)^2+e_-v^2e_-]\phi)=(\psi, \phi)$ so that ρ is a linear isometry. The corresponding map $\rho': L(H)\to L(K)$ is given by

$$\rho'(x) = \begin{bmatrix}(I-e_-)x(I-e_-) & (I-e_-)xe_-v\\ ve_-x(I-e_-) & ve_-xe_-v\end{bmatrix} = \begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix},$$

say. We evaluate $\rho'(s)$:

$$s_{11} = (I-e_-)s(I-e_-) = (I-e_-)s = (e_2+e_++e_1)s = 2e_2+e_1+e_+s$$

$$s_{12} = s_{21} = 0$$

$$s_{22} = ve_-sv = e_+v^2(2I-s) = e_+(2I-s).$$

Putting $2a=e_+s$, $2b=e_+(2I-s)$ we have $e_+\geq a\geq(e_+/2)\geq b\geq 0$, $a+b=e_+$, $\frac{1}{2}$ and 1 do not lie in the point spectrum of a or b, and

$$\rho'(s) = \begin{bmatrix} 2(e\wedge f)+e\wedge f^{\perp}+e^{\perp}\wedge f+2a & 0 \\ 0 & 2b \end{bmatrix}.$$

Lastly we evaluate $\rho'(d)$:

$$d_{11} = (I-e_-)u(2s-s^2)^{1/2}(I-e_-) = (e_++e_1)u(e_++e_1)(2s-s^2)^{1/2}$$

$$= ue_1 = e\wedge f^{\perp}-e^{\perp}\wedge f,$$

$$d_{22} = ve_-u(2s-s^2)^{1/2}e_-v = vue_+e_-(2s-s^2)^{1/2}v = 0,$$

$$d_{12} = (I-e_-)u(2s-s^2)^{1/2}e_-v = (I-e_-)(2s-s^2)^{1/2}uve_+$$

$$= (2s-s^2)^{1/2}(I-e_-)e_+ = 2(ab)^{1/2}.$$

Similarly $d_{21}=2(ab)^{1/2}$. Thus

$$\rho'(d) = \begin{bmatrix} e\wedge f^{\perp}-e^{\perp}\wedge f & 2(ab)^{1/2} \\ 2(ab)^{1/2} & 0 \end{bmatrix}.$$

It now follows from the linearity of ρ' that $\rho'(e)$ and $\rho'(f)$ are as stated in the theorem.

Proof of (ii). Let $p=efe$, $q=e^{\perp}fe$, $r=e^{\perp}fe^{\perp}$. Then $qq^*+r^2=r$, $q^*q+p^2=p$, $rq+qp=q$. We first examine the spectrum of p which, since $I\geq p\geq 0$, lies in the closed interval [0, 1].

LEMMA 4. *If μ is the spectral measure corresponding to $p=efe$ then*

(a) $\mu(\{1\})=e\wedge f$,

(b) $\mu(\{0\})=e\wedge f^{\perp}+e^{\perp}$.

Proof of Lemma 4. (a) If $\psi\in(e\wedge f)H$ then $efe\psi=\psi$ so $\psi\in\mu(\{1\})H$. Conversely, suppose $\psi\in\mu(\{1\})H$ and $\|\psi\|=1$. Then $1=\|efe\psi\|\leq\|fe\psi\|\leq\|e\psi\|\leq 1$. This means $e\psi=\psi$ and so $\|f\psi\|=1$ implying $f\psi=\psi$ so that $\psi\in(e\wedge f)H$.

(b) If $\psi\in(e\wedge f^{\perp}+e^{\perp})H$ then $efe\psi=0$. Thus $\mu(\{0\})H\supseteq(e\wedge f^{\perp}+e^{\perp})H$. Conversely, suppose $efe\psi=0$. Let $\psi_1=e\psi$, $\psi_2=e^{\perp}\psi$. Then $(\psi_1, f\psi_1)=0$ so $f\psi_1=0$, whence $\psi_1\in(e\wedge f^{\perp})H$. Since $\psi_2\in e^{\perp}H$ this means $\psi\in(e\wedge f^{\perp}+e^{\perp})H$.

Replacing e by $e^{\perp}$ in Lemma 2 we obtain $\nu(\{1\})=e^{\perp}\wedge f$ and $\nu(\{0\})=e^{\perp}\wedge f^{\perp}+e$ where ν denotes the spectral projection corresponding to r. Let $e_\mu=\mu((0, 1))$,

$e_\nu = \nu((0, 1))$. Since $pr=0$ the projections $e\wedge f$, e_μ, $e\wedge f^\perp$, $e^\perp\wedge f$, e_ν, $e^\perp\wedge f^\perp$ are mutually orthogonal.

We now examine q. Since $q^*q=p-p^2$ the polar decomposition of q takes the form $q=u(p-p^2)^{1/2}$, where u is a partial isometry and q, u, and $(p-p^2)^{1/2}$ have the common support e_μ. Moreover, $qq^*=(r-r^2)^{1/2}$ so that q^* and u have the support e_ν. Directly from the definition of the polar decomposition we have also $uu^*=e_\nu$, $u^*u=e_\mu$, and $ue_\mu=u=e_\nu u$. Lastly, the identity $rq+qp=q$ gives

$$ru(p-p^2)^{1/2}-u(p-p^2)^{1/2}(I-p) = 0$$

whence $ru-u(I-p)=0$.

Now let $v=u+u^*+(I-e_\mu-e_\nu)$. Then v is unitary, indeed $v=v^*$ and $v^2=I$. Let $\tau: H \to H\oplus H=K$ be given by

$$\tau(\psi) = \begin{bmatrix} e\psi \\ ve^\perp\psi \end{bmatrix}.$$

Clearly, τ is an isometry. The corresponding map $\tau': L(H) \to L(K)$ is given by

$$\tau'(x) = \begin{bmatrix} exe & exe^\perp v \\ ve^\perp xe & ve^\perp xe^\perp v \end{bmatrix}.$$

Clearly $\tau'(e) = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$. We compute $\tau'(f)$:

$$efe = p = e\wedge f+e_\mu p,$$
$$ve^\perp fe^\perp v = e^\perp\wedge f+u^*ru = e^\perp\wedge f+u^*u(I-p) = e^\perp\wedge f+e_\mu(I-p),$$
$$ve^\perp fe = vq = vu(p-p^2)^{1/2} = e_\mu(p-p^2)^{1/2}.$$

This gives

$$\tau'(f) = \begin{bmatrix} e\wedge f+c & (cd)^{1/2} \\ (cd)^{1/2} & d+e^\perp\wedge f \end{bmatrix},$$

where $c=e_\mu p$, $d=e_\mu(I-p)$. Here $c+d=e_\mu$ and 1 does not belong to the point spectrum of c or d.

Corollary 5. *Let e, f be two projections in a Hilbert space $\hat{H}$, such that*

$$e\wedge f = e^\perp\wedge f = e\wedge f^\perp = e^\perp\wedge f^\perp = 0.$$

Then there is a subspace $H \subseteq \hat{H}$ and an isomorphism ρ' of $\hat{H}$ into $K=H\oplus H$, such that

$$\rho'(e) = \begin{bmatrix} a & (a(I-a))^{1/2} \\ (a(I-a))^{1/2} & I-a \end{bmatrix}$$

$$\rho'(f) = \begin{bmatrix} a & -(a(I-a))^{1/2} \\ -(a(I-a))^{1/2} & I-a \end{bmatrix},$$

where $a \in L(H)$, $I/2 \le a \le I$ and $\frac{1}{2}$ and 1 are not eigenvalues of a.

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Proof. Let e_+, e_- and u be defined as in the proof of Lemma 3. Let $H=e_+\hat{H}$ and

$$\rho(\psi) = \begin{bmatrix} e_+ & \psi \\ ue_- & \psi \end{bmatrix}.$$

From this equation it follows that:

$$\rho'(x) = \begin{bmatrix} e_+xe_+ & e_+xe_-u \\ ue_-xe_+ & ue_-xe_+u \end{bmatrix}$$

and in particular

$$\rho'(s) = \begin{bmatrix} 2a & 0 \\ 0 & 2(I-a) \end{bmatrix}$$

since $ue_-se_-u=e_+usue_+=e_+(2I-s)e_+=2e_+-2a$ and e_+ coincides with the identity in $H=e_+\hat{H}$. Q.E.D.

We are indebted to the referee for indicating the following applications of Theorem 2.

THEOREM 6. *Let e, f, $\hat{e}$, $\hat{f}$, be projections in $L(H)$, let λ and $\hat{\lambda}$ be the spectral measures determined by the selfadjoint elements $s=e+f$ and $\hat{s}=\hat{e}+\hat{f}$ respectively, and let*:

$$a = s\lambda((1,2)) \quad \textit{and} \quad \hat{a} = \hat{s}\hat{\lambda}((1,2)).$$

In order that there exists a unitary element $u\in L(H)$, such that simultaneously:

$$\hat{e} = ueu^*$$

and

$$\hat{f} = ufu^*,$$

it is necessary and sufficient that a and $\hat{a}$ are unitarily equivalent and in addition

$$\dim(e\wedge f) = \dim(\hat{e}\wedge\hat{f})$$
$$\dim(e^\perp\wedge f) = \dim(\hat{e}^\perp\wedge\hat{f})$$
$$\dim(e\wedge f^\perp) = \dim(\hat{e}\wedge\hat{f}^\perp)$$
$$\dim(e^\perp\wedge f^\perp) = \dim(\hat{e}^\perp\wedge\hat{f}^\perp).$$

Proof. (i) The condition is necessary. Indeed let $u\in L(H)$ such that

$$\hat{e} = ueu^*$$
$$\hat{f} = ufu^*.$$

From $e\wedge f\leq e, f$ it follows that

$$u(e\wedge f)u^* \leq \hat{e} \quad \text{and} \quad u(e\wedge f)u^* \leq \hat{f}$$

and thus:

$$u(e\wedge f)u^* \leq \hat{e}\wedge\hat{f}.$$

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Similarly:

$$u^*(\hat{e}\wedge\hat{f})u \leq e\wedge f$$

and hence

$$u(e\wedge f)u^* = \hat{e}\wedge\hat{f},$$

which implies dim $(\hat{e}\wedge\hat{f})=$dim $(e\wedge f)$. Similarly for $e^{\perp}\wedge f$, etc.

Moreover:

$$\hat{s} = \hat{e}+\hat{f} = ueu^*+ufu^* = u(e+f)u^* = usu^*.$$

Therefore:

$$\hat{e}_+ = ue_+u^* \quad \text{and} \quad \hat{a} = \tfrac{1}{2}\hat{s}\hat{e}_+ = \tfrac{1}{2}usu^*ue_+u^* = uau^*.$$

(ii) The condition is sufficient. Assume it to be satisfied. Then there exists a unitary $u_0 \in L(H)$, such that

$$\hat{a} = u_0au_0^*.$$

We define a partial isometry u_+ by:

$$u_+ = u_0e_+.$$

Then

$$u_+^*u_+ = e_+u_0^*u_0e_+ = e_+$$

and

$$u_+u_+^* = u_0e_+u_0^* = u_0e(a)u_0^* = e(\hat{a}) = \hat{e}_+.$$

Here we mean by $e(a)$ the projection onto the closure of the range of a which coincides with e_+, since $e_+\leq a\leq 2e_+$.

Thus u_+ is a partial isometry from e_+H onto $\hat{e}_+H$. Now, as in the proof of Theorem 2, let v be the partial isometry obtained by multiplying the partial isometry occurring in the polar decomposition of $d=e-f$ by (e_++e_-) (from the left or from the right), and let $\hat{v}$ be the analogous element of $L(H)$ associated with the pair $\hat{e},\hat{f}$.

Define

$$u_- = \hat{v}u_+v.$$

Then, using the fact that $v=v^*$ and $\hat{v}=\hat{v}^*$, we obtain:

$$\begin{aligned} u_-^*u_- &= vu_+^*\hat{v}\hat{v}u_+v = vu_+^*\hat{e}_+u_+v \\ &= vu_+^*(u_+u_+^*)u_+v = ve_+v = e_- \end{aligned}$$

and

$$\begin{aligned} u_-u_-^* &= \hat{v}u_+vvu_+^*\hat{v} = \hat{v}u_+e_+u_+^*\hat{v} \\ &= vu_+u_+^*u_+u_+^*\hat{v} = \hat{v}\hat{e}_+\hat{v} = \hat{e}_-. \end{aligned}$$

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Finally let us make use of the second part of the condition in Theorem 6: let u_1, u_2, u_3, u_4, be partial isometries such that

$$\begin{aligned} u_1^*u_1 &= e\wedge f, & u_1u_1^* &= \hat{e}\wedge\hat{f}, \\ u_2^*u_2 &= e^\perp\wedge f, & u_2u_2^* &= \hat{e}^\perp\wedge\hat{f}, \\ u_3^*u_3 &= e\wedge f^\perp, & u_3u_3^* &= \hat{e}\wedge\hat{f}^\perp, \\ u_4^*u_4 &= e^\perp\wedge f^\perp, & u_4u_4^* &= \hat{e}^\perp\wedge\hat{f}^\perp, \end{aligned}$$

and define

$$w = u_+ + u_- + u_1 + u_2 + u_3 + u_4.$$

w is obviously unitary. Now applying Theorem 2, we obtain:

$$\begin{aligned} \hat{e} &= \hat{e}\wedge\hat{f} + \hat{e}\wedge\hat{f}^\perp + \hat{a} + (\hat{a}\hat{b})^{1/2}\hat{v} + \hat{v}(\hat{a}\hat{b})^{1/2} + \hat{v}\hat{b}\hat{v} \\ &= u_1(e\wedge f)u_1^* + u_3(e\wedge f^\perp)u_3^* + u_+(ab)^{1/2}u_+^*\hat{v} \\ &\quad + \hat{v}u_+(ab)^{1/2}u_+^* + \hat{v}u_+bu_+^*\hat{v}. \end{aligned}$$

Since $\hat{v}u_+ = u_-v$, the last expression can be rewritten as:

$$\begin{aligned} \hat{e} &= u_1(e\wedge f)u_1^* + u_3(e\wedge f^\perp)u_3^* + u_+(ab)^{1/2}vu_-^* \\ &\quad + u_-v(ab)^{1/2}u_+^* + u_-vbvu_-^* \\ &= w[(e\wedge f) + (e\wedge f^\perp) + (ab)^{1/2}v + v(ab)^{1/2} + vbv]w^* \\ &= wew^*. \end{aligned}$$

In a similar way we obtain:

$$\hat{f} = wfw^*.$$

Q.E.D.

Corollary 7. *Let $b \in L(H)$ be an operator of the form $b = e + if$, where e and f are projections. Let λ be the spectral measure of $e+f$ and $e_+ = \lambda(1, 2)$. Define*:

$$a = \frac{1}{2}(e+f)e_+.$$

Then the multiplicity function of the spectral measure determined by a (cf. Halmos [2]) *together with the cardinal numbers* $\dim(e\wedge f)$, $\dim(e^\perp\wedge f)$, $\dim(e\wedge f^\perp)$, $\dim(e^\perp\wedge f^\perp)$ *constitute a complete set of unitary invariants for b.*

For the formulation and proof of Theorem 8 we shall make use of the following terminology and notation:

(i) Let $\sigma \subseteq L(H)$ be a subset. By the W^*-algebra $\{\sigma\}$ generated by σ we mean the smallest selfadjoint, weakly closed subalgebra of $L(H)$ containing σ. By the von Neumann algebra $N(\sigma)$ generated by σ we mean the W^*-algebra generated by σ

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and the identity I. In [1] it is shown that $N(\sigma)=(\sigma \cup \sigma^*)''$ where $'$ denotes the formation of the commutant. The relation between $\{\sigma\}$ and $N(\sigma)$ is simple and can be described by the equations:

$$N(\sigma) = \{a \in L(H) \mid a = \mu I+b, \mu \in \mathbf{C}, b \in \{\sigma\}\}$$

and

$$\{\sigma\} = \{a \in N(\sigma) \mid e(a) \leq e(\sigma)\}$$

where for any subset $\sigma \subseteq L(H)$, the *support* $e(\sigma)$ of σ is defined by:

$$e(\sigma) = \inf\{e \mid e = \text{projection}, eb = be = b \text{ for all } b \text{ in } \sigma\}.$$

(ii) $\mathbf{M}_n$ denotes the matrix algebra of order n over the complex field $\mathbf{C}$. If $a \in L(H)$ then $\mathbf{M}_n(\{a\})$ denotes the matrix algebra over the W^*-algebra $\{a\}$ generated by the element a.

THEOREM 8. *Let H be a Hilbert space, $e, f \in L(H)$ two projections. Then the von Neumann algebra $N(e,f)$ generated by e and f is a direct sum of two subalgebras*:

$$N(e,f) = W_0 \oplus W$$

where W_0 consists of all linear combinations of the orthogonal projections $e \wedge f$, $e^\perp \wedge f$, $e \wedge f^\perp$, $e^\perp \wedge f^\perp$ and W is isomorphic to $\mathbf{M}_2(\{a\})$. Here $a=(e+f)\lambda(1, 2)$ where λ is the spectral measure of $e+f$, and $\{a\}$ stands for the abelian W^*-algebra generated by a.

The proof depends on two Lemmas:

LEMMA 9. *An alternative set of generators for $N(e,f)$ is given by*:

$$e\wedge f; \quad e^\perp \wedge f; \quad e \wedge f^\perp; \quad e^\perp \wedge f^\perp; \quad \hat{e}; \quad \hat{f},$$

where

$$\hat{e} = e - e\wedge f - e \wedge f^\perp,$$
$$\hat{f} = f - e\wedge f - e^\perp \wedge f,$$

the products of two generators being zero with the exception of $\hat{e}\hat{f}$.

Proof of Lemma 9. The lemma is a consequence of the fact that the projections in a von Neumann algebra are closed under intersection and orthocomplementation.

From Lemma 9 it follows immediately that $N(e,f)$ decomposes according to:

$$N(e,f) = W_0 \oplus W$$

where W_0 consists of all linear combinations of

$$e\wedge f, \quad e^\perp \wedge f, \quad e\wedge f^\perp, \quad e^\perp \wedge f^\perp, \quad \text{and} \quad W = \{\hat{e}, \hat{f}\}.$$

Let $g=I-e\wedge f-e^\perp \wedge f-e\wedge f^\perp-e^\perp \wedge f^\perp$ and let $\hat{H}=gH$. We have $g \in N(e,f)$ and $\hat{e}=eg$ and $\hat{f}=fg$.

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LEMMA 10. *An alternative set of generators for W is given by:*

$$(1)\qquad \begin{aligned} a &= (e+f)e_+ = (e+f)e_+, \\ w &= ue_+, \end{aligned}$$

where $e_+ = \lambda(1, 2)$ and λ stands for the spectral measure defined by $e+f$ and u is the unitary operator in H occurring in the polar decomposition of $\hat{d} = \hat{e} - \hat{f}$ ($\hat{d} = u \mid \hat{d} \mid$).

Proof of Lemma 10. It is clear from (1), that a, $w \in W$. Conversely using the isomorphism ρ' of Corollary 5 we obtain:

$$(2)\qquad \begin{cases} \text{(i)} & \rho'(a) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \\ \text{(ii)} & \rho'(w) = \begin{bmatrix} 0 & 0 \\ e_+ & 0 \end{bmatrix} \end{cases}$$

and hence it follows from Corollary 5 that

$$\hat{e} = a + w(a(I-a))^{1/2} + (a(I-a))^{1/2}w^* + w(I-a)w^*,$$

$$\hat{f} = a - [w(a(I-a))^{1/2} + (a(I-a))^{1/2}w^*] + w(I-a)w^*.$$

This proves the lemma.

Since $aw = w^2 = 0$ and $w^*w = e_+$, the most general element $x \in W$ has the form

$$x = b_{11} + wb_{12} + b_{21}w^* + wb_{22}w^*,$$

where $b_{ik} \in \{a\}$ for $i, k = 1, 2$.

Using equations (2) we obtain the desired isomorphisms.

$$x \mapsto \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

of W onto $\mathbf{M}_2(\{a\})$. Q.E.D.

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