A Maximum Principle With Applications To Subharmonic Functions in n-space

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1. Introduction

Denote points in n dimensional Euclidean space $\mathbf{R}^n$, $n \geq 3$, by $x = (x_1, x_2, \ldots, x_n)$. Let $r = |x|$ and $x_1 = r \cos \theta$, $0 \leq \theta \leq \pi$. A real valued function f defined on a subset E of $\mathbf{R}^n$ is said to be symmetric (with respect to the x_1 axis) if $f(x) = f(y)$ whenever $x, y \in E$ and x and y have the same r, θ coordinates.

For $r > 0$ let $B(r) = \{x: |x| < r\}$, $S(r) = \{x: |x| = r\}$ and $S = S(1)$. For $0 \leq \alpha \leq \pi$ let $C(\alpha) = S \cap \{x: \theta < \alpha\}$. Given a set $E \subset \mathbf{R}^n$, let $\bar{E}$, ∂E, denote the closure and boundary of E in $\mathbf{R}^n$. If $E \subset S(r)$ let $\tilde{\partial} E$ denote the boundary of E relative to $S(r)$. Let H^m denote m dimensional Hausdorff measure in $\mathbf{R}^n$.

If f is defined on a set $E \subset \mathbf{R}^n$ let $\theta(r)$ be defined by

$$H^{n-1}(C(\theta(r))) = H^{n-1}(p(S(r) \cap E))$$

where p denotes the radial projection of $\mathbf{R}^n - \{0\}$ onto S. For $0 \leq \theta \leq \theta(r)$ let

$$\hat{f}(r, \theta) = \sup \int_F f(ry) dH^{n-1} y$$

where the supremum is taken over all measurable sets $F \subset p(S(r) \cap E)$ with $H^{n-1}(F) = H^{n-1}(C(\theta))$.

Let Ω be a bounded region in $\mathbf{R}^n$ of the form

$$\Omega = \bigcup_{r_1 < r < r_2} C(\theta(r))$$

where $0 \leq r_1 < r_2 < \infty$ and $0 < \theta(r) \leq \pi$ for $r_1 < r < r_2$. Let h be a symmetric, bounded, harmonic function in Ω such that, for $r_1 < r < r_2$, $h(r, \theta)$ is a non increasing function of θ for $0 < \theta < \theta(r)$. Then

$$\hat{h}(r, \theta) = \int_{C(\theta)} h(ry)dH^{n-1}y \quad \text{in} \quad \Omega.$$

Let u be a subharmonic function ($\not\equiv -\infty$) in $B(R) \supset \Omega, R > r_2$. In § 3 we will prove

THEOREM 1. *If $\hat{h}$ has a continuous extension to $\bar{\Omega} - \{0\}$ and $\hat{u} \leq \hat{h} + c$ on $\partial\Omega - \{0\}$ where $c \geq 0$, then $\hat{u} \leq \hat{h} + c$ everywhere in Ω.*

We note that Baernstein [2, Theorem A'] has obtained a similar theorem in $\mathbf{R}^2$.

We will give two applications of Theorem 1. The first is to an extremal problem for potentials. Given a real number γ, $1 \leq \gamma < \infty$, let $H(\gamma)$ denote the class of potentials

$$p(x) = \int_S |x - y|^{2-n} d\mu(y), \quad x \in \mathbf{R}^n$$

where μ is a probability measure on S and

$$p(x) \leq \gamma \quad \text{whenever} \quad x \in \mathbf{R}^n.$$

Choose α so that the Newtonian capacity of $S - C(\alpha)$ is γ^{-1} and let $P \in H(\gamma)$ denote the corresponding equilibrium potential. In § 4 we prove

THEOREM 2. *If Φ is a nondecreasing convex function on $(-\infty, \infty)$, then*

$$\int_S \Phi(p(ry))dH^{n-1}y \leq \int_S \Phi(P(ry))dH^{n-1}y$$

whenever $r > 0$ and $p \in H(\gamma)$.

Thus, if $\lambda \geq 1$, $\Phi(u) = u^\lambda$ for $u \geq 0$, and $\Phi(u) = 0$ for $u < 0$, we have

$$\int_S (p(ry))^\lambda dH^{n-1}y \leq \int_S (P(ry))^\lambda dH^{n-1}y$$

whenever $r > 0$ and $p \in H(\gamma)$. It follows that

$$\max\{p(x): x \in S(r)\} \leq \max\{P(x): x \in S(r)\}$$

whenever $r > 0$ and $p \in H(\gamma)$.

We note that the above inequality has been obtained by Davis and Lewis [6].

If u is a subharmonic function in $\mathbf{R}^n$, let $M(r, u) = \max\{u(x): x \in S(r)\}$ whenever $r > 0$ and $M(0, u) = u(0)$. As a second application of Theorem 1 we prove in § 5.

THEOREM 3. *Given $0 \leq \mu < 1$ and $0 < \beta < 1$, there exists $\varrho = \varrho(\mu, \beta, n) > 0$ such that if u is any subharmonic function in R^n with*

$$H^{n-1}(\}x\colon u(x) > \mu M(|x|, u)\} \cap S(r)) \leq \beta H^{n-1}(S(r))$$

whenever $r > 0$, *then either* $u \leq 0$ *everywhere in* $\mathbf{R}^n$ *or* $\lim_{r\to\infty} r^{-\varrho} M(r, u)$ *exists and is positive (possibly* $+\infty$).

For $0 < \beta < 1$ and $\mu = 0$, Dahlberg [4], Hüber [14], and Talpur [15] have all shown the existence of $\varrho^* = \varrho^*(\beta, n) > 0$ for which the conclusion above holds. In § 6 we will show the ϱ we obtain is best possible for $0 \leq \mu < 1$ and $0 < \beta < 1$.

Baernstein [1] has obtained a similar result in $\mathbf{R}^2$.

To prove Theorem 3 for $0 < \mu < 1$ we use Theorem 1 to reduce the problem to one considered by Dahlberg [5] and Essen and Lewis [7]. For $\mu = 0$ we use Theorem 1 and arguments similar to those of Heins [12, p. 114, ex. 11].

2. Spherical symmetrization

Given a closed set $F \subset \mathbf{R}^n$, define the spherical symmetrization F^* of F as follows: If $F \cap S(r) = \phi$, then $F^* \cap' S(r) = \phi$. Otherwise $H^{n-1}(F^* \cap S(r)) = H^{n-1}(F \cap S(r))$ and $F^* \cap S(r)$ is either the point $(r, 0, \ldots, 0)$ or the closed cap on $S(r)$ centered at $(r, 0, \ldots, 0)$. Let u be subharmonic in $B(R)$, $R > 0$. Given t, $-\infty \leq t < \infty$, let $F(t) = \{x\colon u(x) \geq t\}$ and note that $F(t)$ is closed. Define an associated function u^* by letting

$$u^*(x) = \sup\{t\colon x \in F^*(t)\} \quad \text{whenever} \quad x \in B(R).$$

It is easily seen that u^* is symmetric and $\{x\colon u^*(x) \geq t\} = F^*(t)$. It follows that u^* is upper semicontinuous, u and u^* are equimeasurable, and

$$\hat{u}(r, \theta) = \int_{C(\theta)} u^*(ry)\, dH^{n-1}y \tag{2.1}$$

whenever $0 < r < R$, $0 \leq \theta \leq \pi$. We note for later reference that Gehring [10, lemma 4] has shown that u^* is Lipschitz in $B(R)$ whenever u is.

Consider now the restriction of u and u^* (also denoted by u and u^*) to $S(r)$ for fixed r, $0 < r < R$. Assume that u and u^* are Lipschitz functions on $S(r)$. Define a Borel measure $u_{\#}H^{n-1}$ on $\mathbf{R}$ by letting

$$u_{\#}H^{n-1}(E) = H^{n-1}(u^{-1}(E))$$

whenever E is a Borel subset of $\mathbf{R}$. Define $u^*_{\#}H^{n-1}$ analogously.

Let $\tilde{\nabla}$ denote the gradient relative to the sphere $S(r)$, and let G be the subset of $S(r)$ where $\tilde{\nabla} u^*$ exists. Define a function g on $\mathbf{R}$ by letting

$$g(t) = 0 \quad \text{if} \quad (u^*)^{-1}(t) \cap G = \phi$$

and

$$g(t) = |\tilde{\nabla} u^*(x)| \quad \text{for any} \quad x \in (u^*)^{-1}(t) \cap G, \quad \text{otherwise}.$$

Since u^* is symmetric, g is well defined. Note that $g \circ u^*(x) = |\tilde{\nabla} u^*(x)|$ for H^{n-1} almost every $x \in S(r)$. Thus by [8, 2.4.18 (1)].

$$\int_{A^*(t_1, t_2)} |\tilde{\nabla} u^*|^2 dH^{n-1} = \int_{t_1}^{t_2} g^2 du^*_{\#} H^{n-1},$$

where $A^*(t_1, t_2) = \{x: t_1 < u^*(x) < t_2\}$.

Since $u_{\#} H^{n-1} = u^*_{\#} H^{n-1}$ we see by [8, 2.4.18 (2)] that $g \circ u$ is H^{n-1} measurable and

$$\int_{t_1}^{t_2} g^2 du_{\#} H^{n-1} = \int_{A(t_1, t_2)} (g \circ u)^2 dH^{n-1}$$

where $A(t_1, t_2) = \{x: t_1 < u(x) < t_2\}$. Hence

$$\int_{A(t_1, t_2)} (g \circ u)^2 dH^{n-1} = \int_{A^*(t_1, t_2)} |\tilde{\nabla} u^*|^2 dH^{n-1}.$$

Using the coarea formula [8, 3.2.22 (3)] and the spherical isoperimetric inequality for sets of finite perimeter (see [8, 3.243 and 4.5.9 (31)] for a similar inequality in the Euclidean case), we obtain

$$\int_{A^*(t_1, t_2)} |\tilde{\nabla} u^*|^2 dH^{n-1} = \int_{t_1}^{t_2} \left(\int_{(u^*)^{-1}(t)} g \circ u^* dH^{n-2} \right) dt$$
$$\leq \int_{t_1}^{t_2} \left(\int_{u^{-1}(t)} g \circ u dH^{n-2} \right) dt = \int_{A(t_1, t_2)} (g \circ u) |\tilde{\nabla} u| dH^{n-1}.$$

From Holder's inequality, it follows that

$$\int_{A(t_1, t_2)} (g \circ u) |\tilde{\nabla} u| dH^{n-1} \leq$$
$$\leq \left[\int_{A(t_1, t_2)} (g \circ u)^2 dH^{n-1} \right]^{1/2} \left[\int_{A(t_1, t_2)} |\tilde{\nabla} u|^2 dH^{n-1} \right]^{1/2}$$
$$= \left[\int_{A^*(t_1, t_2)} |\tilde{\nabla} u^*|^2 dH^{n-1} \right]^{1/2} \left[\int_{A(t_1, t_2)} |\tilde{\nabla} u|^2 dH^{n-1} \right]^{1/2}.$$

Thus

$$\int_{A^*(t_1, t_2)} |\tilde{\nabla} u^*|^2 dH^{n-1} \leq \int_{A(t_1, t_2)} |\tilde{\nabla} u|^2 dH^{n-1}.$$

Applying the coarea formula again we obtain

$$\int_{t_1}^{t_2} \left(\int_{(u^*)^{-1}(t)} |\tilde{\nabla} u^*| dH^{n-2} \right) dt \leq \int_{t_1}^{t_2} \left(\int_{u^{-1}(t)} |\tilde{\nabla} u| dH^{n-2} \right) dt$$

whenever $t_1 < t_2$. Hence for almost every t (with respect to one dimensional Lebesque measure)

$$\int_{(u^*)^{-1}(t)} |\tilde{\nabla} u^*| dH^{u-2} \leq \int_{u^{-1}(t)} |\tilde{\nabla} u| dH^{n-2} \tag{2.2}$$

The coarea formula also implies that

$$H^{n-2}[u^{-1}(t) - \tilde{\partial}\{x\colon u(x) > t\}] = 0$$

for almost every t. Thus, for almost every t, we can replace $u^{-1}(t)$ by $\tilde{\partial}\{x\colon u(x) > t\}$ in (2.2).

The argument above was suggested by [10, (27)].

3. Proof of Theorem 1

The proof is by contradiction. Suppose there is an $x_0 \in \Omega$ such that $\hat{u}(x_0) > h(x_0) + c$. Let $w(x) = h(x) + \eta|x|^{2-n} + \eta x_1$, where $\eta > 0$ is so small that $\hat{u}(x_0) - \hat{w}(x_0) = c_1 > c$. Clearly w is symmetric, harmonic in Ω, and $\partial w/\partial\theta < 0$ at each point of Ω off the x_1 axis. Also, $\hat{u} \leq \hat{w} + c$, on $\partial\Omega - \{0\}$.

There exists a decreasing sequence $\{u_j\}$ of subharmonic functions in $B(1/2(r_2 + R))$ with continuous second partial derivatives that converges pointwise to u in $B(1/2(r_2 + R))$. Since u_j^* is Lipschitz in $\bar{B}(r_2)$, it follows from (2.1) that $\hat{u}_j$ is continuous in $\bar{B}(r_2) - \{0\}$. Since

$$0 \leq \hat{u}_j(r, \theta) - \hat{u}(r, \theta) \leq \hat{u}_j(r, \pi) - \hat{u}(r, \pi),$$

and $\hat{u}_j(r, \pi), \hat{u}(r, \pi)$ are continuous functions of r on $[\sigma, 1/2(r_2 + R)]$ for $0 < \sigma < 1/2(r_2 + R)$, it follows from Dini's Theorem that $\{\hat{u}_j\}$ converges uniformly to $\hat{u}$ in the closure of $B(r_2) - B(\sigma)$ whenever $0 < \sigma < r_2$. Thus $\hat{u}$ is continuous on $\bar{B}(r_2) - \{0\}$. Choose $\sigma > 0$ so small that $\hat{u} - \hat{w} < c_1$ on the closure of $B(\sigma) \cap \Omega$. Then there exist m and $\varepsilon > 0$ such that

$$\hat{u}_m(x) + \varepsilon H^{n-1}(S)|x|^2 - \hat{w}(x) < c_1$$

whenever $x \in \partial[\Omega - B(\sigma)]$.

Let $v(x) = u_m(x) + \varepsilon|x|^2$ for $x \in \Omega - B(\sigma)$ and note that

$$\hat{v}(r, \theta) - \hat{w}(r, \theta) = \int_{C(\theta)} v^*(ry) dH^{n-1}y - \int_{C(\theta)} w(ry) dH^{n-1}y$$

has a relative maximum at a point in $\Omega - \overline{B(\sigma)}$ with coordinates (r_0, θ_0), $0 < \theta_0 < \pi$. Note also that

$$\triangle v \geq 2n\varepsilon. \tag{3.1}$$

Since v^* and w are continuous in Ω, it follows that $v^*(r_0, \theta_0) = w(r_0, \theta_0)$ and for $\theta - \theta_0 > 0$ and sufficiently small,

$$\int_{C(\theta)-C(\theta_0)} v^*(r_0 y) dH^{n-1}y \leq \int_{C(\theta)-C(\theta_0)} w(r_0 y) dH^{n-1}y.$$

Since v^* is Lipschitz, and $v^*(r_0, \theta)$ and $w(r_0, \theta)$ are nonincreasing and decreasing functions of θ respectively, it follows that

$$|\tilde{\nabla} v^*(r_0, \theta)| \geq |\tilde{\nabla} w(r_0, \theta)| > 0 \tag{3.2}$$

for all θ in a set F with the property: Given any $\tau > 0$, the one dimensional Lebesgue measure of $F \cap [\theta_0, \theta_0, + \tau]$ is positive.

For $\theta - \theta_0$ small and positive, let $E(\theta) \subset S$ be such that

(i) $S \cap \{y: v(r_0 y) > v^*(r_0, \theta)\} \subset E(\theta) \subset S \cap \{y: v(r_0 y) \geq v^*(r_0, \theta)\}$,

(ii) $H^{n-1}(E(\theta)) = H^{n-1}(C(\theta))$,

(iii) $\hat{v}(r_0, \theta) = \int_{E(\theta)} v(r_0 y) dH^{n-1}y = \int_{C(\theta)} v^*(r_0 y) dH^{n-1}y.$

Note that all three sets in (i) have the same H^{n-1} measure whenever $\theta \in F$, and that $E(\theta_0) \subset E(\theta)$ whenever $\theta_0 < \theta \in F$.

Let

$$\psi(r) = \int_{E(\theta_0)} v(ry) dH^{n-1}y - \int_{C(\theta_0)} w(ry) dH^{n-1}y$$

and observe that

$$\psi(r) \leq \hat{v}(r, \theta_0) - \hat{w}(r, \theta_0) \leq \psi(r_0),$$

for r sufficiently close to r_0. Thus ψ has a relative maximum at r_0 and

$$\frac{d}{dr}\left(r^{n-1} \frac{d\psi}{dr}\right)_{r=r_0} \leq 0.$$

Consequently given any $\gamma > 0$, we have

$$\int_{E(\theta)} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial v}{\partial r}\right)(r_0 y) dH^{n-1}y \leq \int_{C(\theta)} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial w}{\partial r}\right)(r_0 y) dH^{n-1}y + \gamma \tag{3.3}$$

whenever $\theta - \theta_0 > 0$ and sufficiently small.

For $\lambda > 0$ let

$$L(\theta, \lambda) = \{sy: r_0 \leq s \leq r_0 + \lambda, y \in E(\theta)\}$$

and $L(\theta) = L(\theta, 0)$. Since $\{v^*(r_0, \theta): \theta \in F \cap [\theta_0, \theta_0 + \tau]\}$ has positive one dimensional measure, whenever $\tau > 0$ there is an $F' \subset F$ containing θ arbitrarily near θ_0 and such that (2.2) holds with $v = u, t = v^*(r_0, \theta)$, and $\tilde{\partial} L(\theta)$ replacing $u^{-1}(t)$ whenever $\theta \in F'$. By [8, 3.2.22 (2)] we can assume that $\tilde{\partial} L(\theta)$ is

$(H^{n-2}, n-2)$ rectifiable whenever $\theta \in F'$ and hence that $\partial L(\theta, \lambda)$ is $(H^{n-1}, n-1)$ rectifiable whenever $\theta \in F'$.

Now, from (3.1),

$$2n\varepsilon\lambda^{-1}H^n(L(\theta, \lambda)) \leq \lambda^{-1} \int_{L(\theta,\lambda)} \triangle v dH^n .$$

Using the Gauss-Green theorem [8, 4.5.6 (5)] and letting $\lambda \to 0$ we obtain for $\theta \in F'$,

$$2n\varepsilon H^{n-1}(L(\theta)) \leq r_0^{1-n} \int_{L(\theta)} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial v}{\partial r}\right)_{r=r_0} dH^{n-1} - \int_{\tilde{\partial} L(\theta)} |\tilde{\nabla} v| dH^{n-2} .$$

Since w is harmonic a similar argument gives

$$r_0^{1-n} \int_{C_1(\theta)} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial w}{\partial r}\right)_{r=r_0} dH^{n-1} = \int_{\tilde{\partial} C_1(\theta)} |\tilde{\nabla} w| dH^{n-2}$$

where $C_1(\theta) = \{r_0 y \colon y \in C(\theta)\}$.

Using (3.2), (2.2), (3.3), and the above inequalities we obtain for $\theta \in F'$,

$$\begin{aligned}\int_{\tilde{\partial} C_1(\theta)} |\tilde{\nabla} w| dH^{n-2} &\leq \int_{\tilde{\partial} C_1(\theta)} |\tilde{\nabla} v^*| dH^{n-2} \leq \int_{\partial L(\theta)} |\tilde{\nabla} v| dH^{n-2} \\ &\leq r_0^{1-n} \int_{L(\theta)} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial v}{\partial r}\right)_{r=r_0} dH^{n-1} - 2n\varepsilon H^{n-1}(L(\theta)) \\ &\leq r_0^{1-n} \int_{C_1(\theta)} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial w}{\partial r}\right)_{r=r_0} dH^{n-1} - 2n\varepsilon H^{n-1}(L(\theta)) + \gamma \\ &= \int_{\tilde{\partial} C_1(\theta)} |\tilde{\nabla} w| dH^{n-2} - 2n\varepsilon H^{n-1}(L(\theta)) + \gamma .\end{aligned}$$

Thus $2n\varepsilon H^{n-1}(L(\theta) \leq \gamma$ whenever $\theta \in F'$ and hence $2n\varepsilon r_0^{n-1} H^{n-1}(C(\theta_0)) \leq \gamma$. Since γ is arbitrary and $\theta_0 > 0$, we have reached a contradiction. Hence Theorem 1 is true.

4. Proof of Theorem 2

Let $\gamma, H(\gamma)$, and $P \in H(\gamma)$ be as in § 1. If $\gamma = 1$, then the conclusion of Theorem 2 is obvious since P is the only member of $H(1)$. Thus we assume that $1 < \gamma < \infty$. Then $0 < \alpha < \pi$ and $h = -P$ is subharmonic in $\mathbf{R}^n$, harmonic in $\mathbf{R}^n - [S - C(\alpha)]$, and $h = -\gamma$ on $S - C(\alpha)$. It is readily seen that h is symmetric and that $h(r, \theta)$ is a nonincreasing function of θ for $0 < \theta < \pi$ and fixed $r > 0$. From the proof of Theorem 1 we see that $\hat{h}$ is continuous in $\mathbf{R}^n - \{0\}$.

Now suppose $p \in H(\gamma)$ and $u = -p$. Clearly u is subharmonic in $\mathbf{R}^n$. Given $\varepsilon > 0$ choose R large enough that $\hat{u} < \hat{h} + \varepsilon$ on $S(R)$.

Let $\Omega \subset B(R)$ denote the bounded symmetric region in $\mathbf{R}^n$ such that $B(R) - \Omega$ consists of the union of $S - C(\alpha)$ and the line segment from the origin to $(-R, 0, \ldots, 0)$. One verifies that $\hat{u}(r, \pi) = \hat{h}(r, \pi)$ for $0 < r < \infty$. Since $u^* \geq h = -\gamma$ on $S - C(\alpha)$, it follows that $\hat{u} \leq \hat{h}$ on $S - C(\alpha)$. Thus $\hat{u} \leq \hat{h} + \varepsilon$ on $\partial\Omega - \{0\}$. By Theorem 1, $\hat{u} \leq \hat{h} + \varepsilon$ in Ω. It follows that $\hat{u} \leq \hat{h}$ in $\mathbf{R}^n - \{0\}$. Note that

$$\hat{u}(r, \theta) = (-\hat{p})(r, \theta) = \hat{p}(r, \pi - \theta) - \hat{p}(r, \pi)$$

with a similar relation holding between $\hat{h}$ and $\hat{P}$. Thus since $\hat{p}(r, \pi) = \hat{P}(r, \pi)$ for $0 < r < \infty$, we have $\hat{p} \leq \hat{P}$ in $\mathbf{R}^n - \{0\}$. It is known [11, p. 170, 249—250] that this inequality implies the conclusion of Theorem 2.

5. Proof of Theorem 3

It sufficies to assume that $u \geq 0$ (otherwise consider max $\{u, 0\}$) and that $u \not\equiv 0$. Let $\alpha, 0 < \alpha < \pi$, be such that $H^{n-1}(C(\alpha)) = \beta H^{n-1}(S)$ and let

$$p(x) = \max\{\mu M(|x|, u), u^*(x)\}$$

whenever $x \in \mathbf{R}^n - \{0\}$. We observe from the hypotheses of Theorem 3 that $p(r, \theta) = \mu M(r, u)$ if $\theta > \alpha$. For $0 < \sigma < \pi$ let

$$K(\sigma) = \{ty\colon 0 < t < \infty \;\; y \in C(\sigma)\}$$

and let $K(\sigma, R) = B(R) \cap K(\sigma)$. Assume henceforth that $M(R, u) > 0$. Note that for $\sigma > \alpha, p$ is upper semicontinuous on $\partial K(\sigma, R) - \{0\}$, and continuous except on a polar set. Thus there is a unique bounded harmonic function h_σ in $K(\sigma, R)$ such that

$$\limsup_{x \to y} h_\sigma(x) \leq p(y) \quad \text{whenever} \quad y \in \partial K(\sigma, R) - \{0\},$$

and $\lim_{x \to y} h_\sigma(x) = p(y)$ except on a polar set in $\partial K(\sigma, R)$ [13, Lemma 8.20]. Since $\mu M(|x|, u)$ is subharmonic in $\mathbf{R}^n (M(0, u) = u(0))$, it follows that $\mu M(|x|, u) \leq h_\sigma(x)$ in $K(\sigma, R)$. From the boundary values of h_σ we see that h_σ is symmetric in $K(\sigma, R)$.

Let

$$q_\sigma(r, \theta) = \sup\{h_\sigma(r, \theta_1)\colon \theta \leq \theta_1 < \sigma\} \quad \text{in} \quad K(\sigma, R).$$

Then q_σ is symmetric and has the same boundary values as h_σ. Using the fact that $q_\sigma(r, \theta) = h_\sigma(r, \theta_1)$ for some $\theta_1, \theta \leq \theta_1 < \sigma$, it is easily checked that q_σ is upper semicontinuous and satisfies a local sub mean-value property in $K(\sigma, R)$.

Thus q_σ is subharmonic in $K(\sigma, R)$ and since it is obvious that $h_\sigma \leq q_\sigma$, it follows that $h_\sigma = q_\sigma$ in $K(\sigma, R)$. Hence $h_\sigma(r, \theta)$ is nonincreasing for $0 < \theta < \sigma$ and fixed r, $0 < r < R$. The proof of this fact is due to Matts Essén (oral communication).

Fix $\sigma > \alpha$ and let $v(x) = h_\sigma(x) + \varepsilon|x|^{2-n}$ for $x \in K(\sigma, R)$ and $\varepsilon > 0$. Observe that v has a continuous extension to $\overline{K(\sigma, R)} - \{0\}$ and that $\hat{u} \leq \hat{v}$ on $S(R) \cap \bar{K}(\sigma)$. Thus, if

$$\sup \{\hat{u}(y) - \hat{v}(y) \colon y \in \partial K(\sigma, R) - \{0\}\} = c > 0,$$

then $\hat{u}(r, \sigma) - v(r, \sigma) = c$ for some r with $0 < r < R$. However since $u^*(r, \theta) \leq \mu M(r, u) < v(r, \theta)$ whenever $\alpha < \theta < \sigma$, it follows that

$$\hat{u}(r, \alpha) - \hat{v}(r, \alpha) > \hat{u}(r, \sigma) - \hat{v}(r, \sigma) = c > 0,$$

which contradicts Theorem 1. Hence $c \leq 0$. Applying Theorem 1 and letting $\varepsilon \to 0$ we have $\hat{u} \leq \hat{h}_\sigma$ in $K(\sigma, R)$ whenever $\sigma > \alpha$.

Let $h_\sigma(x) = \mu M(|x|, u)$ for $x \in B(R) - K(\sigma, R)$. Then h_σ is subharmonic in $B(R)$ and if $\alpha < \sigma_1 < \sigma_2$, then $h_{\sigma_1} \leq h_{\sigma_2}$ in $B(R)$. Thus $h = \lim_{\sigma \to \alpha+} h_\sigma$ is subharmonic in $B(R)$ and harmonic in $K(\alpha, R)$. Clearly $h(x) = \mu M|x|, u)$ in $B(R) - \overline{K(\alpha, R)}$. Since $B(R) - \overline{K(\alpha, R)}$ is not thin at any $x \in \partial K(\alpha) \cap B(R)$ [13, Corollary 10.5], it follows that $h(x) = \mu M(|x|, u)$ on $\partial K(\alpha) \cap B(R)$. Since $\hat{u} \leq \hat{h}_\sigma$ in $K(\alpha, R)$ whenever $\sigma > \alpha$, we have $\hat{u} \leq \hat{h}$ in $K(\alpha, R)$.

Let

$$h = P_R + Q_R \tag{5.1}$$

where P_R and Q_G are bounded harmonic functions in $K(\alpha, R)$ with

$$\lim_{x \to y} P_R(x) = \mu M(|y|, u) \quad \text{whenever} \quad y \in \partial K(\alpha) \cap B(R),$$

$$\lim_{x \to y} P_R(x) = 0 \quad \text{whenever} \quad y \in K(\alpha) \cap S(R),$$

and $Q_R = h - P_R$. Note that

$$\lim_{x \to y} Q_R(x) = 0 \quad \text{for} \quad y \in \partial K(\alpha) \cap B(R),$$

$$\lim_{x \to y} Q_R(x) = p(y) \quad \text{for} \quad y \in K(\alpha) \cap S(R),$$

off of a polar set.

Let $0 < \gamma_1 < \gamma_2 < \ldots$ be the eigenvalues of the boundary value problem

$$\delta\phi + \gamma\phi = 0 \quad \text{on} \quad C(\alpha),$$

$$\phi = 0 \quad \text{on} \quad \tilde{\partial} C(\alpha)$$

where δ is the Beltrami operator defined in terms of the Laplacian Δ by

$$\Delta = r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + r^{-2}\delta.$$

Let $\{\phi_k\}$ denote corresponding symmetric eigenfunctions with continuous second partial derivatives in $C(\alpha)$ and

$$\int_{C(\alpha)} \phi_k^2 dH^{n-1} = 1 \quad \text{for} \quad k = 1, 2, \ldots$$

Let ϱ_k be the positive root of the equation $\varrho_k(\varrho_k + n - 2) = \gamma_k$ for $k = 1, 2, \ldots$ Then as in [9] we have

$$Q_R(r, \theta) = \sum_{k=1}^{\infty} a_k (r/R)^{\varrho_k} \phi_k(1, \theta) \quad \text{in} \quad K(\alpha, R), \tag{5.2}$$

where

$$a_k = \int_{C(\alpha)} P(Ry)\phi_k(y) dH^{n-1} y.$$

Using the estimates in [7, § 8] or [4, Lemma 2.5], the series

$$\sum_{k=1}^{\infty} (r/R)^{\varrho_k - \varrho_1} |\phi_k(1, \theta)|$$

can be seen to converge uniformly in $K(\alpha, sR)$ whenever $0 < s < 1$. Note also that

$$|a_k| \leq M(R, u) H^{n-1}(C(\alpha))^{1/2}.$$

The case $\mu = 0$. In case $\mu = 0$ we have $P_R = 0$, $Q_R = h$, $p = u^*$, and hence

$$a_k = \int_{C(\alpha)} u^*(Ry)\phi_k(y) dH^{n-1} y. \tag{5.3}$$

It is known [3, VI § 6] that ϕ_1 is either positive or negative in $K(\alpha, R)$. Assume $\phi_1 \geq 0$. Since ϕ_1 is symmetric and $\delta\phi_1 = -\gamma_1\phi_1$, it is readily seen that $d\phi_1/d\theta \leq 0$ in $C(\alpha)$. Using this and the fact that $\hat{u} \leq h$ in $K(\alpha, R)$, we have

$$\begin{aligned} m(r) = \int_{C(\alpha)} u^*(ry)\phi_1(y) dH^{n-1} y &= -\int_0^{\alpha} \hat{u}(r, \theta) \frac{d\phi_1}{d\theta}(1, \theta) d\theta \\ &\leq -\int_0^{\alpha} \hat{h}(r, \theta) \frac{d\phi_1}{d\theta}(1, \theta) d\theta = \int_{C(\alpha)} h(ry)\phi_1(y) dH^{n-1} y. \end{aligned}$$

From (5.2) and (5.3)

$$\int_{C(\alpha)} h(ry)\phi_1(y) dH^{n-1} y = a_1 (r/R)^{\varrho_1} = (r/R)^{\varrho_1} \int_{C(\alpha)} u^*(Ry)\phi_1(y) dH^{n-1} y,$$

and hence

$$r^{-\varrho_1}m(r) \leq R^{-\varrho_1}m(R) \quad \text{for} \quad 0 < r < R.$$

Consequently $b = \lim_{r\to\infty} r^{-\varrho_1}m(r)$ exists. We assume that

$$\liminf_{r\to\infty} (r^{-\varrho_1}M(r, u)) < \infty.$$

Otherwise the proof is complete in case $\mu = 0$ with $\varrho = \varrho_1$. Since

$$m(r) \leq M(r, u)H^{n-1}(C(\alpha))^{1/2}$$

we have $b < \infty$.

Now from (5.2) we deduce that for $0 < r < R/2$,

$$r^{-\varrho_1}\hat{u}(r, \theta) \leq r^{-\varrho_1}\hat{h}(r, \theta)$$

$$= R^{-\varrho_1}a_1 \int_{C(\theta)} \phi_1 dH^{n-1} + R^{-\varrho_1}\sum_{k=2}^{\infty} a_k(r/R)^{\varrho k-\varrho_1} \int_{C(\theta)} \phi_k dH^{n-1}$$

$$\leq R^{-\varrho_1}m(R) \int_{C(\theta)} \phi_1 dH^{n-1} + R^{-\varrho_1}M(R, u)g(r/R)$$

where g is continuous on $[0, \frac{1}{2}]$ and $g(0) = 0$. Since $\liminf_{R\to\infty} R^{-\varrho_1}M(R, u) < \infty$, it follows that

$$r^{-\varrho_1}\hat{u}(r, \theta) \leq b \int_{C(\theta)} \phi_1 dH^{n-1} \quad \text{in} \quad K(\alpha).$$

This inequality and the subharmonicity of u imply that

$$r^{-\varrho_1}M(r, u) \leq b\phi_1(1, 0) \quad \text{for} \quad r > 0,$$

and hence that $b > 0$.

Suppose that

$$\liminf_{r\to\infty} r^{-\varrho_1}M(r, u) < b\phi_1(1, 0).$$

Then there exists a sequence $\{r_j\}$ with $r_j \uparrow \infty$ and $\varepsilon > 0$ such that

$$r_j^{-\varrho_1}M(r_j, u) < b\phi_1(1, \theta)$$

For $j = 1, 2, \ldots$ and $0 < \theta < \varepsilon$. Thus,

$$r_j^{-\varrho_1}\hat{u}(r_j, \theta) < b \int_{C(\theta)} \phi_1 dH^{n-1}$$

for $0 < \theta < \varepsilon$ and it follows that

$$r_j^{-\varrho_1}m(r_j) = - \int_0^{\alpha} r_j^{-\varrho_1}\hat{u}(r_j, \theta) \frac{d\phi_1}{d\theta} (1, \theta)d\theta$$

$$< - b \int_0^{\alpha}\left(\int_{C(\theta)} \phi_1 dH^{n-1}\right) \frac{d\phi_1}{d\theta} (1, \theta)d\theta = b.$$

Letting $j \uparrow \infty$ we obtain a contradiction. Hence

$$\lim_{r \to \infty} r^{-\varrho_1} M(r, u) = b\phi_1(1, 0) > 0$$

and the proof is complete in case $\mu = 0$ with $\varrho = \varrho_1$.

The case $0 < \mu < 1$. For $0 < \lambda < 1$ the boundary value problem

$$\begin{aligned} \delta\psi + \lambda\varrho_1(\lambda\varrho_1 + n - 2)\psi &= 0 \quad \text{on} \quad C(\alpha) \\ \psi &= 1 \quad \text{on} \quad \tilde{\partial} C(\alpha) \end{aligned}$$

has a unique symmetric solution. Choose λ so that the corresponding ψ has the value μ^{-1} at $r = 1, \theta = 0$.

Since $\hat{u} \leq \tilde{h}$ in $K(\alpha, R)$ it follows that $M(r, u) \leq h(r, 0)$ for $0 < r < R$ and hence that

$$h(y) = \mu M(|y|, u) \leq \mu h(|y|, 0)$$

for $y \in \partial K(\alpha) \cap B(R)$. Thus, using the arguments of [7, (3.1)],

$$r^{-\lambda\varrho_1} M(r, u) \leq r^{-\lambda\varrho_1} h(r, 0) \leq \mu^{-1} R^{-\lambda\varrho_1} M(r, u) \tag{5.4}$$

for $0 < r < R$. It follows that

$$0 < \limsup_{r \to \infty} r^{-\lambda\varrho_1} M(r, u) \leq \mu^{-1} \liminf_{r \to \infty} r^{-\lambda\varrho_1} M(r, u).$$

Assume that $\limsup_{r \to \infty} r^{-\lambda\varrho_1} M(r, u) < \infty$. Otherwise the proof is complete in case $0 < \mu < 1$ with $\varrho = \lambda\varrho_1$.

For P_R as in (5.1) we note that $P_{R_1} \leq P_{R_2}$ in $K(\alpha, R_1)$ whenever $R_1 \leq R_2$. Also, from (5.4), we have

$$M(r, P_R) \leq h(r, 0) \leq \mu^{-1}(r/R)^{-\lambda\varrho_1} M(R, u)$$

for $0 < r < R$. Since $\liminf_{R \to \infty} R^{-\lambda\varrho_1} M(R, u) < \infty$, it follows that $V = \lim_{R \to \infty} P_R$ is harmonic in $K(\alpha)$ and

$$M(r, V) \leq \mu^{-1} r^{\lambda\varrho_\cdot} \liminf_{R \to \infty} R^{-\lambda\varrho_1} M(R, u). \tag{5.5}$$

From (5.4) and the definition of Q_R we have

$$P_{R_2} - P_{R_1} \leq \mu^{-2}(R_1/R_2)^{\lambda\varrho_1} \frac{M(R_2 u)}{M(R_1, u)} Q_{R_1}$$

in $K(\alpha, R_1)$ whenever $R_1 < R_2$. Letting $R_2 \to \infty$ it follows that

$$0 \leq V - P_{R_1} \leq (\text{constant})\, Q_{R_1} \quad \text{in} \quad K(\alpha, R_1).$$

Thus

$$V(y) = \lim_{x \to y} V(x) = \mu M(|y|, u) \quad \text{on} \quad \partial K(\alpha). \tag{5.6}$$

From (5.2) we have

$$Q_R(r, \theta) \leq A(r/R)^{\varrho_1} M(R, u) \quad \text{for} \quad 0 < r < \frac{R}{2}$$

where A is a positive constant independent of R. Since

$$\limsup_{R \to \infty} R^{-\lambda \varrho_1} M(R, u) < \infty,$$

it follows that $Q_R \to 0$ uniformly on compact subsets of $K(\alpha)$ as $R \to \infty$. Using (5.4), (5.1) and letting $R \to \infty$ we deduce that $M(r, u) \leq V(r, 0)$ for $r > 0$.

This last inequality, (5.5), and (5.6) imply that $\lim_{r\to\infty} r^{-\lambda\varrho_1} M(r, u)$ exists [7, (4.6)] and hence the proof is complete in case $0 < \mu < 1$ with $\varrho = \lambda \varrho_1$.

6. Remark

With ϱ_1 and ϕ_1 as in the proof of the case $\mu = 0$, let

$$u(r, \theta) = r^{\varrho_1} \phi_1(1, \theta) \quad \text{in} \quad K(\alpha)$$

and

$$u(r, \theta) = 0 \quad \text{in} \quad \mathbf{R}^n - K(\alpha)$$

Then u is subharmonic in $\mathbf{R}^n$ and satisfies the hypothesis of Theorem 3. Hence $\varrho = \varrho_1$ is the best possible exponent in case $\mu = 0$.

In case $0 < \mu < 1$, let λ and ψ correspond to μ as in the proof of Theorem 3. It is known [7, (1.5)] that $\psi \geq 1$ in $C(\alpha)$. Let

$$u(r, \theta) = r^{\lambda \varrho_1} \psi(1, \theta) \quad \text{in} \quad K(\alpha)$$

and

$$u(r, \theta) = r^{\lambda \varrho_1} \quad \text{in} \quad \mathbf{R}^n - K(\alpha).$$

Then u is subharmonic in $\mathbf{R}^n$ and satisfies the hypotheses of Theorem 3. Thus the exponent $\varrho = \lambda \varrho_1$ is best possible when $0 < \mu < 1$.

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