Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A Meir-Keeler common type fixed point theorem on partial metric spaces

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Abstract

In this article, we prove a general common fixed point theorem for two pairs of weakly compatible self-mappings of a partial metric space satisfying a generalized Meir-Keeler type contractive condition. The presented theorem extends several well known results in literature.

1 Introduction

Partial metric spaces were introduced by Matthews [1] to study denotational semantics of dataflow networks. In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory. For example, in the research area of computer domains and semantics, partial metric spaces have serious applications potentials (see for example, [2-5]). In 1994, Matthews [1] generalized the Banach contraction principle to the class of complete partial metric spaces: a self mapping T on a complete partial metric space (X, p) has a unique fixed point if there exists $0 \leq k < 1$ such that $p(Tx, Ty) \leq kp(x, y)$ for all $x, y \in X$. After the remarkable contribution of Matthews, many authors have studied on partial metric spaces and its topological properties (see for example, [1-28]).

In the sequel we recall the notion of a partial metric space and some of its properties which will be useful later on. A partial metric is a function $p : X \times X \to [0, \infty)$ satisfying the following conditions

(P1) $p(x, y) = p(y, x)$,

(P2) If $p(x, x) = p(x, y) = p(y, y)$, then $x = y$,

(P3) $p(x, x) \leq p(x, y)$,

(P4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$,

for all $x, y, z \in X$. Then (X, p) is called a partial metric space. If p is a partial metric p on X, then the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X. Also, each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$.

Definition 1.1. [1,7]*Let* (X, p) *be a partial metric space.*

(i) A sequence $\{x_n\}$ *in X converges to* $x \in X$ *whenever* $\lim_{n\to\infty} p(x, x_n) = p(x, x)$,

(ii) A sequence $\{x_n\}$ *in X is called Cauchy whenever* $\lim_{n,m\to\infty} p(x_n, x_m)$ *exists (and finite),*

(iii) (X, p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in X converges, with respect to* τ_p, *to a point* $x \in X$, *that is,* $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

(iv) A mapping $f : X \to X$ *is said to be continuous at* $x_0 \in X$ *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 1.1. [1,7]*Let* (X, p) *be a partial metric space.*

(a) A sequence $\{x_n\}$ *is Cauchy if and only if* $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d_p),

(b) (X, p) *is complete if and only if the metric space* (X, d_p) *is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x). \quad (1)$$

Definition 1.2. [29]*Let X be a non empty set and* $f, g : X \to X$. *If* $w = fx = gx$, *for some* $x \in X$, *then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. If* $w = x$, *then x is a common fixed point of f and g.*

Definition 1.3. [29]*Let f and g be two self-maps defined on a non empty set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.*

Recently, Ćirić et al. [17] established a common fixed point result for two pairs of weakly compatible mappings satisfying generalized contractions on a partial metric space. For this, denote by Φ the set of non-decreasing continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ satisfying:

(a) $0 < \phi(t) < t$ for all $t > 0$,

(b) the series $\sum_{n\geq 1} \phi^n(t)$ converge for all $t > 0$.

The result [17] is the following.

Theorem 1.2. *Suppose that A, B, S, andT are self-maps of a complete partial metric space* (X, p) *such that* $AX \subseteq TX$, $BX \subseteq SX$ *and*

$$p(Ax, By) \leq \varphi(M(x, y)) \quad (2)$$

for all $x, y \in X$, *where* $\phi \in \Phi$ *and*

$$M(x, y) = \max\left\{p(Sx, Ty), p(Ax, Sx), p(By, Ty), \ \frac{1}{2}[p(Sx, By) + p(Ax, Ty)]\right\}.$$

If one of the ranges AX, BX, TX and SX is a closed subset of (X, p), *then*

(i) *A and S have a coincidence point,*

(ii) *B and T have a coincidence point.*

Moreover, if the pairs $\{A, S\}$ *and* $\{B, T\}$ *are weakly compatible, then A, B, T, and S have a unique common fixed point.*

In this manuscript, replacing (2) by some new weaker hypotheses we also establish a common fixed point result for four self maps satisfying a generalized Meir-Keeler type contraction on partial metric spaces. Our theorem generalizes several well known results in the literature.

2 Main results

The following lemmas will be frequently used in the proofs of the main results.

Lemma 2.1. [6,19]*Let* (X, p) *be a partial metric space. Then*

(a) If $p(x, y) = 0$, *then* $x = y$,

(b) If $x \neq y$, *then* $p(x, y) > 0$.

Lemma 2.2. [6,19]*Let* (X, p) *be a partial metric space and* $x_n \to z$ *with* $p(z, z) = 0$. *Then* $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ *for all* $y \in X$.

Now, we are ready to state and prove our main result.

Theorem 2.3. *Let A, B, S, and T be the self maps defined on a complete partial metric space* (X, p) *satisfying the following conditions:*

(C1) $AX \subseteq TX$ *and* $BX \subseteq SX$,

(C2) *for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow p(Ax, By) \leq \varepsilon, \tag{3}$$

where $M(x, y) = \max\{p(Sx, Ty), p(Ax, Sx), p(By, Ty), \frac{1}{2}[p(Sx, By) + p(Ax, Ty)]\}$,

(C3) *for all* $x, y \in X$ *with* $M(x, y) > 0 \Rightarrow p(Ax, By) < M(x, y)$,

(C4) $p(Ax, By) \leq \max\{a[p(Sx, Ty) + p(Ax, Sx) + p(By, Ty)], b[p(Sx, By) + p(Ax, Ty]\}$ *for all* $x, y \in X$, $0 \leq a < \frac{1}{2}$ *and* $0 \leq b < \frac{1}{2}$.

If one of the ranges AX, BX, TX, and SX is a closed subset of (X, p), *then*

(I) A and S have a coincidence point,

(II) B and T have a coincidence point.

Moreover, if A and S, as well as, B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $AX \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$. Since $BX \subseteq SX$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad \forall n \in \mathbb{N}. \tag{4}$$

Suppose $p(y_{2n}, y_{2n+1}) = 0$ for some n. Then $y_{2n} = y_{2n+1}$ implies that $Ax_{2n} = Tx_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, so T and B have a coincidence point. Further, if $p(y_{2n+1}, y_{2n+2}) = 0$ for some n then $Ax_{2n+2} = Tx_{2n+3} = Bx_{2n+1} = Sx_{2n+2}$, so A and S have a coincidence point. For the rest, assume that $p(y_n, y_{n+1}) \neq 0$ for all $n \geq 0$.

If for some $x, y \in X$, $M(x, y) = 0$, then we get that $Ax = Sx$ and $By = Ty$, so we proved (I) and (II).

If $M(x, y) > 0$ for all $x, y \in X$, then by (C3),

$$p(Ax, By) < M(x, y) \quad \text{for all}, x, y \in X. \tag{5}$$

Hence, we have

$$\begin{aligned}
&p(y_{2p}, y_{2p+1}) < M(x_{2p}, x_{2p+1}) = \\
&\max\left\{p(Sx_{2p}, Tx_{2p+1}), p(Ax_{2p}, Sx_{2p}), p(Bx_{2p+1}, Tx_{2p+1}), \frac{1}{2}[p(Sx_{2p}, Bx_{2p+1}) + p(Ax_{2p}, Tx_{2p+1})]\right\} \\
&= \max\left\{p(y_{2p-1}, y_{2p}), p(y_{2p}, y_{2p-1}), p(y_{2p+1}, y_{2p}), \frac{1}{2}[p(y_{2p-1}, y_{2p+1}) + p(y_{2p}, y_{2p})]\right\} \\
&\leq \max\left\{p(y_{2p-1}, y_{2p}), p(y_{2p+1}, y_{2p}), \frac{1}{2}[p(y_{2p-1}, y_{2p}) + p(y_{2p}, y_{2p+1})]\right\} \\
&= \max\{p(y_{2p-1}, y_{2p}), p(y_{2p}, y_{2p+1})\}
\end{aligned}$$

since

$$p(y_{2p-1}, y_{2p+1}) + p(y_{2p}, y_{2p}) \leq p(y_{2p-1}, y_{2p}) + p(y_{2p}, y_{2p+1}).$$

It is easy that max $\{p(y_{2p-1}, y_{2p}), p(y_{2p}, y_{2p+1})\} = p(y_{2p}, y_{2p+1})$ is excluded. It follows that

$$p(y_{2p}, y_{2p+1}) < M(x_{2p}, x_{2p+1}) \leq p(y_{2p-1}, y_{2p}) \quad \text{for all } p \geq 1. \tag{6}$$

Similarly, one can find

$$p(y_{2p+2}, y_{2p+1}) < M(x_{2p+2}x_{2p+1}) \leq p(y_{2p+1}, y_{2p}) \quad \text{for all } p \geq 0. \tag{7}$$

We deduce that

$$p(y_n, y_{n+1}) < p(y_{n-1}, y_n) \quad \text{for all } n \geq 1.$$

Thus, $\{p(y_n, y_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence which is bounded below by 0. Hence, it converges to some $L \in [0, \infty)$, that is,

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = L. \tag{8}$$

We claim that $L = 0$. If $L > 0$, then from (8), there exist $\delta > 0$ and a natural number $m \geq 1$ such that for each $n \geq m$ $L < d(y_m, y_{n+1}) < L + \delta$. In particular, from this and (6)

$$L < M(x_{2m}, x_{2m+1}) < L + \delta.$$

Now by using (3), we get that $p(Ax_{2m}, Bx_{2m+1}) = p(y_{2m}, y_{2m+1}) \leq L$ which is a contradiction. Thus, $L = 0$, that is,

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{9}$$

We claim that $\{y_n\}$ is a Cauchy sequence in the partial metric space (X, p). From Lemma 1.1, we need to prove that $\{y_n\}$ is Cauchy in the metric space (X, d_p). We argue by contradiction. Then there exist $\varepsilon > 0$ and a subsequence $\{y_{n(i)}\}$ of $\{y_n\}$ such that $d_p(y_{n(i)}, y_{n(i+1)}) > 4\varepsilon$. Select δ in (C2) as $0 < \delta \leq \varepsilon$. By definition of the metric d_p,

$$d_p(x, y) \leq 2p(x, y) \quad \text{for all } x, y \in X,$$

so $p(y_{n(i)}, y_{n(i+1)}) > 2\varepsilon$. Since $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$, hence there exists $N \in \mathbb{N}$ such that

$$p(y_n, y_{n+1}) < \frac{\delta}{6} \quad \text{whenever } n \geq N.$$

Let $n(i) \geq N$. Then, there exist integers $m(i)$ satisfying $n(i) < m(i) < n(i + 1)$ such that

$$p(y_{n(i)}, y_{m(i)}) \geq \varepsilon + \frac{\delta}{3}.$$

If not, then by triangle inequality (which holds even for partial metrics)

$$\begin{aligned} p(y_{n(i)}, y_{n(i+1)}) &\leq p(y_{n(i)}, y_{n(i+1)-1}) + p(y_{n(i+1)-1}, y_{n(i+1)}) \\ &< \varepsilon + \frac{\delta}{3} + \frac{\delta}{6} < 2\varepsilon, \end{aligned}$$

it is a contradiction. Without loss of generality, we can assume $n(i)$ to be odd. Let $m(i)$ be the smallest even integer such that

$$p(y_{n(i)}, y_{m(i)}) \geq \varepsilon + \frac{\delta}{3}. \tag{10}$$

Then

$$p(y_{n(i)}, y_{m(i)-2}) < \varepsilon + \frac{\delta}{3},$$

and

$$\begin{aligned}\varepsilon + \tfrac{\delta}{3} \leq p(y_{n(i)}, y_{m(i)}) &\leq p(y_{n(i)}, y_{m(i)-2}) + p(y_{m(i)-2}, y_{m(i)-1}) + p(y_{m(i)-1}, y_{m(i)}) \\ &< \varepsilon + \tfrac{\delta}{3} + \tfrac{\delta}{6} + \tfrac{\delta}{6} = \varepsilon + 2\tfrac{\delta}{3}.\end{aligned} \tag{11}$$

Also, $p(y_{n(i)}, y_{m(i)}) \leq M(x_{n(i)+1}, x_{m(i)+1}) < \varepsilon + 2\frac{\delta}{3} + \frac{\delta}{6} < \varepsilon + \delta$, that is,

$$\varepsilon < \varepsilon + \frac{\delta}{3} \leq M(x_{n(i)+1}, x_{m(i)+1}) < \varepsilon + \delta.$$

In view of (C2), this yields that $p(y_{n(i)+1}, y_{m(i)+1}) \leq \varepsilon$. But then

$$\begin{aligned}p(y_{n(i)}, y_{m(i)}) &\leq p(y_{n(i)}, y_{n(i)+1}) + p(y_{n(i)+1}, y_{m(i)+1}) + p(y_{m(i)+1}, y_{m(i)}) \\ &< \frac{\delta}{6} + \varepsilon + \frac{\delta}{6} = \varepsilon + \frac{\delta}{3},\end{aligned}$$

which contradicts (10). Hence $\{y_n\}$ is a Cauchy sequence in the metric space (X, d_p), so also in the partial metric (X, p) which is complete. Thus, there exists a point y in X such that from Lemmas 1.1, 2.2, and (9)

$$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = 0. \tag{12}$$

This implies that

$$\lim_{n\to+\infty} p(y_{2n}, y) = \lim_{n\to+\infty} p(y_{2n-1}, y) = 0. \tag{13}$$

Thus from (13) we have

$$\lim_{n\to+\infty} p(Ax_{2n}, y) = \lim_{n\to+\infty} p(Tx_{2n+1}, y) = 0 \tag{14}$$

and

$$\lim_{n\to+\infty} p(Bx_{2n-1}, y) = \lim_{n\to+\infty} p(Sx_{2n}, y) = 0. \tag{15}$$

Now we can suppose, without loss of generality, that SX is a closed subset of the partial metric space (X, p). From (15), there exists $u \in X$ such that $y = Su$. We claim that $p(Au, y) = 0$. Suppose, to the contrary, that $p(Au, y) > 0$.

By (P4) and (C4) we get

$$\begin{aligned}p(y, Au) &\leq p(y, Bx_{2n+1}) + p(Au, Bx_{2n+1}) - p(Bx_{2n+1}, Bx_{2n+1}) \\ &\leq p(y, Bx_{2n+1}) + p(Au, Bx_{2n+1}) \\ &\leq p(y, Bx_{2n+1}) + \max\{a[p(y, y_{2n}) + p(Au, y) + p(y_{2n+1}, y_{2n})], \\ &\quad b[p(y, y_{2n+1}) + p(Au, y_{2n})]\} \\ &\leq p(y, Bx_{2n+1}) + \max\{a[p(y, y_{2n}) + p(Au, y) + p(y_{2n+1}, y_{2n})], \\ &\quad b[p(y, y_{2n+1}) + p(Au, y) + p(y, y_{2n}) - p(y, y)]\}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (12)-(15), we obtain

$$0 < p(y, Au) \le \max\{ap(Au, y), bp(Au, y)\} < p(Au, y)$$

it is a contradiction since $0 \le a < \frac{1}{2} < 1$ and $0 \le b < \frac{1}{2} < 1$. Thus, by Lemma 2.1, we deduce that

$$p(Au, y) = 0 \text{ and } y = Au. \tag{16}$$

Since $y = Su$, then $Au = Su$, that is, u is a coincidence point of A and S. So we proved (I).

From $AX \subseteq TX$ and (16), we have $y \in TX$. Hence we deduce that there exists $v \in X$ such that $y = Tv$. We claim that $p(Bv, y) = 0$. Suppose, to the contrary, that $p(Bv, y) > 0$. From (C4) and (16), we have

$$\begin{aligned} 0 < p(y, Bv) = p(Au, Bv) &\le \max\{a[p(Su, Tv) + p(Au, Su) + p(Bv, Tv)],\\ &\quad b[p(Su, Bv) + p(Au, Tv)]\}\\ &= \max\{a[p(y, y) + p(y, y) + p(Bv, y)],\\ &\quad b[p(y, Bv) + p(y, y)]\}\\ &= \max\{ap(Bv, y),\ bp(Bv, y)\} \end{aligned}$$

as $y = Su = Au = Tv$ and $p(y, y) = 0$. Since $0 \le a < 1$ and $0 \le b < 1$, this implies that

$$p(Bv, y) < p(Bv, y),$$

which is a contradiction. Then, we deduce that

$$p(Bv, y) = 0 \text{ and } y = Bv = Tv, \tag{17}$$

that is, v is a coincidence point of B and T, then (II) holds.

Since the pair $\{A, S\}$ is weakly compatible, from (16), we have $Ay = ASu = SAu = Sy$. We claim that $p(Ay, y) = 0$. Suppose, to the contrary, that $p(Ay, y) > 0$. We have

$$\begin{aligned} p(Ay, y) &\le p(Ay, y_{2n+1}) + p(y_{2n+1}, y)\\ &= p(Ay, Bx_{2n+1}) + p(y_{2n+1}, y)\\ &\le p(y_{2n+1}, y) + \max\{a[p(Sy, Tx_{2n+1}) + p(Ay, Sy) + p(Bx_{2n+1}, Tx_{2n+1})],\\ &\quad b[p(Sy, Bx_{2n+1}) + p(Ay, Tx_{2n+1})]\}\\ &= p(y_{2n+1}, y) + \max\{a[p(Ay, y_{2n}) + p(Ay, Ay) + p(y_{2n+1}, y_{2n})],\\ &\quad b[p(Ay, y_{2n+1}) + p(Ay, y_{2n})]\}. \end{aligned}$$

Using (12) and (p2), we get letting $n \to +\infty$

$$0 < p(Ay, y) \le \max\{2ap(Ay, y), 2bp(Ay, y)\} < p(Ay, y)$$

a contradiction. Then we deduce that

$$p(Ay, y) = 0 \text{ and } Ay = Sy = y. \tag{18}$$

Since the pair $\{B, T\}$ is weakly compatible, from (17), we have $By = BTv = TBv = Ty$. We claim that $p(By, y) = 0$. Suppose, to the contrary, that $p(By, y) > 0$, then by (C4) and (18), we have

$$\begin{aligned}0 < p(y, By) = p(Ay, By) &\le \max\{a[p(Sy, Ty) + p(Ay, Sy) + p(By, Ty)],\\ &\quad b[p(Sy, By) + p(Ay, Ty)]\}\\ &= \max\{a[p(y, By) + p(y, y) + p(By, By)], b[p(y, By) + p(y, By)]\}\\ &\le \max\{2a, 2b\}p(By, y),\end{aligned}$$

since $p(y, y) = 0$. Thus, we get

$$p(y, By) = 0 \text{ and } By = Ty = y. \tag{19}$$

Now, combining (18) and (19), we obtain

$$y = Ay = By = Sy = Ty,$$

that is, y is a common fixed point of A, B, S, and T with $p(y, y) = 0$.

Now we prove the uniqueness of a common fixed point. Let us suppose that $z \in X$ is a common fixed point of A, B, S, and T such that $p(z, y) > 0$. Using (iv), we get

$$\begin{aligned}p(y, z) &= p(Ay, Bz)\\ &\le \max\{a[p(Ay, Bz) + p(Ay, Ay) + p(Bz, Bz)], b[p(Ay, Bz) + p(Az, By)]\}\\ &= \max\{a[p(y, z) + p(y, y) + p(z, z)], 2bp(y, z)\}\\ &\le \max\{2a, 2b\}p(y, z)) < p(y, z),\end{aligned}$$

which is a contradiction. Then we deduce that $z = y$. Thus the uniqueness of the common fixed point is proved. The proof is completed.

Repeating the proof of Theorem 2.3, we get easily the following.

Corollary 2.4. *Let A, B, S, and T be the self maps defined on a partial metric space (X, p) satisfying the following conditions:*

(C1) $AX \subseteq TX$ *and* $BX \subseteq SX$,

(C2) *for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow p(Ax, By) \le \varepsilon,$$

where $M(x, y) = \max\{p(Sx, Ty), p(Ax, Sx), p(By, Ty), \frac{1}{2}[p(Sx, By) + p(Ax, Ty)]\}$,

(C3) *for all* $x, y \in X$ *with* $M(x, y) > 0 \Rightarrow p(Ax, By) < M(x, y)$,

(C4) $p(Ax, By) < k[p(Sx, Ty) + p(Ax, Sx) + p(By, Ty) + p(Sx, By) + p(Ax, Ty]$ *for all* $x, y \in X$ *and* $0 \le k < \frac{1}{3}$.

If one of AX, BX, SX, or TX is a complete subspace of X, then

(I) A and S have a coincidence point,

(II) B and T have a coincidence point.

Moreover, if A and S, as well as, B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

3 Some equivalence statements of Meir-Keeler contraction

Jachymski [30] proved the following important lemma.

Lemma 3.1. *Let* $\mathbb{Q}$*be a subset of* $[0, \infty) \times [0, \infty)$. *Then the following statements are equivalent:*

($J1$) *There exists a function* $\delta : (0, \infty) \to (0, \infty)$ *such that for any* $\varepsilon > 0$, $\delta(\varepsilon) > \varepsilon$ *and*

($J1a$) $\sup\{\delta(s) : s \in (0, \varepsilon)\} \ge \delta(\varepsilon)$ *and*

($J1b$) $(s, t) \in \mathbb{Q}$*and* $0 \le s < \delta(\varepsilon)$ *imply* $t < \varepsilon$.

($J2$) *There exist functions* $\beta, \eta : (0, \infty) \to (0, \infty)$ *such that, for any* $\varepsilon > 0$, $\beta(\varepsilon) > \varepsilon$, $\eta(\varepsilon) < \varepsilon$, *and* $(s, t) \in \mathbb{Q}$*and* $0 \le s < \beta(\varepsilon)$ *imply* $t < \eta(\varepsilon)$.

($J3$) *There exists an upper semi continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *such that* φ *is non-decreasing,* $\varphi(s) < s$ *for* $s > 0$, *and* $(s, t) \in \mathbb{Q}$*implies* $t \le \varphi(s)$.

($J4$) *There exists a lower-semi continuous function* $\delta : (0, \infty) \to (0, \infty)$ *such that for any* δ *is non-decreasing, for any* $\varepsilon > 0$, $\delta(\varepsilon) > \varepsilon$, *and* $(s, t) \in \mathbb{Q}$*and* $0 \le s < \delta(\varepsilon)$ *imply* $t < \varepsilon$.

($J5$) *There exists a lower-semi continuous function* $\omega : [0, \infty) \to [0, \infty)$ *such that for any* ω *is non-decreasing,* $\omega(s) > s$ *for* $s > 0$ *and* $(s, t) \in \mathbb{Q}$*implies* $w(t) \le s$.

Theorem 3.2. *Let* (X, p) *be a partial metric space, and* S, T, $A_i (i \in \mathbb{N})$ *be self-mappings on* X. *For* $x, y \in X$ *and for* $i, j \in \mathbb{N}$, *we define*

$$M_{ij}(x, y) = \left\{ p(Sx, Ty), p(Sx, A_i x), p(Ty, A_j y), \frac{[p(Sx, A_j y) + p(Ty, A_i x)]}{2} \right\}.$$

Then the following statements are equivalent.

($JT1$) *There exists a lower-semi continuous function* $\delta : (0, \infty) \to (0, \infty)$ *such that, for any* $\varepsilon > 0$, $\delta(\varepsilon) > \varepsilon$ *and for any* $x, y \in X$ *and distinct* $i, j \in \mathbb{N}$

$$\varepsilon \le M_{ij}(x, y) < \delta(\varepsilon) \text{ implies } p(A_i x, A_j y) < \varepsilon.$$

($JT2$) *There exists an upper-semi continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *such that,* φ *is non-decreasing,* $\varphi(t) < t$, *and*

$$p(A_i x, A_j y) \le \phi(M_{ij}(x, y)).$$

for any $x, y \in X$ *and distinct* $i, j \in \mathbb{N}$.

($JT3$) *There exists a lower-semi continuous function* $\omega : [0, \infty) \to [0, \infty)$ *such that,* ω *is non-decreasing,* $\omega(s) > s$ *for* $s > 0$, *and*

$$\omega(p(A_i x, A_j y)) \le M_{ij}(x, y)$$

for any $x, y \in X$ *and distinct* $i, j \in \mathbb{N}$.

Proof. It follows immediately from Lemma 3.1.

Remark 3.1. *In Theorem* 1.2, *Ćirić et al. assumed that the hypothesis* $p(Ax, By) \le \varphi(M(x, y))$ *is satisfied for all* $x, y \in X$ *with* $\varphi \in \Phi$ *and obtained a common fixed point result.*

In particular from the assumptions on that φ, ($JT2$) *holds for* $A_1 = A$ *and* $A_2 = B$. *So, by Theorem* 3.2, ($JT1$) *holds, that is; for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$

$$\varepsilon \le M(x, y) < \varepsilon + \delta \Rightarrow p(Ax, By) < \varepsilon, \tag{20}$$

By Lemma 3.1 *of Jachymski* [31], (20) *implies (as in metric cases) that the conditions (C2) and (C3) are satisfied, but nothing on the condition (C4). Conversely, in Theorem* 2.3 *we have assumed that (C2) and (C3) hold, but we added another condition which is (C4) in order to get a common fixed point result.*

Remark 3.2. *Theorem* 2.3 *is the analogous of Theorem* 1 *of Rana et al.* [32]*on partial metrics, except that the conditions* (20) *and the fact that* $a, b \in [0, 1]$, *are replaced by the weaker conditions (C2), (C3) and* $a, b \in [0, \frac{1}{2}]$. *The condition on a and b is modified due to the fact that* $p(x, x)$ *may not equal to* 0 *for* $x \in X$. *Also, Corollary* 2.4 *extends Theorem* 2.1 *of Bouhadjera and Djoudi* [33]*on partial metric cases. Note that Theorem*

2.1 *in* [33]*was improved recently by Akkouchi* [[34], Corollary 4.4]. *Indeed, the Lipschitz constant k is allowed to take values in the interval* $[0, \frac{1}{2}]$*instead of the case studied in* [33], *where the constant k belongs to the smaller interval* $[0, \frac{1}{3}]$.

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Authors' contributions
The authors have contributed in obtaining the new results presented in this article. All authors read and approve the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 28 November 2011 Accepted: 24 February 2012 Published: 24 February 2012

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doi:10.1186/1687-1812-2012-26
Cite this article as: Aydi and Karapinar: **A Meir-Keeler common type fixed point theorem on partial metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:26.