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A meshfree weak-strong form method

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Abstract

A novel meshfree weak-strong (MWS) form method is proposed based on a combined formulation of both the local weak form and the strong form. In the MWS method, the problem domain and its boundary is represented by a set of distributed nodes. The strong form or the collocation method is used for all internal nodes and the nodes on the essential boundaries. The local weak form (Petrov-Galerkin weak form) is used for nodes on or near the natural boundaries. The natural boundary conditions can then be easily imposed to produce stable and accurate solutions. The MWS method has advantages of both meshfree methods based on strong forms and weak forms. In the entire problem, only local integration "meshes" for nodes on or near the natural boundary are required.

1 Introduction

More and more researchers are devoting themselves to the research of the meshfree methods, due to the fact that there are still many difficult issues to be solved to fully realize the dream of the meshless method. Detailed descriptions of many meshfree methods can be found in the recent monograph by Liu [1]. Meshfree methods can be largely categorized into two major categories: meshfree methods based on strong forms (or short for meshfree strong-form methods) and meshfree methods based on weak forms (or short for meshfree weak-form methods), such as the element-free Galerkin (EFG) method[2], the point interpolation method (PIM)[3][4], etc. A software package, MFree 2D©, has also been developed based on these three meshfree weak-form methods [1]. There are also meshfree methods based on the integral representation method for functional approximations, such as the particle methods, many of which are briefly introduced in the book of Liu and Liu[5].

The meshfree strong-form methods have a relatively longer history of development. A typical meshfree strong-form method is the meshfree collocation method [6]. Compared with meshfree weak-form methods, meshfree strong-form methods have following attractive advantages:

- The algorithms are simple.
- They are computationally efficient.
- They are truly meshless methods without using any mesh for both field variable approximation and integration.

Because of the above advantages, meshfree strong-form methods have been successfully used in fluid mechanics. However, they are often unstable and less accurate for problems governed by partial differential equations with Neumann (derivative) boundary conditions, especially for solid mechanics problems with stress (natural) boundary conditions. In the direct meshfree collocation methods, Neumann boundary conditions are implemented using a series of separate equations, which are different from the governing equations. The error induced from the boundaries, therefore, cannot be efficiently controlled.

Meshfree weak-form methods, such as the EFG method [2], have following advantages:

1) They have very good stability and excellent accuracy.
2) The Neumann boundary conditions can be naturally satisfied through the use of the weak form.

Therefore, meshfree weak-form methods have been successfully applied in problems of solid mechanics. However, the numerical integration makes them computationally expensive, and the background mesh used for the integration of the weak form is responsible for not being "truly" meshless. In order to alleviate the global integration background mesh, meshfree methods based on the local Petrov-Galerkin weak forms have been developed, such as the meshless local Petrov-Galerkin (MLPG) method originated by Atluri et al.[7], and the local point interpolation method (LPIM)[8][9]. In these local meshfree methods, local weak forms integrated in a regular-shaped local domain are used. However, the local numerical integration could still be a burden, especially for nodes close to the boundary of complex shapes.

The meshfree strong-form methods and meshfree weak-form methods have their own advantages and shortcomings, and they are complementary. The question is "can we couple the week form with the strong form together in a proper manner to fully take their advantages and avoid their disadvantages and how?". Liu and Gu [10] have tried to find a answer to this question. This paper addresses the same question in a greater detail.

Close examination of the meshfree methods based on strong forms and local weak forms, reviews the following facts. The implementation scheme of these two types of meshfree methods is, in fact, very similar. If the delta function is used as the weight function, the meshfree method based on local weak forms

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becomes a meshfree strong-form method. In meshfree strong-form methods, the instability and computational error is mainly induced by the presence of the natural boundary condition. On the contrary, the natural boundary condition can be easily and exactly enforced using the local weak form.

The above observations provide us a possibility to combine the local weak form and the strong form together to fully take their advantages and avoid their disadvantages. In this paper, a novel meshfree method, the meshfree weak-strong (MWS) form method, is proposed based on a combined formulation of both the local weak form and the strong form. In the MWS method, the strong form or collocation method is used for all internal nodes and the nodes on the essential boundaries. The local weak form is only used for nodes on or near the natural boundaries. There is no need at all for numerical integrations for all the internal nodes and the nodes on the essential boundaries due to the use of the strong form. The natural boundary conditions can also be easily imposed to produce stable and accurate solutions due to the use of the local weak form.

2 The idea of the Meshfree Weak-Strong (MWS) form method

Consider a 2-D solid mechanics problem with a problem domain Ω shown in Figure 1. The problem domain and boundaries are represented by a set of scattered field nodes. The key idea of the MWS method is that in establishing the discrete system equations, both the strong form and the local weak form are used for the same problem. In Figure 1, Ω_q is the local quadrature domain for a filed node. If Ω_q does not intersect with the natural boundaries, the strong form is used for this node. Otherwise, the local weak form is used.

2.1 Strong form

For an internal node or a node on the essential boundary, whose local quadrature domain does not intersect with the natural boundary, the following standard strong form of 2D elasticity is used.

$$\begin{cases} \dfrac{E}{1-v^2}\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{1-v}{2}\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 v}{\partial x \partial y}\right)+b_x=0 \\ \dfrac{E}{1-v^2}\left(\dfrac{\partial^2 v}{\partial y^2}+\dfrac{1-v}{2}\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 u}{\partial x \partial y}\right)+b_y=0 \end{cases} \tag{1}$$

where E and v are Young's modulus and the Poisson ratio. b_x and b_y are body forces at x direction and y direction, respectively. The collocation method is used directly to discretize equation (1).

2.2 Local weak form

A generalized local weak form of 2-D solids , over a local quadrature domain Ω_q bounded by Γ_q, can be obtained using the weighted residual method [7]

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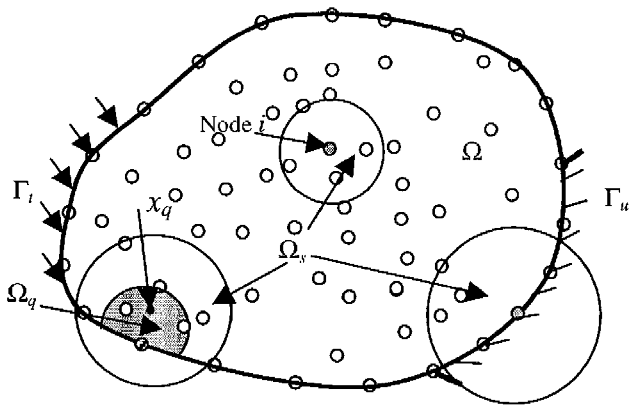
$$\int_{\Omega_q} w_i(\sigma_{ij,j} + b_i)\mathrm{d}\Omega - \int_{\Gamma_{qu}} \alpha w_i(u_i - \bar{u}_i)d\Gamma = 0 \tag{2}$$

where w_i is the weight function that can be the 4th-order quartic spline weight function or other weight functions [1]. It should note here that the last term in (2) is to enforce the essential boundary condition. If radial PIM (RPIM) shape functions [1] (with delta function property) are used, this term is not needed. However, if MLS shape functions (no delta function property) are used, this term is necessary. The first term on the left hand side of equation (2) can be integrated by parts. It can be found that the boundary Γ_q for the local quadrature domain usually comprises three parts: the internal boundary Γ_{qi} that is located within the problem domain, the essential boundaries Γ_{qu} that intersects with the global essential boundary Γ_u, and the natural boundary Γ_{qt} that intersects with the global natural boundary Γ_t. Imposing the natural boundary condition, the local weak form is then obtained

$$\int_{\Gamma_{si}} w_i t_i \mathrm{d}\Gamma + \int_{\Gamma_{su}} w_i t_i \mathrm{d}\Gamma + \int_{\Gamma_{st}} w_i \bar{t}_i \mathrm{d}\Gamma - \int_{\Omega_s} (w_{i,j}\sigma_{ij} - w_i b_i)\mathrm{d}\Omega - \int_{\Gamma_{qu}} \alpha w_i(u_i - \bar{u}_i)d\Gamma = 0 \tag{3}$$



Ω_q: the local quadrature domain for integration of the weak form
Ω_s: the local support domain for field variables interpolation

Figure 1: A problem domain represented with a set distributed nodes for implementing the MWS method.

2.3 Discrete formulations

For a field node, x_i, or a quadrature point, x_q, the local support domains, Ω_s , are used to construct shape functions. Using radial point interpolation [1] or the MLS approximation into the strong form equation (1) and local weak form equation (3) for all nodes leads to the following discrete equations

$$\mathbf{KU} = \mathbf{F} \tag{4}$$

where **U** is the vector of displacements for all nodes in the entire problem domain. **K** and **F** are defined as

$$\mathbf{K}_{ij} = \begin{cases} \int_{\Omega_q} \mathbf{v}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j \mathrm{d}\Omega - \int_{\Gamma_{qi}} \mathbf{w}_i \mathbf{N} \mathbf{D} \mathbf{B}_j \mathrm{d}\Gamma - \int_{\Gamma_{qu}} \mathbf{w}_i \mathbf{N} \mathbf{D} \mathbf{B}_j \mathrm{d}\Gamma \\ \qquad + \alpha \int_{\Gamma_{qu}} \mathbf{w}_i \mathbf{\Phi}_j \mathrm{d}\Gamma, \qquad \Omega_q(\mathbf{x}_i) \cap \Gamma_t \neq \varnothing \\ \mathbf{L}_i^{\mathrm{T}} \mathbf{D} \mathbf{L}_i \mathbf{\Psi}_j \ , \qquad \Omega_q(\mathbf{x}_i) \cap \Gamma_t = \varnothing \end{cases} \tag{5}$$

$$\mathbf{f}_i(t) = \int_{\Gamma_{qt}} \mathbf{w}_i \bar{\mathbf{t}} \mathrm{d}\Gamma + \int_{\Omega_q} \mathbf{w}_i \mathbf{b} \mathrm{d}\Omega + \alpha \int_{\Gamma_{qu}} \mathbf{w}_i \bar{\mathbf{u}} \mathrm{d}\Gamma, \qquad \Omega_q(\mathbf{x}_i) \cap \Gamma_t \neq \varnothing \tag{6}$$

with $\mathbf{w}_i$ being the value of the weight function matrix, corresponding to node *i*, evaluated at the point **x**, $\mathbf{\Phi}_j$ is the matrix of shape functions, and

$$\mathbf{N} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}, \quad \mathbf{B}_j = \begin{bmatrix} \phi_{j,x} & 0 \\ 0 & \phi_{j,y} \\ \phi_{j,y} & \phi_{j,x} \end{bmatrix}, \quad \mathbf{v}_i = \begin{bmatrix} w_{i,x} & 0 \\ 0 & w_{i,y} \\ w_{i,y} & w_{i,x} \end{bmatrix} \tag{7}$$

where (n_x, n_y) is the unit outward normal to the boundary Γ_q, **L** is the differential operator matrix, **D** is the matrix of elastic constants of material.

3 Numerical examples

3.1 Standard patch test

The first numerical example is the standard patch test. In the patch test, the displacements are prescribed on all outside boundaries by a linear function of *x* and *y*. Satisfaction of the patch test requires that the displacement of any interior node should be given by the same linear functions and that the strains and stresses should be constant in the patch. It has been found that the MWS method can exactly pass all standard patch tests. If RPIM shape functions (with the linear polynomial terms) are used[1], the relative displacements error less than 10^{-15}. If MLS shape functions are used, the relative displacements error, which is mainly affected by the penalty coefficient chosen for the enforcement of the essential boundary conditions, is less than 10^{-6}.

3.2 Higher-order patch test

As shown in Figure 2, a rectangular patch is subjected to two types of loading at the right end.

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1) Case 1: a uniform axial stress of unit intensity is applied on the right edge. The exact solution for this problem with E=1 and ν=0.25 is: $u_i=x_i$, $v_i=y_i/4$.
2) Case 2: a linearly varying normal stress is applied on the right edge. The exact solution for this problem is: $u_i=2xy/3$, $v_i=-(x^2+y^2/4)/3$.

Both regularly and irregularly distributed nodes are used. It can be found that Case 1 is passed exactly by the presented MWS method using both RPIM (with the linear polynomial terms) and MLS. In Case 1, it demonstrates that the MWS method exactly implement the natural (force) boundary condition and lead to an exact solution for this problem whose analytical displacement solution is a linear function.

It can be seen that there exist error in solving Case 2 by the MWS method using both RPIM and MLS shape functions (Table 1). The error is due to the fact that the exact solution of the displacement field is of 2nd order, and the basis functions using in the current method do not contain such high order terms.

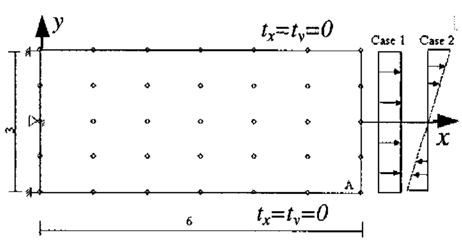

Figure 2: High-order patch test.

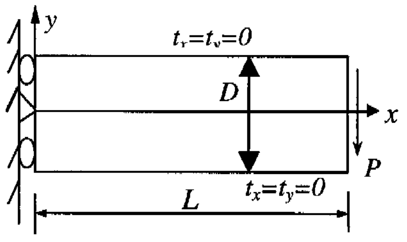

Figure 3: A cantilever beam.

Table 1: Relative errors (%) of u_x at point A for higher-order patch test Case 2 (using irregular nodes).

	Exact	MWS (RPIM)	LRPIM	Collocation (RPIM)	MWS (MLS)	MLPG
u	-6.00	-6.389	-5.951	-8.786	-5.976	-5.982
Error	/	6.491%	-0.808%	46.6%	-0.396%	-0.291%
v	-12.19	-13.234	-12.020	-16.202	-12.168	-12.172
Error	/	8.586%	-1.408%	49.3%	-0.160%	-0.159%

For comparison, results by the local radial point interpolation method (LRPIM) and the MLPG method, which are local meshfree methods using local weak forms for all nodes, are also obtained. It can be seen that LRPIM and MLPG usually lead to more accurate results than the MWS method because the local weak form is used for the entire problem domain. The meshfree collocation method that uses strong forms for all nodes is also used to get results for Case 2. It has been found that the collocation method can also get satisfactory results for Case 1, whose natural boundary condition is simple. However, the error for Case 2 is as high as 20% for regular nodes and 49% for irregular nodes (Table 1). In fact, the solution of the meshfree collocation method is basically not stable. Compared with the meshfree collocation method, the present MWS method has far better accuracy for this high order patch test. The error and the instability of the meshfree collocation method mainly induced by the presence of the complex natural boundary condition. The solutions for all the methods that use weak forms including the present MWS method are very stable. This is because the use of weak forms controls well the possible error from the natural boundaries.

3.3 Cantilever beam

A cantilever beam shown in Figure 3 is considered. The analytical solution for this problem is available. Both regularly and irregularly distributed nodes are employed. For this problem, the MWS method gives very accurate results for both regular nodes and irregular nodes. Figure 4 illustrates the comparison between the shear stress τ_{xy} at the cross-section $x=L/2$ calculated analytically and using the MWS methods. Very good agreement is observed.

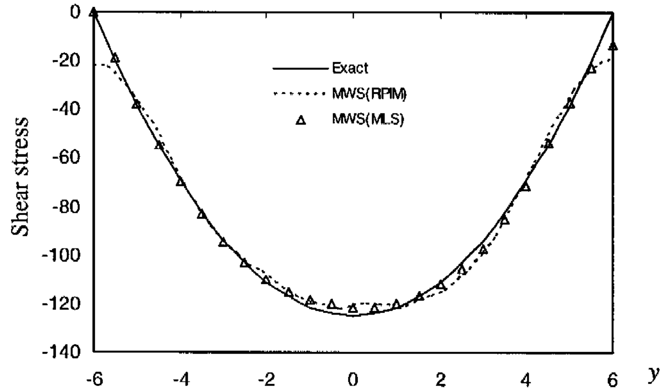

Figure 4: Stress (τ_{xx}) of the beam the cross-section of along $x=L/2$ obtained using 189 irregular nodes.

The meshfree collocation method that uses strong forms for all nodes is also used to get results for this problem. It has been found that the error in the solution of the meshfree collocation method is very big. The solution of the meshfree collocation method is also unstable. The computation even fails when irregular nodes are used. Compared with the meshfree collocation method, the present MWS method has far better accuracy and stability for this problem.

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A convergence study has also been carried and the results are shown in Figure 5. For comparison, the convergence curves for LRPIM and MLPG are also plotted in the same figure. From Figure 5, we can find:

1) LRPIM and MLPG have better accuracy than the MWS method.
2) Using MLS, the MWS method has very good convergence rate and the accuracy.
3) The convergence process of MWS using RPIM is not very good although the accuracy is acceptable. It is because that locally supported radial basis function (RBF) usually has a bad *h* convergence due to possibly the lack of linear reproducibility of the RBF [11].

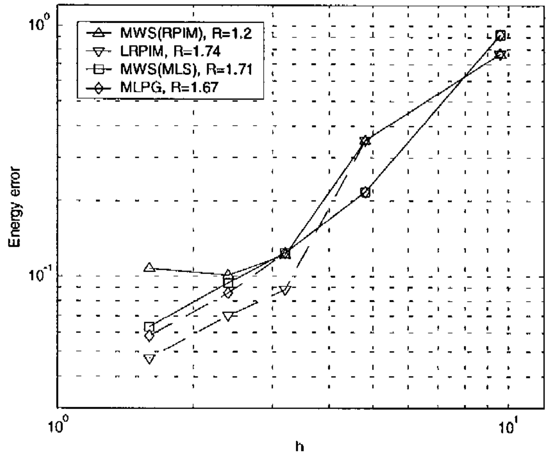

Figure 5: Convergence of energy error (*R*: convergence rate).

Table 2. CPU time (s)*.

	MWS(RPIM)	LRPIM	MWS(MLS)	MLPG
55 nodes	43.710	50.060	2.060	2.120
189 nodes	66.730	310.630	7.270	9.650
403 nodes	123.160	822.710	13.840	24.760

* Performed on a DataMini PC with an Intel Pentium 4 CPU 1.80 GHz processor.

In the efficiency study, regularly distributed 55, 189 and 403 nodes are used. The CPU time incurred by the MWS method, LRPIM and MLPG are listed in Table 2. Form this table, it can be found that MWS method uses much more less CPU time than LRPIM and MLPG, respectively.

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To be fair, the computational cost must be considered together with the accuracy of the results. A successful numerical method should obtain high accuracy at a lower computational cost. The performance curves of error vs. computation time for the present MWS method, LRPIM and MLPG are obtained and plotted in Figure 6. From Figure 6, the following remarks can be made:

a) The MWS method with MLS and MLPG have better efficiency than MWS with RPIM and LRPIM, respectively. It is because the MLS approximation has better efficiency than the RPIM interpolation[1].
b) For a desired accuracy (such as 10^{-1} error), the cost of MWS methods is lower than corresponding local meshfree methods. It is because, in the MWS method, a big part numerical integration is saved by the use of the strong form.
c) For a given cost (say 20s or 100s), the performance of the MWS method is the better than corresponding local meshfree methods.

Summarizing the above discussions, one can conclude that the efficiency of the MWS method is better than corresponding local meshfree methods.

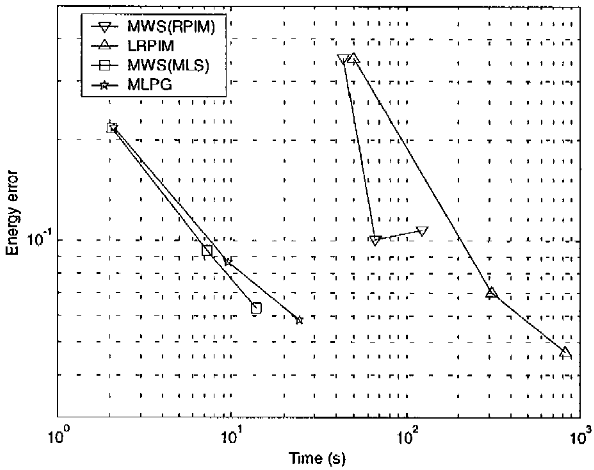

Figure 6: Comparison of the performance of meshfree methods.

4 Conclusions

A novel meshfree method, the meshfree weak-strong (MWS) form method, is proposed based on a combined formulation of both local weak forms and strong forms. The strong form or collocation method is used for all nodes whose local quadrature domains do not intersect with natural boundaries. Therefore, there are no numerical integrations for these nodes. The local weak form, which needs the local numerical integration, is used for nodes on or near the natural boundaries. The natural boundary conditions can be easily imposed to produce stable and accurate solutions. Numerical examples demonstrate that the present MWS method is very easy to implement, and very flexible and efficient for calculating displacements and stresses in solids.

In the MWS method, the local weak form and the strong form are combined together. It is a stable meshfree method that uses the least mesh in the entire simulation. No mesh at all is required for the field variable approximation, and only local cells for integration are required for nodes near the natural boundaries. The MWS method takes fully advantages of strong forms and weak forms to achieve the better efficiency. It is much more accurate and stable than meshfree strong-form methods. In the meantime, it is more efficient than meshfree weak-form methods for the entire problem domain.

As an efficient meshfree method, the present MWS method opens an alternative avenue to develop adaptive meshfree codes for stress analysis in solids and structures. Of course, further research work is needed to improve it. For example, the local background cells could give difficulties in the modelling when the geometry of the natural boundary is too complex.

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