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A METHOD FOR COMPUTING THE EVOLUTION OF STAR CLUSTERS

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SUMMARY

We describe a method for computing the evolution of a spherical stellar system, using a fluid-dynamical approach based on the numerical solution of moment equations derived from the Boltzmann equation. Moments of the velocity distribution up to the fourth order are included in this treatment. In order to represent relaxation effects, ' collision terms ' are included and are evaluated using the Fokker–Planck equation. The principal assumption of the method is that the velocity distribution deviates by only a small amount from a Maxwellian distribution; however, it has been attempted to allow for a large anisotropy of the velocity distribution in the outer part of the system. To illustrate the application of the method, we apply it to the problem of the 'gravothermal catastrophe ' of an isothermal sphere, as recently discussed by Lynden-Bell & Wood (1968). The results confirm the predictions of Lynden-Bell & Wood concerning the instability and thermal runaway of an enclosed isothermal sphere when the central condensation becomes too high; however, the time scale for the ensuing ' gravothermal catastrophe ' is found to be much longer than the relaxation time. A discussion of the results shows that the rate of evolution is intimately related to the overall structure of the system; the more closely it approaches an isothermal structure with a Maxwellian velocity distribution, the slower is the evolution.

1. INTRODUCTION

The dynamical evolution of a large system of stars (i.e., one containing $\gg 100$ stars) is a long-standing problem in stellar dynamics, which has not been completely solved except with the help of certain fairly strong simplifying assumptions. Hénon (1961, 1965) obtained solutions for the structure and evolution of a globular cluster by assuming that the evolution is homologous and that the velocity distribution always remains isotropic. Michie (1961) studied the evolution of a cluster model having an anisotropic velocity distribution, but specified by a distribution function containing only three parameters; the change with time of these three parameters was computed using the Fokker–Planck equation, but only one time step was obtained. Von Hoerner (1963, 1968) has proposed a theory for the evolution of a stellar system which is based on a number of simplifying assumptions, including the assumption of ' local virial equilibrium ', and he has obtained some predictions in agreement with the results of n-body calculations for small clusters ($n = 25$). However, the degree of validity of the various simplifying assumptions which have been made by different authors is still not entirely clear; for example, it is still not clear whether anisotropy of the velocity distribution is an important effect which must be taken into account.

In the present paper we shall consider a fluid-dynamical approach to stellar dynamics, based on the numerical solution of moment equations derived from the Boltzmann equation. Such a method has been used, for example, by Larson (1969) to study the initial collapse and formation of a spherical galaxy. Since the fluid-dynamical approach is capable in principle of dealing with a system of arbitrary structure, it is of interest to see whether relaxation effects can be incorporated in this method; this would allow the computation in detail of such expected evolutionary effects as the increasing central condensation of a stellar system, the loss of stars from the system, and the tendency toward an increasing degree of anisotropy of the velocity distribution in the outer parts of the system.

It is clear from the work of Hénon (1961) and Lynden-Bell & Wood (1968) that the evolution of a stellar system may be understood as being caused primarily by an outward flow of energy from the centre of the system to the outer regions, analogous to the flow of heat in a gas. In terms of the velocity distribution, this corresponds to a tendency for the most energetic stars to be moving preferentially outwards relative to the bulk of the stars, and this corresponds in turn to a non-zero third moment of the velocity distribution. Thus it is clear that the third moments of the velocity distribution will play an essential role in the fluid-dynamical calculations, and that the moment equations must be carried to at least third order. In fact, it turns out on examination of the equations (see Section 2 and the discussion in Section 5) that in order to be able to calculate the third moments properly, it is necessary to know also the fourth moments and to carry the moment equations to at least fourth order; the system of equations may then be closed by assuming a relation between the fifth moments and the lower order moments.

It then remains to incorporate the effects of encounters between the stars, which are responsible for producing the energetic stars which escape from the central part of the cluster. The effect of encounters on the velocity distribution is usually described by means of the Fokker-Planck equation, which is approximately valid for systems with a sufficiently large number of stars (Cohen, Spitzer & Routly 1950). The Fokker–Planck equation provides an expression for the 'encounter term' in the Boltzman equation, which may be integrated over the velocity distribution to yield the rates of change of the various moments of the velocity distribution as caused by encounters.

In Section 2 we shall give the derivation of the moment equations, and in Section 3 we present the calculation of the relaxation effects using the Fokker–Planck equation. In Section 4 we describe the application of this method to the problem of the 'gravothermal catastrophe' for an isothermal sphere, as recently studied by Lynden-Bell & Wood (1968). The application of the method to some more realistic models of stellar systems will be described in a later paper.

2. DERIVATION OF THE MOMENT EQUATIONS

We consider a system with spherical symmetry described by the spherical polar coordinates r, θ, and ϕ, and we denote by u, v, and w the corresponding velocity components. We denote by $\rho(r)$ the mass density of stars as a function of r, and by $\langle u \rangle (r)$ the mean velocity in the radial direction. For a system with spherical symmetry, the mean tangential velocity components $\langle v \rangle$ and $\langle w \rangle$ must vanish, as must all other moments of odd order in v or w. Also, since the two tangential coordinate directions are equivalent, the value of any moment is unaffected by interchanging v and w.

In addition to the density of stars ρ and the mean radial velocity $\langle u \rangle$, we define the following higher moments of the velocity distribution, which we adopt as the fundamental parameters characterizing the velocity distribution:

$$\left.\begin{aligned} \alpha &\equiv \langle (u - \langle u \rangle)^2 \rangle \\ \beta &\equiv \langle v^2 \rangle = \langle w^2 \rangle \\ \epsilon &\equiv \langle (u - \langle u \rangle)^3 \rangle \\ \zeta &\equiv \langle (u - \langle u \rangle)^4 \rangle . \end{aligned}\right\} \tag{1}$$

Instead of using the fourth moment ζ itself, it will be convenient to write the moment equations in terms of a variable ξ, defined as the difference between ζ and the value which ζ would have for a Maxwellian velocity distribution, viz. $3\alpha^2$:

$$\xi \equiv \zeta - 3\alpha^2. \tag{2}$$

In equation (1), α and β are just the mean squared random velocities in the radial and transverse directions, and the third moment ϵ represents an energy transport or 'heat flow' in the radial direction, as was discussed in Section 1. The quantity ξ defined in equation (2) may be thought of as representing an excess or deficiency of high velocity stars relative to a Maxwellian velocity distribution; a positive value of ξ corresponds to an excess of high velocity stars, while a negative ξ corresponds to a deficiency. As will be seen later (Section 5), the existence of a finite ξ is of fundamental importance, and the tendency of ξ to relax toward zero is what provides the 'driving force' for the evolution of the system.

The moment equations for ρ, $\langle u \rangle$, α, β, ϵ, and ξ may be derived by the usual procedure of multiplying the Boltzmann equation by successively higher powers of the velocity components u, v, w, and integrating over the velocity distribution. In the resulting set of equations, the following moments are found to occur in addition to those already defined: $\langle (u - \langle u \rangle) v^2 \rangle$, $\langle (u - \langle u \rangle)^2 v^2 \rangle$, $\langle (u - \langle u \rangle)^5 \rangle$, $\langle (u - \langle u \rangle)^3 v^2 \rangle$, and the corresponding moments with w^2 in place of v^2. In order to close the system of equations, some approximate way must be found of relating these moments to the ones already defined. In order to do this, we introduce the following approximate representation of the velocity distribution, which will also be used in Section 3 in evaluating relaxation effects from the Fokker–Planck equation.

2.1 *The assumed form of the velocity distribution*

Let V denote the magnitude of the random velocity vector, and μ the cosine of the angle between this vector and the r-direction. We then have

$$(u - \langle u \rangle) = V\mu, \qquad v^2 + w^2 = V^2(1 - \mu^2).$$

Let $f(V, \mu)$ be the normalized distribution of random velocities; the normalization condition is

$$2\pi \int_0^\infty V^2 dV \int_{-1}^{+1} d\mu f(V, \mu) = 1. \tag{3}$$

Since by definition we have $\langle (u - \langle u \rangle) \rangle = 0$, $f(V, \mu)$ must satisfy the further condition that

$$\langle V\mu \rangle = 2\pi \int_0^\infty V^3 dV \int_{-1}^{+1} \mu d\mu f(V, \mu) = 0. \tag{4}$$

In statistical equilibrium the velocity distribution approaches a Maxwellian distribution, which in normalized form may be written

$$g(V) = (2\pi b)^{-3/2} e^{-V^2/2b}. \tag{5}$$

Since the actual velocity distribution $f(V, \mu)$ is expected to be roughly approximated by a Maxwellian distribution, at least in the central part of a cluster where the relaxation effects are most important, we shall adopt the Maxwellian distribution (5) as a zero-order approximation to $f(V, \mu)$. We choose the value of b such that $g(V)$ has the same kinetic energy of random motions as the actual velocity distribution $f(V, \mu)$; this gives

$$b = \frac{\alpha + 2\beta}{3}. \tag{6}$$

We assume for the moment that the deviation of $f(V, \mu)$ from a Maxwellian distribution is small and may be adequately represented by an expansion in Legendre polynomials, retaining only the first three terms in the series. We then write

$$f(V, \mu) = g(V) + \sum_{n=0}^{2} a_n(V) P_n(\mu), \tag{7}$$

where the $P_n(\mu)$ are Lengendre polynomials and the coefficients $a_n(V)$ are imagined to be small ' corrections ' to the Maxwellian distribution $g(V)$. Equation (7) may be thought of as providing a first-order approximation to the velocity distribution, valid in the limit of small deviations from a Maxwellian distribution. As will be seen (equations (9) and (10)), $a_0(V)$ is related to the quantity ξ defined in equation (2), $a_1(V)$ is related to the energy flux parameter ϵ, and $a_2(V)$ is related to the anisotropy parameter $(\alpha - \beta)$; thus three terms in the Legendre expansion is the minimum number required to adequately represent the essential features of the velocity distribution.

We now assume that the functions $a_n(V)$ may be approximated as the product of $g(V)$ by a power series in V, where only the first few terms in the power series are retained. Since the velocity distribution must be of the form

$$f(V, \mu) = f'(u - \langle u \rangle, v^2 + w^2) = f'(V\mu, V^2(1-\mu^2)), \tag{8}$$

i.e. must be a function of V^2 and $V\mu$ only, it is clear that $a_0(V)$ and $a_2(V)$ must contain only even powers of V, whereas $a_1(V)$ must contain only odd powers of V. The simplest functions of this form which give non-zero values for ξ, ϵ, and $(\alpha - \beta)$ and still satisfy conditions (3), (4), (6), and (8) are as follows:

$$\left.\begin{aligned} a_0(V) &= c_0 \left(1 - \frac{2V^2}{3b} + \frac{V^4}{15b^2}\right) g(V) \\ a_1(V) &= c_1 \frac{V}{b^{1/2}} \left(-1 + \frac{V^2}{5b}\right) g(V) \\ a_2(V) &= c_2 \frac{V^2}{b} g(V). \end{aligned}\right\} \tag{9}$$

If we now evaluate the moments α, β, ϵ, and ξ for the velocity distribution defined by equations (7) and (9), we obtain the following relations giving the constants

c_0, c_1, and c_2 in terms of the moments of the velocity distribution:

$$\left.\begin{aligned} c_0 &= \frac{5}{8}\frac{\xi+\frac{4}{3}(\alpha-\beta)^2}{b^2} \\ c_1 &= \frac{5}{6}\frac{\epsilon}{b^{3/2}} \\ c_2 &= \frac{\alpha-\beta}{3b}. \end{aligned}\right\} \tag{10}$$

Using equations (7), (9), and (10), we can now evaluate the unknown moments required to close the system of moment equations. We obtain

$$\left.\begin{aligned} \langle(u-\langle u\rangle)v^2\rangle &= \epsilon/3 \\ \langle(u-\langle u\rangle)^2v^2\rangle &= \zeta/3-b(\alpha-\beta) \\ \langle(u-\langle u\rangle)^5\rangle &= 10b\epsilon \\ \langle(u-\langle u\rangle)^3v^2\rangle &= 2b\epsilon. \end{aligned}\right\} \tag{11}$$

Equations (11) are expected to be valid in the limit of a nearly Maxwellian velocity distribution; in particular, they are expected to be valid if ϵ and $(\alpha-\beta)$ are small. It is found in practice, however, that the velocity distribution always tends to become strongly anisotropic in the outer parts of a cluster, in the sense that β becomes $\ll\alpha$ (ϵ remains small). In an attempt to provide a better approximation for such circumstances, we assume that the effect of a large anisotropy of the velocity distribution may be represented as a scale change by a factor $(\beta/\alpha)^{1/2}$ in the v and w directions. The moments containing v^2 in equations (11) then become equal to β/α times the values which they would have for $\alpha = \beta$. Thus we finally adopt

$$\left.\begin{aligned} \langle(u-\langle u\rangle)v^2\rangle &= \frac{\beta}{\alpha}\frac{\epsilon}{3} \\ \langle(u-\langle u\rangle)^2v^2\rangle &= \frac{\beta}{\alpha}\frac{\zeta}{3} \\ \langle(u-\langle u\rangle)^5\rangle &= 10\alpha\epsilon \\ \langle(u-\langle u\rangle)^3v^2\rangle &= 2\beta\epsilon. \end{aligned}\right\} \tag{12}$$

2.2 *The moment equations*

For a system with spherical symmetry and a distribution function of the form $f(r, u, v, w, t)$, the Boltzmann equation may be written (cf. Larson 1969)

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial r}+\dot{u}\frac{\partial f}{\partial u}+\dot{v}\frac{\partial f}{\partial v}+\dot{w}\frac{\partial f}{\partial w} = \left(\frac{\partial f}{\partial t}\right)_c, \tag{13}$$

where

$$\left.\begin{aligned} \dot{u} &= -\frac{\partial\Phi}{\partial r}+\frac{v^2+w^2}{r} \\ \dot{v} &= -\frac{uv}{r}+\frac{w^2}{r\tan\theta} \\ \dot{w} &= -\frac{uw}{r}-\frac{vw}{r\tan\theta}. \end{aligned}\right\} \tag{14}$$

In equation (13) the term $(\partial f/\partial t)_c$ represents the rate of change of f due to the effects of encounters between the stars.

The required moment equations may be derived by multiplying equation (13) successively by 1, u, $(u-\langle u\rangle)^2$, v^2, $(u-\langle u\rangle)^3$ and $(u-\langle u\rangle)^4$, and integrating over the velocity distribution, making use of the definitions (1) and (2) and the assumed relations (12). After some reductions, the following equations are obtained:

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho\langle u\rangle = 0 \tag{15}$$

$$\frac{\partial\langle u\rangle}{\partial t}+\langle u\rangle\frac{\partial\langle u\rangle}{\partial r}+\frac{1}{\rho}\frac{\partial}{\partial r}\rho\alpha+\frac{2}{r}(\alpha-\beta)+\frac{\partial\Phi}{\partial r} = 0 \tag{16}$$

$$\frac{\partial\alpha}{\partial t}+\langle u\rangle\frac{\partial\alpha}{\partial r}+2\alpha\frac{\partial\langle u\rangle}{\partial r}+\frac{1}{\rho}\frac{\partial}{\partial r}\rho\epsilon+\frac{2\epsilon}{r}\left(1-\frac{2}{3}\frac{\beta}{\alpha}\right) = \left(\frac{\partial\alpha}{\partial t}\right)_c \tag{17}$$

$$\frac{\partial\beta}{\partial t}+\langle u\rangle\frac{\partial\beta}{\partial r}+2\beta\frac{\langle u\rangle}{r}+\frac{1}{3\rho}\frac{\partial}{\partial r}\frac{\beta}{\alpha}\rho\epsilon+\frac{4}{3}\frac{\beta}{\alpha}\frac{\epsilon}{r} = \left(\frac{\partial\beta}{\partial t}\right)_c \tag{18}$$

$$\frac{\partial\epsilon}{\partial t}+\langle u\rangle\frac{\partial\epsilon}{\partial r}+3\epsilon\frac{\partial\langle u\rangle}{\partial r}+3\alpha\frac{\partial\alpha}{\partial r}+\frac{1}{\rho}\frac{\partial}{\partial r}\rho\xi+\frac{2\xi}{r}\left(1-\frac{\beta}{\alpha}\right) = \left(\frac{\partial\epsilon}{\partial t}\right)_c \tag{19}$$

$$\frac{\partial\xi}{\partial t}+\langle u\rangle\frac{\partial\xi}{\partial r}+4\xi\frac{\partial\langle u\rangle}{\partial r}+6\epsilon\frac{\partial\alpha}{\partial r}+4\alpha\frac{\partial\epsilon}{\partial r} = \left(\frac{\partial\xi}{\partial t}\right)_c. \tag{20}$$

The terms $(\partial\alpha/\partial t)_c$, etc., representing the effects of encounters, will be evaluated in the following section.

3. EVALUATION OF RELAXATION EFFECTS

In treating the effects of encounters between the stars, the masses of the stars play an important role, and it seems clear (Wielen 1968) that in order to obtain realistic results it is necessary to include a realistic mass spectrum. This could be incorporated in the present method by considering several ‘ populations ’ of stars with different masses, each described by its own set of moment equations. While this refinement would pose no difficulty in principle, the computational requirements would be increased considerably; therefore, in the present exploratory investigation we shall consider only the case of equal masses.

In computing the effect of encounters on the velocity distribution, we return to the representation of the velocity distribution given in equation (7), and we again assume that the velocity distribution is nearly Maxwellian. It is hoped that this will provide an adequate approximation for computing the relaxation effects, since it is expected that large deviations from a Maxwellian velocity distribution will occur only in the outer parts of a cluster where relaxation effects are unimportant anyway.

The form of the Fokker–Planck equation has been worked out in detail by Rosenbluth, MacDonald & Judd (1957) for the case of an axially symmetric velocity distribution which is given as an expansion in Legendre polynomials. In order to make use of their result, it will be convenient to rewrite equation (7) slightly as follows:

$$f(V,\mu) = \sum_{n=0}^{2} A_n(V)P_n(\mu), \tag{21}$$

where $A_0(V) = g(V) + a_0(V)$, $A_1(V) = a_1(V)$, and $A_2(V) = a_2(V)$. If equation (21) is substituted into the equations of Rosenbluth *et al.*, the resulting expression for the encounter term $(\partial f/\partial t)_c$ in the Boltzmann equation also takes the form of an expansion in Legendre polynomials:

$$\left(\frac{\partial f}{\partial t}\right)_c = \sum_{n=0}^{2} \left(\frac{\partial A_n}{\partial t}\right)_c P_n(\mu). \tag{22}$$

Here the coefficients $(\partial A_n/\partial t)_c$ are given by formulas involving integration and differentiation with respect to V only. In general these formulas are exceedingly complicated and unwieldy, even with only 3 terms in the Legendre expansion, and in order to make the calculations more tractable it is necessary to introduce some simplifications.

The Fokker–Planck equation specifies the rate at which the distribution function f of a population of 'test particles' changes as they encounter a background population of 'target particles'. In the present case the target particles are identical with the test particles, and they have the same velocity distribution, as given by equation (21). To simplify the calculations, however, we shall assume that the target particles have an isotropic velocity distribution equal to $A_0(V)$, the isotropic term in the Legendre expansion (21). This simplification should still allow a reasonably accurate determination of the most important relaxation effect, which involves $a_0(V)$ and ξ. With this assumption, the equations simplify considerably and we obtain, using equations (31), (41), and (46) of Rosenbluth *et al.* (1957),

$$\left(\frac{\partial A_n}{\partial t}\right)_c = \frac{C}{V^2}\left\{\frac{d}{dV}\left[A_n I_0 + \frac{1}{3V}(J_0 + V^3 K_0)\frac{dA_n}{dV}\right] - \frac{n(n+1)}{6V^3}(3V^2 I_0 - J_0 + 2V^3 K_0)A_n\right\} \tag{23}$$

where

$$\left.\begin{aligned} I_0(V) &= \int_0^V v^2 A_0(v)\, dv \\ J_0(V) &= \int_0^V v^4 A_0(v)\, dv \\ K_0(V) &= \int_V^\infty v A_0(v)\, dv. \end{aligned}\right\} \tag{24}$$

In the present situation, where we have gravitational interactions between stars of mass m, the constant C is given by

$$C = 16\pi^2 G^2 m\rho \ln\left(\frac{D_{\max}\langle V^2\rangle}{2Gm}\right), \tag{25}$$

where $D_{\max}$ is the maximum impact parameter for an encounter. Cohen *et al.* (1950) showed that $D_{\max}$ should be taken not as the mean interparticle distance, as was originally assumed by Chandrasekhar (1942), but as the largest distance over which the particles can interact; in the case of a stellar system, $D_{\max}$ should therefore be set equal to the size of the system, or perhaps more accurately to the size of the central core which contains most of the stars. Fortunately, the logarithmic factor in equation (25) is quite insensitive to its argument, and to sufficient accuracy it may be set equal to a constant value for the whole cluster.

We can now evaluate the relaxation terms $(\partial\alpha/\partial t)_c$, etc., by forming the moments α, β, ϵ, and ζ of the velocity distribution (21) and differentiating with respect to time, using equations (22), (23), and (24). In performing these calculations we assume that $a_0(V)$, $a_1(V)$, and $a_2(V)$ are all small compared with $g(V)$, and we neglect all terms of second order in $a_0(V)$, $a_1(V)$, and $a_2(V)$. After some rather lengthy manipulation of integrals, we obtain the following results:

$$\left(\frac{\partial\alpha}{\partial t}\right)_c = -\frac{8\pi C}{15}\int_0^\infty \left(5VI-\frac{J}{V}\right) a_2(V)\, dV \tag{26}$$

$$\left(\frac{\partial\beta}{\partial t}\right)_c = +\frac{4\pi C}{15}\int_0^\infty \left(5VI-\frac{J}{V}\right) a_2(V)\, dV \tag{27}$$

$$\left(\frac{\partial\epsilon}{\partial t}\right)_c = -\frac{8\pi C}{15}\int_0^\infty (6V^2I-2J-5V^3K)\, a_1(V)\, dV \tag{28}$$

$$\left(\frac{\partial\zeta}{\partial t}\right)_c = -\frac{16\pi C}{105}\int_0^\infty \{7[3V^3I-7VJ-7V^4K+3V^2(2b^2+bV^2)\,g(V)]\, a_0(V) + [21V^3I-11VJ-14V^4K]a_2(V)\}\, dV \tag{29}$$

where

$$\left.\begin{aligned} I(V) &= \int_0^V v^2 g(v)\, dv \\ J(V) &= \int_0^V v^4 g(v)\, dv \\ K(V) &= \int_V^\infty v g(v)\, dv. \end{aligned}\right\} \tag{30}$$

Finally, if we substitute the assumed forms of the functions $a_0(V)$, $a_1(V)$, and $a_2(V)$ as given by equations (9) and (10) into equations (26)–(29), we obtain, after considerable reduction,

$$\left(\frac{\partial\alpha}{\partial t}\right)_c = -\frac{4C(\alpha-\beta)}{45(\pi b)^{3/2}} \tag{31}$$

$$\left(\frac{\partial\beta}{\partial t}\right)_c = +\frac{2C(\alpha-\beta)}{45(\pi b)^{3/2}} \tag{32}$$

$$\left(\frac{\partial\epsilon}{\partial t}\right)_c = -\frac{29C\epsilon}{480(\pi b)^{3/2}} \tag{33}$$

$$\left(\frac{\partial\xi}{\partial t}\right)_c = -\frac{C[7\xi-15\alpha(\alpha-\beta)]}{105(\pi b)^{3/2}}, \tag{34}$$

where a term containing $(\alpha-\beta)^2$ has been neglected in equation (34).

We note that, by combining equations (31) and (32), we may write

$$\left(\frac{\partial(\alpha-\beta)}{\partial t}\right)_c = -\frac{2C(\alpha-\beta)}{15(\pi b)^{3/2}}. \tag{35}$$

Also, if $(\alpha-\beta) = 0$, equation (34) becomes

$$\left(\frac{\partial\xi}{\partial t}\right)_c = -\frac{C\xi}{15(\pi b)^{3/2}}. \tag{36}$$

From equations (33), (35), and (36) it is evident that, in the absence of other effects, the effect of encounters between the stars is to make the three quantities $(\alpha-\beta)$, ϵ, and ξ decay exponentially with different time constants. These time constants are related to the relaxation time T defined by Chandrasekhar (1942, equation 2.379) and often used as a reference time in stellar dynamics:

$$T = \frac{1}{16}\left(\frac{3}{\pi}\right)^{1/2} \frac{\langle V^2\rangle^{3/2}}{G^2 m\rho \ln(D_{\max}\langle V^2\rangle/2Gm)}. \tag{37}$$

From equations (25) and (37) we have, putting $\langle V^2\rangle = 3b$,

$$T = \frac{9(\pi b)^{3/2}}{C}. \tag{38}$$

Using equation (38), we can now write equations (35), (33) and (36) as follows:

$$\left(\frac{\partial \ln|\alpha-\beta|}{\partial t}\right)_c^{-1} = -\frac{5}{6}T \tag{39}$$

$$\left(\frac{\partial \ln|\epsilon|}{\partial t}\right)_c^{-1} = -\frac{160}{87}T \tag{40}$$

$$\left(\frac{\partial \ln|\xi|}{\partial t}\right)_c^{-1} = -\frac{5}{3}T. \tag{41}$$

Equations (39)–(41) allow a more precise interpretation of the relaxation time T in terms of the time constants for decay of the quantities $(\alpha-\beta)$, ϵ, and ξ, which represent different types of deviations from a Maxwellian velocity distribution. We note that, as expected, the decay times are of the same order as the relaxation time T, but there appears to be a tendency for the higher moments of the velocity distribution to relax more slowly than the lower ones. This may be attributable to the fact that the higher moments give more weight to the stars with high velocities, for which the relaxation effects are weaker than for low velocity stars.

Substitution of equations (31)–(34) into equations (15)–(20) now completes the system of equations required to compute the time evolution of a spherical stellar system, given appropriate initial conditions and boundary conditions. The equations have been solved using numerical techniques similar in concept to methods which have often been used for hydrodynamical problems with spherical symmetry. An Eulerian grid structure is chosen, and the grid points are spaced at equal intervals of 0·1 in log r. The odd-order moments $\langle u\rangle$ and ϵ and the mass m inside radius r are considered to be defined at the grid points, and the even-order moments ρ, α, β, and ξ are considered to be defined at a second set of points halfway between the grid points. To ensure stability, the difference equations are written in implicit form with backward time differences, and they are solved iteratively by the Newton–Raphson technique. The method of constructing difference expressions is quite similar to that which was used by Larson (1969) in studying the formation of a spherical galaxy. The space and time steps have been chosen such that numerical accuracies are generally of the order of 10–20 per cent, which is probably better than the intrinsic accuracy of the theory.

4. APPLICATION TO THE 'GRAVOTHERMAL CATASTROPHE' FOR AN ISOTHERMAL SPHERE

In order to obtain some insight into the evolutionary behaviour of stellar systems, Lynden-Bell & Wood (1968) considered an idealized example consisting of a system of self-gravitating particles enclosed in a rigid spherical container with perfectly reflecting walls. In the presence of relaxation effects which tend to produce a Maxwellian velocity distribution, such a system can be in equilibrium only if its structure is that of an isothermal sphere, for which there is a one-parameter family of possibilities depending on the central concentration ρ_c/ρ_b (ρ_c = central density, ρ_b = boundary density). The stability of a bounded isothermal sphere in the presence of a mechanism capable of producing heat flow effects was investigated by Lynden-Bell & Wood (LW), and they showed that there is a critical central concentration $\rho_c/\rho_b = 709$ such that for $\rho_c/\rho_b < 709$ the isothermal sphere is in stable equilibrium, whereas for $\rho_c/\rho_b > 709$ the equilibrium becomes unstable in the sense that if the system is perturbed it will evolve away from an isothermal sphere and will continue evolving indefinitely with ever increasing central concentration. This phenomenon was referred to by LW as the 'gravothermal catastrophe' of an isothermal sphere.

It happens that the idealized problem studied by LW (i.e., a system enclosed by rigid reflecting walls) is also the easiest problem to compute by the technique developed in this paper; because of this, and because of its current interest, we shall use it as an example of the application of the present method.

As boundary conditions for the numerical calculations, it is necessary to specify the values of the moments $\langle u \rangle$ and ϵ at the outer boundary $r = R$. In the present case the existence of a perfectly reflecting wall at the outer boundary requires that all odd moments of the velocity distribution must vanish at this point; thus the required boundary conditions are $\langle u \rangle = 0$ and $\epsilon = 0$ at $r = R$. These boundary conditions imply that the total mass M and the total energy E of the system remain constant as it evolves.

In the calculations to be described, we have adopted a total mass of $10^6 M_\odot$ and a radius of 100 pc, roughly the values appropriate for a globular cluster. The logarithmic factor in equation (25) has been set equal to a constant value of 10 throughout the calculations. We have computed the evolution of a number of examples starting from artificially constructed non-isothermal initial configurations with central concentrations ρ_c/ρ_b smaller than that for an isothermal sphere of the same energy. As expected, the evolution always proceeds in the direction of increasing central concentration, and the structure approaches that of an isothermal sphere. The results confirm the prediction of LW that for a system whose energy E is greater than the minimum energy for an isothermal sphere of mass M and radius R ($E_{\min} = -0.335\, GM^2/R$), the system evolves into an isothermal sphere and remains thereafter in a stable isothermal equilibrium configuration. However, if the total energy is less than $E_{\min}$, a strictly isothermal state is never attained, and the central concentration continues to increase indefinitely at an ever accelerating rate. From such calculations the limit of stability for an isothermal sphere may be determined empirically; for the limiting stable configuration we obtain a central density of about $47\, M_\odot/\text{pc}^3$ and a value for α of about $22.5\ (\text{pc}/10^6\ \text{yr})^2$. The central concentration ρ_c/ρ_b is about 730, in good agreement with the theoretical value of 709 (the difference is attributable to numerical inaccuracies in the present calculations).

To illustrate the density runaway or ' gravothermal catastrophe ' qualitatively described by LW, we have taken as a starting model the limiting stable isothermal sphere described above, and we have perturbed it slightly by decreasing α by about 0·2 per cent, thereby decreasing E below $E_{\min}$ and making the system unstable. The evolution has been followed through an increase of about nine orders of magnitude in the central density, corresponding to a time interval of about 4×10^{13} yr. Because the time scale becomes greatly compressed as the evolution proceeds, it will be convenient to measure time from the instant t_0 when the central density becomes infinite; this instant is readily determined by extrapolation. We then define $\tau \equiv t_0 - t$, and we hereafter use τ as the time variable in place of t.

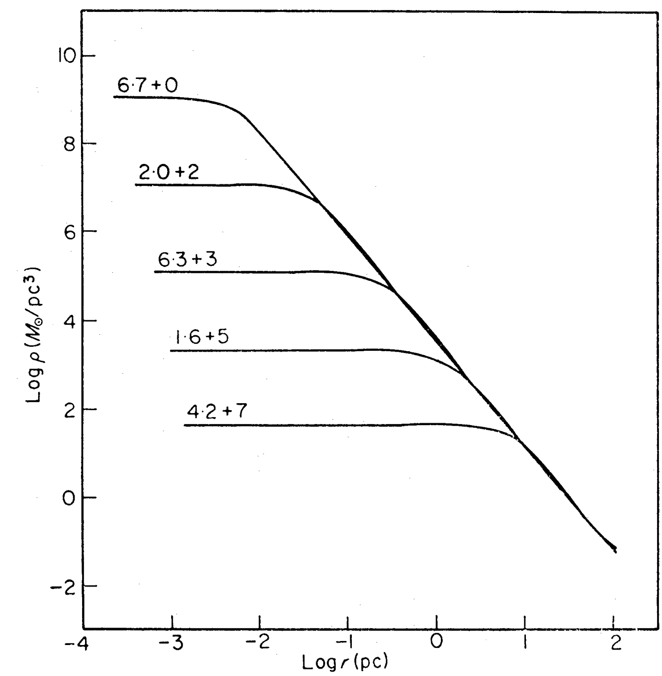

FIG. 1. *The evolution of the density distribution during the ' gravothermal catastrophe '. The curves are labelled with the corresponding values of* $\tau(\equiv t_0 - t)$ *in units of* 10^6 *yr.*

The evolution of the density distribution is illustrated in Fig. 1, which shows the density distribution for several values of τ. It is evident first of all that the density distribution never deviates very greatly from that of an isothermal sphere; it evolves mainly by an increase in central density, while maintaining nearly the same shape in a logarithmic plot. Throughout the evolution the density distribution over the major part of the system is approximately represented by the law

$$\rho(r) \propto r^{-2\cdot 4}. \qquad (42)$$

The rapid acceleration of the evolution with increasing central density is evident from the values of τ marked on the curves in Fig. 1. However, it is also to be noted that the evolution remains quite slow in comparison with other time scales of interest; even at the latest time shown in Fig. 1 ($\tau = 6\cdot 7\times 10^6$ yr, $\rho_c = 1\cdot 2\times 10^9\, M_{\odot}/\text{pc}^3$), the time scale for the evolution is still three orders of magnitude longer than the relaxation time at the centre ($7 \cdot 5 \times 10^3$ yr) and four-and-a-half orders of magnitude longer than the free-fall time ($2 \cdot 3 \times 10^2$ yr). Thus the evolution is perhaps less ' catastrophic ' than was imagined by Lynden-Bell & Wood.

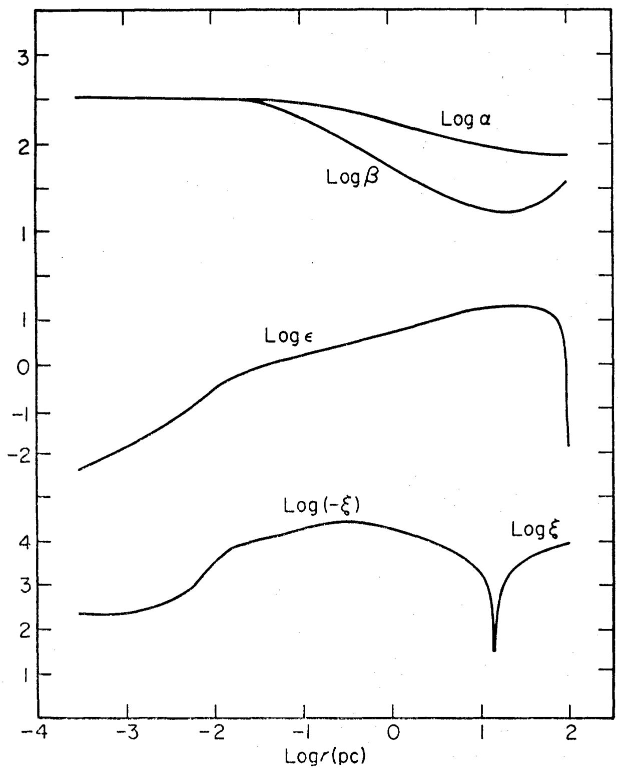

FIG. 2. *The variation with radius of* α, β, ϵ, *and* ξ *at* $\tau = 6 \cdot 7 \times 10^6$ *yr. The unit of velocity is* 1 *pc*/10^6 *yr* = 0·978 *km* s^{-1}.

Fig. 2 shows the variation with radius of α, β, ϵ, and ξ at the latest time shown in Fig. 1 ($\tau = 6 \cdot 7 \times 10^6$ yr). We note first that α decreases outward, as expected, but its variation is quite small compared with the variation in the density—only about a factor of 4·5, compared with a factor of about 2×10^{10} in the density. We note also that the velocity distribution has developed a significant anisotropy in the outer part of the cluster; the ratio β/α reaches a minimum value of about 0·22 at $r \simeq 15$ pc. The third moment ϵ is positive throughout the cluster, as expected, indicating an outward flux of energy. The quantity ξ is negative in the inner part of the system, indicating a deficiency of high velocity stars relative to a Maxwellian distribution, and is positive in the outermost part of the system, indicating an excess of high velocity stars. The deficiency of high velocity stars in the inner part of the system is expected from classical theories of stellar dynamics, which predict that the high velocity ' tail ' of a hypothetical Maxwellian velocity distribution should be depleted due to the escape of the most energetic stars from the central part of the system. In the present case the ' escaping stars ' accumulate in the outermost part of the system, where they contribute to producing an excess of high velocity stars.

The time development of the central density and temperature is illustrated in Fig. 3, which shows log ρ_c and log α_c plotted vs. log τ. The curves start out with a small slope, since the system is initially very close to an equilibrium isothermal structure and therefore evolving quite slowly. As the system becomes less nearly

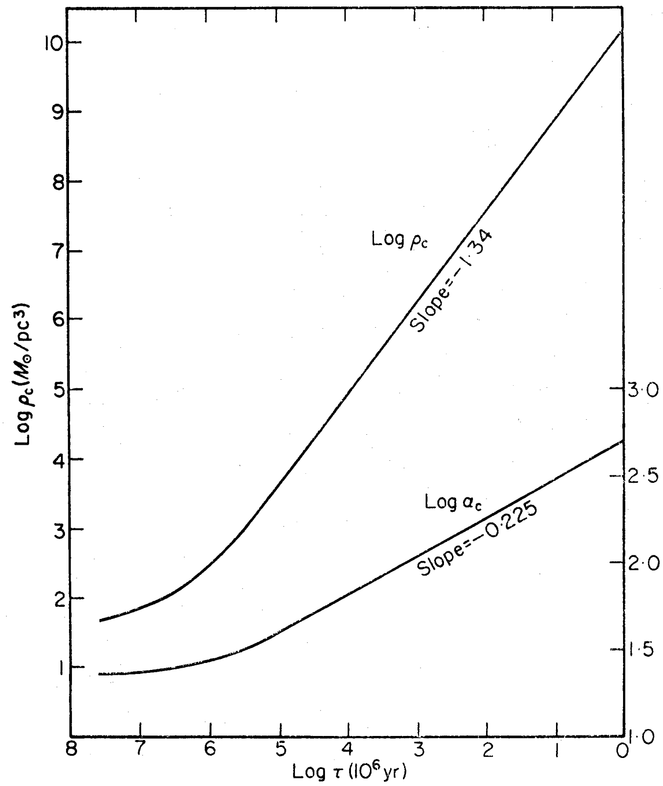

FIG. 3. *The time variation of the central density ρ_c and the central ' temperature ' α_c.*

isothermal the evolution speeds up, and for $\tau \lesssim 10^{10}$ yr the slopes $d \log \rho_c / d \log \tau$ and $d \log \alpha_c / d \log \tau$ approach constant values of $-1{\cdot}34$ and $-0{\cdot}225$ respectively; thus for the later stages of the evolution we have

$$\left.\begin{aligned} \rho_c &\propto \tau^{-1\cdot 34} \\ \alpha_c &\propto \tau^{-0\cdot 225}. \end{aligned}\right\} \tag{43}$$

It is also of interest to note the relation between τ and the central relaxation time T_c. At the initial instant, T_c is $3{\cdot}6 \times 10^9$ yr and the ratio τ / T_c is about $1{\cdot}2 \times 10^4$; as the evolution proceeds this ratio decreases, and for $\tau \lesssim 10^{10}$ yr it levels off at a constant value of

$$\frac{\tau}{T_c} = 8{\cdot}9 \times 10^2. \tag{44}$$

The constancy of $d \log \rho_c / d \log \tau$, $d \log \alpha_c / d \log \tau$, and τ / T_c for $\tau \lesssim 10^{10}$ yr is related to the fact that during the later stages of the evolution the central part of the system appears to settle into a state of ' homologous evolution ' in which the structure of the central region remains invariant except for scale factors which vary with time.

5. DISCUSSION

Since the evolution of the system is caused primarily by a flow of energy outward from the centre, it is clear that the rate of evolution depends on the magnitude of the energy flux parameter ϵ. One might think that the magnitude of ϵ would be determined primarily by equation (19), but this is not the case, since it turns out on examination of the results that ϵ is so small that the terms containing ϵ in equation (19) are always negligibly small in comparison with the other terms. Thus equation (19) serves primarily as a relation between ξ, α, β, ρ. As a simple example of the implications of equation (19) when ϵ is negligibly small, we note that if ξ is also negligibly small we must have $\partial\alpha/\partial r = 0$; this is just the well-known result that a Maxwellian velocity distribution implies an isothermal temperature distribution. As a second example, if $\alpha-\beta = 0$ and $\xi = -[3/(n+2)]\alpha^2$, then it can be shown from equations (16) and (19) that in hydrostatic equilibrium the system must be a polytropic sphere of index n. Thus it is clear that the magnitude of ξ and its variation with r are closely related to the overall structure of the system, as determined by the functions $\rho(r)$ and $\alpha(r)$.

Coming back to the question of what determines ϵ, it appears from the results that ϵ is primarily determined by equation (20). Near the centre of the system all of the terms in equation (20) turn out to be negligibly small except the terms $4\alpha(\partial\epsilon/\partial r)$ and $(\partial\xi/\partial t)_c$. Since $\alpha-\beta = 0$ near the centre, $(\partial\xi/\partial t)_c$ is given by equation (36); equation (20) then becomes

$$4\alpha\,\frac{\partial\epsilon}{\partial r} = -\frac{C\xi}{15(\pi b)^{3/2}}. \tag{45}$$

From equation (45) we see that the values of $\partial\epsilon/\partial r$ and hence ϵ near the centre of the system are proportional to the value of ξ at the centre; thus, as might be expected, the rate of evolution of the system depends on the extent of the deviation from a Maxwellian velocity distribution, as indicated by the magnitude of the parameter ξ. The magnitude of ξ, in turn, is tied up with the overall structure of the system, as we have seen in the preceding paragraph; thus the rate of evolution ultimately depends on the structure of the whole system, or at least on the structure of an extended central region. The slow rate of evolution indicated by equation (44) may then be understood as being related to the fact that in the 'homologous evolution' attained after $\tau \sim 10^{10}$ yr, the central region is nearly isothermal (see Fig. 2) and the velocity distribution is very nearly Maxwellian ($\xi/\alpha^2 = -2\cdot1\times10^{-3}$ at the centre).

The size of the central isothermal region is presumably limited by the tendency of an isothermal sphere to become unstable and evolve away from an isothermal sphere if the central concentration becomes too large, as was discussed by Lynden-Bell & Wood (1968) for the case of a bounded isothermal sphere. Thus the structure and evolution of the system may be thought of as resulting from a balance between two counteracting effects—the tendency to relax toward an isothermal structure, and the fundamental instability of an isothermal sphere when the density contrast becomes too large.

It is interesting to note that the results given in equation (43) are nearly the same as the corresponding relations predicted by the theory of Von Hoerner (1968):

$$\rho_c \propto \tau^{-4/3}$$

$$\alpha_c \propto \tau^{-2/9}.$$

However, the ratio τ/T_c as given in equation (44) is much larger than the value 20 adopted by Von Hoerner on the basis of some n-body calculations. This may be because Von Hoerner's n-body calculations have not reached the nearly isothermal, slowly evolving state obtained in the present results. Also, the relaxation time T is shortest at the centre of the system and becomes progressively longer as we go outward; thus a representative relaxation time for the central core of the system would be somewhat longer than T_c, and this would lead to a smaller value of τ/T than that given in equation (44), perhaps by as much as an order of magnitude. However, it is difficult to define uniquely either the size of the central core or a representative relaxation time for it, so this will not be attempted here.

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