

A METHOD FOR CREATING MATERIALS WITH A DESIRED REFRACTION COEFFICIENT

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It is proposed to create materials with a desired refraction coefficient in a bounded domain $D \subset \mathbb{R}^3$ by embedding many small balls with constant refraction coefficients into a given material. The number of small balls per unit volume around every point $x \in D$, i.e., their density distribution, is calculated, as well as the constant refraction coefficients in these balls. Embedding into D small balls with these refraction coefficients according to the calculated density distribution creates in D a material with a desired refraction coefficient.

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1. Introduction

In Refs. 6–15, it was proposed to create material with a desired refraction coefficient by embedding into a given material small particles with suitably chosen boundary impedances. It was proved that any desired refraction coefficient $n^2(x)$, $\Im n^2(x) \geq 0$, can be created in such a way in an arbitrary given bounded domain $D \subset \mathbb{R}^3$. Preparing small particle with a prescribed large boundary impedance may be a technologically challenging problem. In Refs. 1 and 2 numerical results are given. These results illustrate the efficiency of the author's method for solving many-body scattering problem in the case of small scatterers embedded in an inhomogeneous medium.

By this reason we propose in this paper a new method for creating materials with a desired refraction coefficient. We use wave scattering by many small particles (see Ref. 10).

This method for creating materials with a desired refraction coefficient $n^2(x)$ consists of embedding into a given material small particles (balls) with a suitably chosen density distribution of the embedded particles and a suitably chosen constant refraction coefficients of each of the embedded particles. No boundary impedances

are necessary to use in this method. Therefore, one hopes that the new method may be easier to implement in practice.

The density of the distribution of the embedded particles D_m and their constant refraction coefficients $\nu^2(x_m)$ are calculated given the desired refraction coefficient $n^2(x)$ and the refraction coefficient $n_0^2(x)$ of the original material in D .

In the literature there are many papers and books (see Ref. 5 and references therein) in which homogenization formulas of Bruggeman, Maxwell Garnett, and their numerous modifications are used to derive approximate formulas for the dielectric and magnetic parameters of various composite materials. Various bounds, for example, Hashin-Shtrikman bounds, are found for these formulas (see Ref. 5). These formulas are, for the most part, such that the homogenized material is characterized by constant parameters. The inhomogeneities are assumed, in most cases, randomly distributed in the medium. There are also many mathematical papers and books dealing with homogenization theory.^{3,4} Our approach differs from the published in several respects: we do not assume periodic structure of the medium, the small parameter is not entering the coefficients of the equations, the problems we are studying are non-selfadjoint and the operators involved have continuous, rather than discrete, spectrum.

The problem, discussed in our paper, is also different: the small inhomogeneities are not distributed randomly and our goal is not to derive the properties of the homogenized medium. On the contrary, we prescribe the desired “property of the medium”, which in our paper is the desired potential, and we give a method for creating a medium with the *a priori* prescribed “properties”. This method consists of embedding into a given medium many small inhomogeneities (particles) with constant parameters which vary from particle to particle. We prove that the density of the distribution of the embedded particles and their parameters, as functions of the positions of the embedded particles, can be chosen so that the limiting medium will have the desired “properties”, i.e., in our case, the desired potential. This is a “synthesis” problem, rather than an “analysis” problem. Our results are rigorous, and not approximate. They do not require the assumption, made in Bruggeman’s and Maxwell Garnett’s theories, about smallness of the relative volume of the embedded inhomogeneities.

Let us compare the new method with the method originally proposed in Refs. 6–11. Let a denote the characteristic size of the small particles D_m . We assume that all the particles have the same characteristic size. The physical properties of these particles D_m of the characteristic size a are described in Refs. 6–8 by the boundary impedances ζ_m of the particles D_m . The order of magnitude of ζ_m is $O(a^{-\kappa})$, as $a \rightarrow 0$, where $\kappa \in (0, 1]$ is a parameter a physicist can choose as he/she wishes, and the total number \mathcal{N} of the embedded particles is of the order $O(a^{-(2-\kappa)})$. For $\kappa = 1$, for example, this order is $O(a^{-1})$.

In the new method, proposed in this paper, the physical properties of the embedded small particles D_m are described by their constant refraction coefficients $\nu^2(x_m)$, where $x_m \in D_m$ is a point inside D_m . Since D_m is small, it does not matter

what point $x_m \in D_m$ is chosen. The boundary impedances are not used in the new method. The total number \mathcal{N} of the embedded particles in the new method is of the order $O(a^{-3})$, as $a \rightarrow 0$. This number is much larger than $O(a^{-(2-\kappa)})$, as $a \rightarrow 0$.

A possible disadvantage of the new method, compared with the original one, is the increase of the number of embedded small particles as $a \rightarrow 0$.

An advantage of the new method, compared with the original one, is, possibly, that it is easier technologically to prepare small particles with constant refraction coefficients than small particles with desired boundary impedances ζ_m .

Only experiments can show which of the two methods and for what practical goals is better to use.

In the recent paper¹⁴ a method, similar to the one proposed in this paper, has been used for creating non-relativistic quantum-mechanical potentials of a desired form.

Let us formulate the problem precisely. Assume that a bounded domain $D \subset \mathbb{R}^3$ is filled with a material with known refraction coefficient $n_0^2(x)$, $\Im n_0^2(x) \geq 0$, $n_0^2(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$, $n_0^2(x)$ is piecewise-continuous. Throughout this paper by piecewise-continuous function we mean a bounded function with the set of discontinuities of Lebesgue measure zero in \mathbb{R}^3 , and do not repeat this.

The waves satisfy the equation:

$$L_0 u_0 := [\nabla^2 + k^2 n_0^2(x)] u_0 = 0 \quad \text{in } \mathbb{R}^3, \quad k = \text{const} > 0, \tag{1}$$

$$u_0 = e^{ik\alpha \cdot x} + v. \tag{2}$$

Here v is the scattered field, satisfying the radiation condition:

$$v_r - ikv = o(r^{-1}) \quad r := |x| \rightarrow \infty, \tag{3}$$

where $\alpha \in S^2$ is the direction of the incident plane wave, and S^2 is the unit sphere in \mathbb{R}^3 . It is proved in Ref.⁶ that this scattering problem under the stated assumptions on $n_0^2(x)$ has a unique solution, and the function $G(x, y)$, satisfying the equation

$$L_0 G(x, y) = -\delta(x - y) \quad \text{in } \mathbb{R}^3, \tag{4}$$

and the radiation condition (3), does exist and is unique.

One can write

$$L_0 = \nabla^2 + k^2 - q_0(x),$$

where

$$q_0(x) := q_0(x, k) := k^2 - k^2 n_0^2(x), \quad q_0(x) = 0 \quad x \in D'. \tag{5}$$

Let $n^2(x)$ be a desired refraction coefficient in D . We assume that $n^2(x)$ is piecewise-continuous, $\Im n^2(x) \geq 0$, and $n^2(x) = 1$, $x \in D'$.

We wish to create material with the refraction coefficient $n^2(x)$ in D by embedding into D many small non-intersecting balls B_m , $1 \leq m \leq M$, of radius a , centered at the points $x_m \in D$, with constant refraction coefficients n_m^2 in B_m .

Smallness of the particles means that $ka \ll 1$.

Let $\Delta \subset D$ be any subdomain of D . We assume that the number of small particles, embedded in Δ , is given by the formula:

$$\mathcal{N}(\Delta) := \sum_{x_m \in \Delta} 1 = V_a^{-1} \int_{\Delta} N(x) dx [1 + o(1)], \quad a \rightarrow 0, \tag{6}$$

where $N(x) \geq 0$ is a piecewise-continuous function in D , and

$$V_a := \frac{4\pi a^3}{3}$$

is the volume of a ball of radius a .

Formula (6) gives the density distribution of the centers of the embedded small balls in D . The total number of these balls tends to infinity as $O(V_a^{-1}) = O(a^{-3})$ when $a \rightarrow 0$.

We assume that the total volume $V(D)$ of the embedded particles (balls) is not greater than $|D|$, where $|D|$ is the volume of D , i.e.,

$$V(D) = V_a \mathcal{N}(D) = \int_D N(x) dx [1 + o(1)] \leq |D|, \quad a \rightarrow 0.$$

This means physically that although $N(x)$ can be large in some subdomains of D , its average over D is not greater than unity.

The scattering problem in the case of the embedded into D particles is:

$$\left(L_0 + k^2 \sum_{m=1}^M n_m^2 \chi_m \right) U = 0 \quad \text{in } \mathbb{R}^3, \tag{7}$$

where χ_m is the characteristic function of the ball B_m , i.e., $\chi_m = 1$ in B_m , $\chi_m = 0$ in $B'_m := \mathbb{R}^3 \setminus B_m$, and

$$U = u_0 + V, \tag{8}$$

where V satisfies the radiation condition (3), u_0 solves the scattering problem in the absence of the embedded particles, i.e., when $M = 0$, and

$$n_m^2 = \nu^2(x_m).$$

Here $\nu^2(x)$ is some piecewise-continuous function in D , $\Im \nu^2(x) \geq 0$.

The solution $U(x) = U_a(x)$ to problem (7) and (8) depends on the parameter a , and the number M of the embedded particles depends also on a ,

$$M = O(V_a^{-1}) = O(a^{-3}),$$

so $M \rightarrow \infty$ at a prescribed rate as $a \rightarrow 0$. We are interested in the limiting behavior of $U(x) = U_a(x)$ as $a \rightarrow 0$. Our basic result, Theorem 1, below, says that the limit

$$\lim_{a \rightarrow 0} U_a(x) := u_e(x), \tag{9}$$

does exist and satisfies the integral equation (17) in Theorem 1.

From (7) and (8) one gets

$$U(x) = u_0(x) + k^2 \sum_{m=1}^M n_m^2 \int_{B_m} G(x, y)U(y)dy. \tag{10}$$

This integral equation we rewrite as

$$U(x) = u_0(x) + k^2 \sum_{m=1}^M n_m^2 \int_{B_m} G(x, y)dyU(x_m)[1 + o(1)] \quad a \rightarrow 0. \tag{11}$$

Here the continuity of U in B_m , $1 \leq m \leq M$, was used. This continuity implies

$$U(y) = U(x_m)[1 + o(1)] \quad a \rightarrow 0; \quad y \in B_m.$$

The function U is twice differentiable in \mathbb{R}^3 , as follows from (11), so it is continuous in D .

We need three lemmas.

Lemma 1. *The following relations hold:*

$$\lim_{|x-y| \rightarrow 0} |x-y|G(x, y) = \frac{1}{4\pi}, \tag{12}$$

$$\sup_{|x-y| \geq a} |x-y||G(x, y)| \leq c. \tag{13}$$

By $c > 0$ we denote various estimation constants.

Proof of Lemma 1 is given in Sec. 2.

Lemma 2. *The following relations hold:*

$$\int_{|y-x_m| \leq a} |x-y|^{-1}dy = V_a|x-x_m|^{-1}, \quad |x-x_m| \geq a, \tag{14}$$

$$\int_{|y-x_m| \leq a} |x-y|^{-1}dy = 2\pi \left(a^2 - \frac{|x-x_m|^2}{3} \right), \quad |x-x_m| \leq a.$$

Proof of Lemma 2 consists of a direct routine calculation and is therefore omitted. The result of Lemma 2 is known from the potential theory.

Lemma 3. *If f is piecewise-continuous and bounded in D and the points x_m are distributed in D by formula (6), then the following limit exists:*

$$\lim_{a \rightarrow 0} V_a \sum_{m=1}^M f(x_m) = \int_D f(x)N(x)dx, \tag{15}$$

where $N(x)$ is defined in (6).

This Lemma is proved in Refs. 6, 17 and 18.

This Lemma was recently generalized by the author to allow the function f to be unbounded at some points or sets \mathcal{S} of Lebesgue measure zero and of dimension less than the dimension of the space. For such f one considers the set

$D_\delta := \{x : x \in D, \text{dist}(x, \mathcal{S}) \geq \delta\}$ in which f is piecewise-continuous and bounded, and defines the sum (15) as

$$\lim_{a \rightarrow 0} V_a \sum_{m=1}^M f(x_m) := \lim_{\delta \rightarrow 0} \lim_{a \rightarrow 0} V_a \sum_{x_m \in D_\delta} f(x_m).$$

With this definition the conclusion of Lemma 3 remains valid for f piecewise-continuous in D with the set \mathcal{W} of discontinuities of Lebesgue measure zero and the subset $\mathcal{S} \subset \mathcal{W}$, at which $f = \infty$ and satisfies the following estimate $|f(x)| \leq c[\text{dist}(x, \mathcal{S})]^{-\rho}$, where $c = \text{const} > 0$ and $0 \leq \rho < 3$, and we assume that the integral $\int_D f(x)N(x)dx := \lim_{\delta \rightarrow 0} \int_{D_\delta} f(x)N(x)dx$ exists as an improper integral or the Cauchy principal value singular integral.

One has:

$$\int_{B_m} G(x, y)dy = V_a G(x, x_m)[1 + o(1)], \quad a \rightarrow 0; \quad |x - x_m| \geq a. \tag{16}$$

From (11), (15) and (16) our basic result follows:

Theorem 1. *There exists the limit (9) and*

$$u_e(x) = u_0(x) + k^2 \int_D G(x, y)N(y)\nu^2(y)u_e(y)dy. \tag{17}$$

Physically the limiting field u_e is interpreted as the effective (self-consistent) field in D .

Corollary 1. *The functions $U(x)$ and $u_e(x)$ are twice differentiable in \mathbb{R}^3 . The function $u_e(x)$ solves the equation:*

$$Lu_e(x) = 0, \quad L := L_0 + k^2N(x)\nu^2(x), \tag{18}$$

so

$$n^2(x) = n_0^2(x) + N(x)\nu^2(x). \tag{19}$$

To prove this Corollary one applies the operator L_0 to Eq. (17) and uses Eq. (4).

Conclusion: *To construct a material with a desired refraction coefficient $n^2(x)$ one embeds small balls with radius a , centered at the points $x_m, 1 \leq m \leq M$, distributed by formula (6), and chooses $N(x)$ and $\nu^2(x)$ so that relation (19) holds.*

The choice of $N(x)$ and $\nu^2(x)$ is therefore nonunique, because the relation (19) can be satisfied by infinitely many ways. For example, one may fix $N(x) > 0$ in D and then choose

$$\nu^2(x) = \frac{n^2(x) - n_0^2(x)}{N(x)}.$$

If $n^2(x) = n_0^2(x)$ in a subdomain $\Delta \subset D$, then one can take $N(x) = 0$ in Δ .

In Sec. 2 proof of Lemma 1 is given.

2. Proofs

Proof of Lemma 1. We start with the equation:

$$G(x, y) = g(x, y) - \int_D g(x, z)q_0(z)G(z, y)dz := g - TG, \tag{20}$$

where q_0 is defined in (5), and

$$g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{21}$$

Equation (20) is of Fredholm-type in the space X of functions $\psi(x, y)$ of the form $\psi(x, y) = \phi(x, y)/|x - y|$, where $\phi(x, y)$ is a continuous function of its arguments, and the norm in X is defined as $\|\psi\| = \sup_{x,y \in \mathbb{R}^3} (|x - y| |\psi(x, y)|)$.

We have $\|g\| = 1/4\pi$. The homogeneous equation (20) has only the trivial solution (see Ref. 6), so the operator $(I + T)^{-1}$ is bounded in X . Therefore, $\|G\| \leq c\|g\| = c/4\pi$. This implies estimate (13).

To prove (12), let us multiply (20) by $|x - y|$ and let $|x - y| \rightarrow 0$. One has $\lim_{|x-y| \rightarrow 0} g = 1/4\pi$. The integral TG is bounded for all $x, y \in D$, so $\lim_{|x-y| \rightarrow 0} (|x - y|TG) = 0$. Thus, relation (12) follows.

Lemma 1 is proved. □

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