INFORMATION TO USERS

This dissertation was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

University Microfilms

300 North Zeeb Road
Ann Arbor, Michigan 48106

A Xerox Education Company

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

A METHOD FOR THE SOLUTION OF THE DISTRIBUTION PROBLEM OF STOCHASTIC LINEAR PROGRAMMING

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

By

JAMES B. EWBANK

Norman, Oklahoma

1972

A METHOD FOR THE SOLUTION OF THE DISTRIBUTION

PROBLEM OF STOCHASTIC LINEAR PROGRAMMING

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF INDUSTRIAL ENGINEERING

By

Hillel Kumin

LaVerne L Hoag

Michael D. Devine

Harold C. Kay

B L Foote

ACKNOWLEDGEMENTS

I wish to express my sincere appreciation to my chairman, Dr. Hillel Kumin, and committeemen Dr. Bob Foote and Dr. Michael Devine who spent a great amount of time and effort in the preparation of this dissertation. Not only were their suggestions and constructive criticisms very valuable, but their considerable encouragement was also an immense help in my completion of this work.

I also want to thank my other committee members, Dr. David Kay, Dr. Raymond Lutz, and Dr. Laverne Hoag.

Finally, I wish to thank Mrs. Fran Olney, who typed the first draft, and Mrs. Jaquine Littell, who typed the final copy.

ABSTRACT

Although a great amount of research has been done in the area of stochastic programming, no really satisfactory method has been found for finding the cumulative distribution (CDF) of the maximum value of the objective function. This dissertation describes a method for finding the CDF that often simplifies the computation.

The CDF for a given basis, as well as the probability that a certain basis is optimal (and feasible) is found by integrating over the space for which the given basis is optimal (and feasible). To simplify the space of integration, and often the integration as well, a change of variables is made using the Jacobian Theorem. The CDF is then found by summing the contributions of CDF for a given basis over all appropriate bases.

TABLE OF CONTENTS

DISTRIBUTION OF THE MAXIMUM OF THE OBJECTIVE FUNCTION

CHAPTER I

INTRODUCTION

Almost since linear programming began in the late 1940's, researchers have had to grapple with a related but tricky problem that has come up in practice as well as theory: what to do if various coefficients take on values that are only known probabilistically. Such coefficients--those whose values can only be given by a known or assumed probability distribution--will henceforth be denoted as stochastic coefficients, vectors containing such coefficients are to be called stochastic vectors, and the programming problems using such coefficients will be lumped together and called probabilistic programming. Many applications of probabilistic programming have been found: for example, multiperiod allocation when demand coefficients are stochastic [10,11], transportation problems [28,23], management decision problems [18], and capital budgeting [16].

In order to classify the various probabilistic programming problems which have been investigated, first consider the standard linear programming problem:

$$\max z(x) = \sum_{i=1}^{n} c_i x_i \tag{1}$$

subject to

$$\sum_{i=1}^{n} a_{ji} x_i + x_{n+j} = b_j \quad j = 1, 2, \ldots m \tag{2}$$

$$x_k \geq 0 \quad k = 1, 2, \ldots m, m+1, \ldots m+n \tag{3}$$

We then have the following categories of probabilistic programming. The terminology comes mainly from Sengupta et al [20].

Chance-Constrained programming: In this case, the constraints (2) become

$$\Pr\left\{\sum_{i=1}^{n} a_{ji} x_i \leq b_j\right\} \geq \alpha_j \quad j = 1, 2, \ldots m$$

where either the a_{ji} or the b_j are given by some probability distribution and $0 \leq \alpha_j \leq 1$ $j = 1, 2, \ldots m$. See, for instance, [8], [13], [15], and [22].

Programming under uncertainty: In this problem (1) and (2) become

$$\max\left\{\sum_{i=1}^{k} c_i x_i + E_y[F(y)]\right\},$$

where $E_y[F(y)]$ designates taking the expected value with respect to y of a function F(y), subject to

$$\sum_{i=1}^{k} a_{ji}x_i + \sum_{i=k+1}^{n} a_{ji}y_i \leq b_j \quad j=1,2,\ldots m$$

where the a_{ji} i=k+1,...n or the b_j's may be stochastic. See, for example, [9], [14], [25], [26], [27], [29], and [30].

Aspiration criterion programming (term due to [12]): In this problem (1) becomes

$$\max \Pr[z(x) = \sum_{i=1}^{n} c_i x_i \geq z_o] \text{ where } z_o \text{ is constant.}$$

See, for instance, [12] and [19].

Stochastic programming: In this problem, the objective is to find the distribution of max z(x), where z(x) is given in (1), and denotes the maximum value of the objective function given values for the stochastic coefficients. Several approaches to stochastic programming have been considered.

Active Approach: In this case, a basis is selected and fixed before the values of the stochastic coefficients are observed or known. If the c_i's are stochastic, one may have

feasibility, but a change in the c_i's may cause the basis to be non-optimal. If the b_j's are stochastic, one may have optimality, but changes in the b_j's may cause infeasibility, and in [20] this would add a cost to the objective function.

Passive Approach: This is the approach of this dissertation. An optimal (thus feasible) basis is selected after the values of the stochastic coefficients are known--that is, the basis can change with the change in the b_j's or c_i's, so that one is neither infeasible nor nonoptimal.

Sengupta, Tintner, and Millham [20] and Sengupta, Tintner, and Morrison [21] give relationships between max z(x) for the active and max z(x) for the passive approaches.

The method given in [20] and [21] for finding the distribution of max z(x) is called the "method of sample points." A (finite) number of sets of fixed values of the stochastic coefficients, known as sample points, are substituted for the stochastic coefficients in the objective function and the constraints. For each set of sample points, the resulting linear programming problem is solved, and a probability density function is then fitted to the resulting functional values of max z(x). The authors of [21] mention

that Kolmogorov-Smirnov non-parametric tests can be used empirically to fit the data.

There are several drawbacks to this approach. One is determining how many sample points to choose in order to make a "good enough fit". Another is having (in general) to resolve the problem for each set of sample points using the simplex method, because different sets of sample points may produce different optimal bases. Finally, there is the problem of finding the best probability density function to fit the functional values of the sample points and of max $z(x)$.

A similar article by Bracken and Soland [6] simplifies the theory by only letting the c's be stochastic and by letting them have only discrete or multivariate distributions. In either case, in finding the distribution of the maximum of the objective function, the determination of a number of extreme points is made. The chief drawbacks to using the above method is (1) the restrictions on the c's and (2) the difficulty in finding the extreme points in sufficient number to get a reliable estimate of the distribution.

Babbar in [1] considers the stochastic coefficients to be normally distributed and the matrix $A=(a_{ji})$ to be square.

Then, given any set of values for the stochastic coefficients, he proceeds by means of determinants, to find the values of the decision variables and of the objective function. He also finds the probability and cumulative distributions of the decision variables and of the objective function. Since the "A-matrix" is not generally square, and distributions other than normal occur in practice, these results have limited applicability.

Prekopa in [17] expands $u-u_o$ into a Taylor series where $u = \max z(x)$ and $u_o = \max[z(x)|$ the stochastic coefficients are fixed]. He then shows under the appropriate conditions that if $\Pr\{f(\underline{c}_n, \underline{b}_n, A_n) < x\}$ tends toward some function $\phi(x)$ for large n, where c_n is a sequence of vectors $(c_1^{(n)}, \ldots c_\ell^{(n)})$, b_n a sequence of vectors $(b_1^{(n)}, \ldots b_m^{(n)})$, and A_n is a sequence of matrices of a_{ij}, and f is a real-valued function, then $\Pr\{\frac{1}{\sigma}(u-u_o) \leq x\}$ also tends toward $\phi(x)$ where σ^2 is the variance of $u-u_o$. A disadvantage of the method is that no technique is given for computing $\phi(x)$.

Finally, Bereanu does obtain a closed form expression for the distribution of max z(x). In [3] he considers linear programming problems having a single random variable. For example, in one case (1) becomes

$$\min_x f(x) = \min_x [c' + t(\omega)d']x$$

where c' and d' are vectors of constants and $t(\omega)$ is the only random variable. He then finds the cumulative distribution of the optimal value of the objective function to be a piecewise linear sum of functional values of T(z), where T(z) is the cumulative distribution of $t(\omega)$. The other cases are similar. A disadvantage of the method is that it is only shown for those problems in which any stochastic coefficient is a linear combination of any other.

In [4] Bereanu considers either the b's or the c's to be stochastic and defines what he calls decision regions of stochastic linear programming. Letting $c(t) = c^0 + t_1 c^1 + \ldots + t_r c^r$, where c^0, c^1, $\ldots$ c^r are constants and t_1, $\ldots$ t_r are random variables, he defines a decision region to be the set

$$S_\ell = \{t \mid c'(t)x^\ell = \min_x [c'(t)x, \quad x \text{ is in the feasible space}]\}$$

where x^ℓ is an optimal basic feasible solution for at least one value of t. Under the assumption that the c's are stochastic he finds the minimum of the objective function, denoted by $\gamma(t)$ to be a linear function over S_ℓ for all ℓ. He then finds the mean, variance, and probabilities of $\gamma(t)$ in terms of integrals over the sets S_ℓ. Similar

results are obtained in the case of stochastic b's. Although basic properties of the decision regions are given, the method of sample points must still be used to obtain the distribution of max z(x).

In [5] Bereanu utilizes the Laplace transform to find the distribution of $\phi(t)$, the optimal value of the objective function. To use his method, the decision regions as mentioned above must be found, and then the probability that the stochastic coefficients are within each decision region must be computed. Given these probabilities and also the Laplace transform of the distribution of $\phi(t)$ over each decision region, the Laplace transform of the distribution function of $\phi(t)$ is found. Unfortunately, though, not every distribution has a Laplace transform of simple form.

In order to obtain a closed-form distribution of the maximum value of the objective function, one may need to find the set of bases that can be optimal. In [32], Zinn describes an algorithm that generates all those primal simplex bases that satisfy the optimality criterion and are feasible and no others. A stopping rule is formulated which determines when all bases that can be optimal and feasible have been found.

Statement of Problem

The problem to be considered in this thesis is that which J. K. Sengupta et al [20] refer to as the passive approach to stochastic programming. It resembles the ordinary linear programming problem:

$$\max z(x) = \sum_{i=1}^{n} c_i x_i$$

subject to

$$\sum_{i=1}^{n} a_{ji} x_i + x_{n+j} = b_j, \quad j = 1,2,\ldots m$$

$$x_k \geq 0, \quad k = 1, 2, \ldots m, m+1, \ldots m+n$$

except that one or more of the coefficients are stochastic--that is, their values can only be given in terms of random variables--and the distribution of max z(x) as a function of all values of the stochastic coefficients is desired. The problem is narrowed by letting only the b_j's or the c_i's be stochastic. Furthermore, while they may have a joint probability distribution, it is defined everywhere and piecewise continuous. Moreover, the probability of the feasible region being non-empty and bounded is positive. Finally, it is assumed that the set of possible optimal and feasible solutions is known, or can be found, such that no two of them

are optimal and feasible simultaneously. The method given in [32] is a good way to enumerate them. Instead of attempting to find the distribution of max z(x) by means of sample points as used by J. K. Sengupta and others in [20] and in [21], the approach taken here is, given the distribution of the b_j's alone or of the c_i's alone, to find this distribution directly or, given the joint distribution of both the b_j's and c_i's, at least use a simplified version of the Monte Carlo method over the method of sample points.

The method is similar to that given by Bereanu in [4] but uses a transformation of variables to determine the distribution of max[z(x)]. Moreover, the dissertation will give a specific method of finding the probability that a basis is optimal and feasible and the probability that the objective function does not exceed a certain value whereas [4] only suggests the sample point method for finding these probabilities.

For notational purposes, define $\underline{b}$ to be the column vector $(b_1, b_2, \ldots b_m)^T$, $\underline{c}$ to be the row vector $(c_1, c_2, \ldots c_n)$, A the matrix of the a_{ji}'s, and $\underline{c}_B = (\hat{c}_1, \ldots \hat{c}_m)$ is defined such that $\hat{c}_i = c_j$ if x_j is the $i^{\underline{th}}$ element of the basis for $j \leq n$ and $\hat{c}_i = 0$ if x_j is in the basis for $j > n$.

CHAPTER II

CASE OF STOCHASTIC b's

In the problem

$$\text{Max } z(x) = \sum_{i=1}^{n} c_i x_i \ , \tag{1}$$

subject to

$$\sum_{i=1}^{n} a_{ji} x_i + x_{n+j} = b_j, \quad j = 1, 2, \ldots m \tag{2}$$

$$x_k \geq 0, \quad k = 1, 2, \ldots m, m+1, \ldots m+n \tag{3}$$

consider first the case of the b_j's being stochastic. Denote $f(\underline{b})$ as the joint continuous probability distribution of $\underline{b} = (b_1, b_2 \ldots b_m)^T$. Let $\{b_j^o \mid j = 1, 2, \ldots m\}$ be a particular set of values for the b_j's. Assuming that a bounded solution exists, then solving (1) subject to (2) and to (3) with b_j^o replacing b_j for $j = 1, 2, \ldots m$, an optimal solution is found with $\{x_k^o \mid k = 1, 2, \ldots m\}$ being the set of basic variables of the solution. This basic feasible solution, denoted E^o, represents an extreme

point of the solution set. Define A_i to be the column vector of A whose elements are coefficients of x_i: $A_i = (a_{1i}, a_{2i}, \ldots a_{mi})^T$ and define $B = (A_1^o, A_2^o, \ldots A_m^o)$ to be an m×m square matrix such that A_i^o corresponds to x_i^o. (B is commonly referred to as the basis matrix.)

If the b_j's are then allowed to vary over their range, the point E^o changes position in m-space. First to be investigated are the conditions under which E^o remains optimal. Since only the b_j's vary, and since changes in the b_j's do not effect the optimality of the solution so long as it is feasible, the set $\{x_k^o \mid k = 1, 2, \ldots m\}$ remains feasible, and thus optimal, if and only if the following holds:

$$B^{-1}\,\underline{b} = \underline{r} \geq 0. \tag{4}$$

From the above, one obtains

$$B(B^{-1}\underline{b}) = \underline{b} = B\underline{r}. \tag{5}$$

The probability that E^o remains the optimal extreme point is

$$P = \int\int_S \ldots \int f(\underline{b})db_1db_2 \ldots db_m\,, \tag{6}$$

where $S = \{\underline{b} \mid (4) \text{ holds}\}$. Substituting (5) in for the b_j's, (6) becomes (see [2], page 335),

$$P = \int\int \ldots \int_{\underline{r} \geq 0} f(B\underline{r}) \mid J_r \mid dr_1 dr_2 \ldots dr_m , \quad (7)$$

where J_r is the Jacobian:

$$J_r = \det\left[\frac{\partial b_k}{\partial r_i}\right] .$$

Because $b_k = (B\underline{r})_k = \sum_{j=1}^{m} B_{kj} r_j$, and $\frac{\partial b_k}{\partial r_i} = B_{ki}$, this implies

$$J_r = \det[B_{ki}] = \det(B) . \quad (8)$$

Note that J_r is nonzero because the columns of B, the A_k^o's must be linearly independent.

However, not every set of m x_k's for k = 1, 2, ... m+n necessarily forms an optimal basis for any selection of values of the b_j's. Letting G be an m-element subset of the set $\{x_k \mid k = 1, 2, \ldots m, m+1, \ldots m+n\}$, G forms an optimal basis if and only if (4) is satisfied and the objective function is maximized. It can be shown that if G satisfies the optimality criterion for one selection of the b_j's, whether or not G is feasible, then G is an optimal basis whenever it is feasible, because the optimality conditions are independent of the b_j's (see [24], page 95). For this reason there is no need to explicitly choose values for the b_j's. Given any G, (7) only gives the probability that G

is feasible.

Using the above results, define

$$\alpha_G = \begin{cases} 1, \text{ if the set G satisfies the optimality criterion.} \\ 0, \text{ otherwise.} \end{cases}$$

To determine whether $\alpha_G = 1$ or $\alpha_G = 0$, one may only need to consider a portion of the constraints and a portion of the slack variables. For let $G = \{x_{n_1}, x_{n_2}, \ldots x_{n_m}\}$, where $x_{n_{k+1}}$, $x_{n_{k+2}}$, ... and x_{n_m} are all of the slack variables. The usual method to check the optimality criterion is to compute $Q_j = c_B B^{-1} A_j - c_j$ and see if $Q_j \geq 0$ for all j such that $1 \leq j \leq m+n$ and $x_j \notin G$. To ease the computation, establish a new set of equations (1'), (2'), and (3'), like those of (1), (2), and (3) except that if a slack variable appears in G, then its corresponding constraint does not appear in (2') and the variable itself does not appear in (3') (or, of course, in (1')). Let J be the set of indices $\{j | x_j$ appears in (3')$\}$. For the new set of equations, A'_k is the column vector of coefficients of x_k in (2'), $c'_B = (c_{n_1}, \ldots c_{n_k})$, and $B' = (A'_{n_1}, A'_{n_2}, \ldots A'_{n_k})$. To check for optimality in (1') subject to (2') and (3'), simply compute $Q'_j = c'_B B'^{-1} A'_j - c_j$ and see if $Q'_j \geq 0$ for all j such that

$j \in J$ and $x_j \notin \{X_{n_1}, \ldots x_{n_k}\}$.

If G is optimal in (1') subject to (2') and (3'), then G satisfies the optimality criterion for (1) subject to (2) and (3). This is because (2) just represents new addition of constraints onto (2') (and (3) additional slack variables to (3')), and the addition of constraints effects only the feasibility of a basis, not the optimality (see [24], page 101).

If two or more non-coincident optimal extreme points exist for the same set of b_j's, one can change some of the c_i's, say $(c_{i_1}, c_{i_2}, \ldots c_{i_k})$ by "arbitrarily small" numbers $\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots \varepsilon_{i_k}$ so that each choice of the b_j's gives only one optimal extreme point and hence only one optimal basis. To understand what is being done, if non-coincident and optimal extreme points, say $\underline{x}^1, \underline{x}^2, \ldots \underline{x}^h$, exist for certain b_j's then they can be simultaneously contained in a hyperplane representing the objective function. However, by a slight change of the c_i's, a hyperplane like that given above can contain only one point from $\{\underline{x}^1, \underline{x}^2, \ldots \underline{x}^h\}$ and it is the optimal one. Then proceed to define the α_G's as given above.

Thus, given a general set of m variables G, the probability that G is an optimal basis can be given by

$$P_G = \int\int \ldots \int_{\underline{r} \geq 0} f(B\underline{r})\alpha_G \, |\det(B)| \, dr_1, dr_2, \ldots dr_m \quad (9)$$

(note that equation (8) is used to replace J_r).

This leads to the following theorem:

Theorem 1: $\sum_H P_G = 1 - P^-$, where P_G is as defined in (9), H is the set of all m-element subsets, of $\{x_1, x_2, \ldots x_m, x_{m+1}, \ldots x_{m+n}\}$, and P^- = Pr {no G is feasible}.

Proof: It is clear from (9) that

$$P_G = \alpha_G \text{ Pr } \{G \text{ is an optimal basis}\}.$$

Because it follows from the hypothesis that if G is an optimal basis, then it is unique and, since the sets of values of the b_j's at which each $G \in H$ is an optimal basis intersect only in sets of zero content (see Appendix A),

$$\sum_H \text{Pr}\{G \text{ is an optimal basis}\} = \text{Pr}\Big(\bigcup_H \{G \text{ is an optimal basis}\}\Big)$$

$$= \text{Pr } \{\text{there is an optimal basis}\}$$

$$= 1 - P^-$$

Hence $\sum_H P_G = 1 - P^-$. q.e.d.

Theorem 2: If z(x) is unbounded for one set of b_j's, then z(x) is unbounded or non-existent for all sets of b_j's.

Proof: Suppose that z(x) is unbounded for a selection of the b_j's forming the vector $\underline{b}$. Then, after a series of

simplex operations (see [24], page 59), there is a column r such that $\underline{C}_B B^{-1}\underline{A}_r - c_r < 0$ (indicating that x_r is a candidate to enter the basis) and $B^{-1}\underline{A}_r \leq \underline{0}$. Let there be a change in the b_j's, changing the vector $\underline{b}$ to $\underline{b}'$. No change occurs in the values of $C_B B^{-1}\underline{A}_r - c_r$ or of the column vector $B^{-1}\underline{A}_r$, but the right-hand side, $B^{-1}\underline{b}'$, may change. Suppose that for some j, $(B^{-1}\underline{b}')_j < 0$. If $(B^{-1}\underline{A}_r)_j < 0$, then by simply increasing the variable x_r, the revised $j^{\underline{th}}$ constraint,

$$\sum_{r=1}^{m+n} (B^{-1}\underline{A}_r)_j x_r = (B^{-1}\underline{b}')_j$$

is satisfied for $x_r \geq 0$, r = 1, 2, ... m+n, and the objective function can increase without bound. If $(B^{-1}\underline{A}_r)_j = 0$, then try a dual simplex multiplication on the revised $j^{\underline{th}}$ constraint. If there is an s such that $(B^{-1}\underline{A}_s)_j < 0$, then the dual simplex method can be applied, and, since the values of $\underline{C}_B B^{-1}\underline{A}_r - c_r$ and of $B^{-1}\underline{A}_r$ remain unchanged after the method is used, the solution stays unbounded. If $(B^{-1}\underline{A}_s)_j \geq 0$ for all s, then the solution is non-existent. q.e.d.

Thus, if the solution turns out to satisfy the criterion of unboundedness ($\underline{C}_B B^{-1}\underline{A}_r - c_r < 0$ and $B^{-1}\underline{A}_r \leq 0$ for some r and matrix B), then due to the above theorem, there is no point in trying to find the distribution of max z(x).

It is noteworthy that in order to find the probability

that G is feasible (but not an optimal basis), one only needs to know the A_k's corresponding to the elements of G.

At this point comes the investigation of the distribution of $Z(\underline{b}) = \max\{z(x)|\underline{b}\}$. Consider the probability that $Z(\underline{b}) \leq \phi$, where ϕ is some finite real number. Then

$$\Pr\{Z(\underline{b}) \leq \phi \text{ and } G \text{ is an optimal basis}\}$$

$$= P_G(\phi) = \alpha_G \int\int_{S_\phi} \ldots \int f(\underline{b}) db_1 db_2 \ldots db_m , \qquad (10)$$

where $S_\phi = \{\underline{b}|$ (4) holds and $Z(\underline{b}) \leq \phi\}$.

Given the values of the b_j's, one has $Z(\underline{b}) = \underline{c}_B B^{-1} \underline{b}$, and hence for $\underline{b} \varepsilon S_\phi$, $\underline{c}_B B^{-1} \underline{b} \leq \phi$.

Since $\underline{b} = B\underline{r}$, it then follows that

$$Z(\underline{b}) = \underline{c}_B \underline{r} = \sum_{j=1}^{m} \hat{c}_j r_j .$$

Changing the variables of integration in (10) from the b_j's to the r_j's as was done before, one has

$$P_G(\phi) = \alpha_G \int\int_{S'_\phi} \ldots \int f(B\underline{r}) |J_r| dr_1, dr_2, \ldots dr_m \qquad (11)$$

where $S'_\phi = \{\underline{r}|\underline{r}>0$ and $Z(\underline{b}) = \sum_{j=1}^{m} \hat{c}_j r_j \leq \phi\}$.

Before specifying the limits of the integration in equation (11), the following should be noted: if $\hat{c}_k \leq 0$ and if $\sum_{j=1}^{m} \hat{c}_j r_j = \underline{c}_B \underline{r} \leq \phi$ for $r_k = \hat{r}_k \geq 0$, then $\sum_{j=1}^{m} \hat{c}_j r_j \leq \phi$

for $r_k \geq \hat{r}_k$. Thus r_k is allowed to vary from zero (because r_k may equal zero) to infinity. Next, reorder the r_j's so that all of those r_j's whose corresponding coefficients in the vector $\underline{c}_B$ are less than or equal to zero come at the end of the $\underline{r}$ vector. A_j's, c_i's, b_j's, and rows and columns of A and B are also to be reordered accordingly. Let p be the number of remaining r_j's. After such a reordering, one can deduce from the equation $\underline{c}_B\underline{r} \leq \phi$ that

$$\sum_{j=1}^{p} \hat{c}_j r_j \leq \phi - \sum_{j=p+1}^{m} \hat{c}_j r_j = \phi' \qquad (12)$$

(defining the right-hand sum of (12) to be zero if p-m).

Then considering the limits of (11) as defined by S'_ϕ, one can deduce from (12) that r_1 varies from 0 to

$$\frac{1}{\hat{c}_1}\left[\phi' - \sum_{j=2}^{p} \hat{c}_j r_j\right] = \frac{1}{\hat{c}_1}\left[\phi - \sum_{j=2}^{m} \hat{c}_j r_j\right] .$$

After r_1 is integrated out, it follows that r_2 then varies from 0 to

$$\frac{1}{\hat{c}_2}\left[\phi' - \sum_{j=3}^{p} \hat{c}_j r_j\right] = \frac{1}{\hat{c}_2}\left[\phi - \sum_{j=3}^{m} \hat{c}_j r_j\right] .$$

Finally, r_p varies from 0 to

$$\frac{1}{\hat{c}_p}(\phi') = \frac{1}{\hat{c}_p}\left[\phi - \sum_{j=p+1}^{m} \hat{c}_j r_j\right] ,$$

(where the sum from p+1 to m is zero for p=m).

Thus, (11) and therefore (10), can be written in the following definite form:

$$P_G(\phi) = \alpha_G \int_0^{\infty} \cdots \int_0^{\infty} \int_0^{h_p(r')} \cdots \int_0^{h_k(r')} \cdots \int_0^{h_1(r')} f(B\underline{r}) \quad (13)$$

$$|\det(B)| dr_1, dr_2, \ldots dr_k, \ldots dr_p, dr_{p+1}, \ldots dr_m ,$$

where $h_k(r') = \frac{1}{\hat{c}_k} [\phi - \sum_{j=k+1}^{m} \hat{c}_j r_j]$.

Clearly, $\lim_{\phi \to \infty} P_G(\phi) = P_G$. Thus, due to theorem 1,

$$F(\phi) = \sum_H \frac{P_G(\phi)}{(1-P^-)} , \text{ and from the definition of S, } F(-\infty) = 0. \quad (14)$$

Moreover, it follows from (13) that $F(\phi)$ is non-decreasing, so $F(\phi)$ is the distribution of the maximum value of the objective function--in other words, the probability that $Z(\underline{b}) \leq \phi$.

This leads to the following theorem:

Theorem 3: If the only stochastic vector is $\underline{b}$, then the probability that a certain choice of m variables, G, is an optimal basis is given by (9). Also, the cumulative distribution of $Z(\underline{b})$ given that G is an optimal basis is $P_G(\phi)/P_G$, where $P_G(\phi)$ is as given by (13). Moreover $F(\phi)$ defined in (14) yields the (cumulative) distribution of the maximum of the objective function.

Equations (9) and (13) give a theoretical method for

computing P_G and $P_G(\phi)$ provided that the function $f(\underline{b})$ is defined over all of $\mathbb{R}^m$, where $\mathbb{R}^m$ is the Cartesian product of the real line m times. However if $f(\underline{b})$ changes functional form over different portions of $\mathbb{R}^m$, there may be one or more multiple integrations for each of the regions and the limits to these integrations may be complicated. Consider three kinds of functions for f(b):

(1) A single functional form over all of $\mathbb{R}^m$, for example the multivariate normal,

(2) A single functional form over the region $R^p = \{\underline{b} \in \mathbb{R}^m \mid \underline{b} \geq 0\}$ and zero outside R^p, as for example, the negative exponential, and

(3) Neither of the first two cases, as for example, the uniform.

If $f(\underline{b})$ is in case (1), then $f(B\underline{r})$ is also of single functional form for all $\underline{r} \geq 0$. Hence, in computing P_G by (9) there is no need to set up more than one multiple integral and $P_G = \alpha_G \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(B\underline{r})|\det(B)|dr_1dr_2\ldots dr_m$. $P_G(\phi)$ is likewise computed with no need to set up another multiple integral for a different region of $\mathbb{R}^m$.

If $f(\underline{b})$ is in case (2) and if all $a_{ij} \geq 0$, then every entry of the matrix B is also non-negative. Hence $B\underline{r} \geq 0$

for $\underline{r} \geq 0$ and so $f(B\underline{r})$ is of single functional form for $\underline{r} \in R^p$. Thus P_G and $P_G(\phi)$ are computed just as in case (1). Examples of this are given in the Appendix.

On the other hand, if $a_{ij}<0$ for some values of i and j and if the a_{ij}'s appear in B then for some $r \geq 0$, $(B\underline{r})_i<0$ and thus $f(B\underline{r}) \equiv 0$. Hence, the actual computation of (9) and of (13) involves finding the limits implicit in the equation $B\underline{r} \geq 0$.

If $f(\underline{b})$ is in case (3), the computation of (9) and of (13) may be very complex. Consider the uniform:

$$f(\underline{b}) = \begin{cases} \dfrac{1}{d_1 d_2 \cdots d_m}, & \ell_i \leq b_i \leq u_i (d_i = u_i - \ell_i), \text{ for } i=1,2,\ldots m \\ 0, \text{ otherwise.} \end{cases}$$

After a change of variables, one has in the computation of P_G, for example, that the space of integration for positive $f(B\underline{r})$ (which equals $\dfrac{1}{d_1 d_2 \cdots d_m}$) is $\ell_i \leq (B\underline{r})_i \leq u_i$ for $i = 1, 2, \ldots m$, and $(B\underline{r})_i$ may be a sum of m terms for each i.

CHAPTER III

CASE OF STOCHASTIC c's

Now consider the problem of (1) given (2) and (3) when only the c_i's are the stochastic coefficients. Analogous to before, let $f(\underline{c})$ be the joint probability density function of the c_i's. Letting $\left\{c_i^o \mid i=1,2,\ldots n\right\}$ be a set of fixed values for the c_i's define the set $\left\{x_k^o \mid k=1,2,\ldots m\right\}$ similarly as before: the basis after the resulting deterministic linear programming problem is solved. A_i and B are defined exactly as before. Before proceeding, denote those c_i^o's that are coefficients of an x_i^o as basic coefficients and $h \leq \min(m,n)$ as the number of basic coefficients. Then reorder the c_i's so that the first h of them are basic, and reorder the A_i's correspondingly. Likewise, reorder the rows of A (and hence those of (2)) so that the first h constraints are tight--that is, the slack variables (x_{n+j}, where $j>0$) are equal to zero. The rows of A that are to be tight can easily be shown to be those rows in which the corresponding slack variable is <u>not</u> in

the set $\left\{x_k^o \mid k=1,2,\ldots m\right\}$. The matrices B and A to be used later are those matrices resulting from this reordering, and it can then be shown that $\underline{c}_B = (\hat{c}_1, \ldots \hat{c}_h, 0, \ldots 0)$.

First to be determined are the conditions under which the x_k^o's remain in solution. Since only the c_i's vary and since changes in the c_i's do not affect the feasibility of the solution so long as it is optimal, $\left\{x_k^o \mid k=1,2,\ldots m\right\}$ remains in solution if and only if the following conditions of optimality hold:

$$\underline{c}_B B^{-1} A - \underline{c} = \underline{s} \geq 0 \tag{15}$$

$$\underline{c}_B B^{-1} = \underline{t} \geq 0 \tag{16}$$

The probability that the x_k^o's remain in solution is thus given by

$$P = \int\int_R \ldots \int f(\underline{c}) dc_1, dc_2, \ldots dc_n , \tag{17}$$

where $R = \{\underline{c} \mid (15) \text{ and } (16) \text{ hold}\}$.

In order to change the variables of $\underline{c}_B$ to those of $\underline{s}$ and of $\underline{t}$, note the following: (a) due to the laws of simplex multiplications (see [24], page 90) for $i \leq h$,

$$B^{-1} A_i = \underbrace{(0, \ldots 0, 1,}_{i^{\underline{th}} \text{ position}} 0, \ldots 0)^T$$

and hence for $i \leq h$ the $i^{\underline{th}}$ entry of (15) is simply $c_i - c_i = s_i = 0$,

and (b) for i>h the $i^{\underline{th}}$ entry of (16) equals zero as will be shown in the following lemma:

Lemma: In equation (16), $(\underline{c}_B B^{-1})_i = 0$ for i>h.

Proof: By the definition of B the last m-h columns of B correspond to the last m-h columns of the identity matrix. Let b_{ij} be the entries of B and c_{ij} those of B^{-1}. Then the entries of BB^{-1} are

$$\sum_{j=1}^{m} c_{ij} b_{jk} = \delta_{ik}, \quad \text{where } \delta_{pq} = 1$$

for p=q and $\delta_{pq}=0$ for $p \neq q$. Consider k>h. Then $b_{ik} = \delta_{ik}$, and so

$$\delta_{ik} = \sum_{j=1}^{m} c_{ij} b_{jk} = \sum_{j=1}^{m} c_{ij} \delta_{jk} = c_{ik},$$

hence the last m-h columns of B^{-1} are also identical to the last m-h columns of the identity matrix. Because of this and of the fact that the $i^{\underline{th}}$ entry of $\underline{c}_B$ equals zero for i>h, $(\underline{c}_B B^{-1})_i =$

$$\sum_{j=1}^{n} \hat{c}_j c_{ji} = \sum_{j=1}^{h} \hat{c}_j \cdot 0 + \sum_{j=h+1}^{n} 0 \cdot c_{ji} = 0. \quad \text{q.e.d.}$$

The above lemma shows that the simplex multipliers for the slack variables in the basis are zero.

Because of the results above, $t_i=0$ for i>h and $s_i=0$ for $i \leq h$. Defining $\underline{\hat{c}}_B = (\hat{c}_1, \ldots \hat{c}_h)$, $\underline{\hat{t}} = (t_1, \ldots t_h)$ and

$\hat{B}$ to be the first h rows and columns of B, one can derive from (16) $\underline{\hat{c}}_B = \underline{\hat{t}}\hat{B}$. Furthermore, by equation (15),

$$\underline{s} = \underline{c}_B B^{-1} A - \underline{c} = \underline{t}A - \underline{c} ,$$

and denoting $\underline{\bar{s}} = (s_{h+1}, \ldots s_n)$, $\underline{\bar{c}} = (c_{h+1}, \ldots c_n)$, and $\bar{A}$ to be the submatrix consisting of the last n-h columns and of the first h rows of A, one can deduce from (15) that $\underline{\bar{s}} = \underline{\hat{t}}\bar{A} - \underline{\bar{c}}$.

The preceding suggests the following substitutions:

$$c_i = (\underline{\hat{t}}\hat{B})_i, \quad i = 1, 2, \ldots h \tag{18}$$

$$c_i = (\underline{\hat{t}}\bar{A})_i - s_i, \quad i = h+1, \ldots n. \tag{19}$$

In actual practice, $(\hat{t}\hat{B})_i$ replace the basic coefficients and $(\underline{\hat{t}}\bar{A})_i - s_i$ replace the other coefficients of $\underline{c}$.

At this point, the Jacobian, J_c, is computed:

$$J_c = \det \left[\frac{\partial c_i}{\partial r_j} , \frac{\partial c_p}{\partial s_q} \right]$$

$$= \det \begin{bmatrix} \frac{\partial c_1}{\partial t_1} & \cdots & \frac{\partial c_1}{\partial t_h} & \frac{\partial c_1}{\partial s_{h+1}} & \cdots & \frac{\partial c_1}{\partial s_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial c_h}{\partial t_1} & \cdots & \frac{\partial c_h}{\partial t_h} & \frac{\partial c_h}{\partial s_{h+1}} & \cdots & \frac{\partial c_h}{\partial s_n} \\ \frac{\partial c_{h+1}}{\partial t_1} & \cdots & \frac{\partial c_{h+1}}{\partial t_h} & \frac{\partial c_{h+1}}{\partial s_{h+1}} & \cdots & \frac{\partial c_{h+1}}{\partial s_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial c_n}{\partial t_1} & \cdots & \frac{\partial c_n}{\partial t_h} & \frac{\partial c_n}{\partial s_{h+1}} & \cdots & \frac{\partial c_n}{\partial s_n} \end{bmatrix}$$

$$= \det \begin{bmatrix} \frac{\partial c_1}{\partial t_1} & \cdots & \frac{\partial c_1}{\partial t_h} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial c_h}{\partial t_1} & \cdots & \frac{\partial c_h}{\partial t_h} & 0 & \cdots & 0 \\ \frac{\partial c_{h+1}}{\partial t_1} & \cdots & \frac{\partial c_{h+1}}{\partial t_h} & -1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial c_n}{\partial t_1} & \cdots & \frac{\partial c_n}{\partial t_h} & 0 & \cdots & -1 \end{bmatrix}$$

$$= (-1)^{n-h} \det \begin{bmatrix} \frac{\partial c_1}{\partial t_1} & \cdots & \frac{\partial c_1}{\partial t_h} \\ \vdots & & \vdots \\ \frac{\partial c_h}{\partial t_1} & \cdots & \frac{\partial c_h}{\partial t_h} \end{bmatrix}$$

and because for $i \leq h$, $c_i = \sum_{j=1}^{h} t_j B_{ji}$, $\frac{\partial c_i}{\partial t_k} = B_{ki}$; hence

$$J_c = (-1)^{n-h} \det(\hat{B}^T) = (-1)^{n-h} \det(\hat{B}) \ . \qquad (20)$$

It follows from (18) and (19) that $f(\underline{c}) = f(\underline{\hat{t}}\hat{B}, \underline{\hat{t}}A - \underline{\bar{s}})$. Thus, substituting (18) and (19) into (17) and applying the change of variables theorem (see [2], pages 335-336), one has

$$P = \int\int \cdots \int_{\underline{\hat{t}} \geq 0, \underline{s} \geq 0} f(\underline{\hat{t}}\hat{B}, \underline{\hat{t}}\bar{A} - \underline{\bar{s}}) \mid J_c \mid dt_1, \ \ldots \ dt_h, \ ds_{h+1}, \ \ldots \qquad (21)$$

ds_n, where J_c is as given in (20).

However, just as before, not every m-element subset of $\{x_1, x_2, \ldots x_m, \ldots x_{m+n}\}$ is necessarily a basis. Similar to the case of the b_j's being stochastic, one may say that if G (defined as before) is feasible, then it satisfies the feasibility criterion. So, much as before, define

$$\alpha_G = \begin{cases} 1, & \text{if the set satisfies the feasibility criterion.} \\ 0, & \text{otherwise.} \end{cases}$$

Using the above definition (and formula 20), the probability that G forms a basis is found to be

$$P_G = \int\int \ldots \int_{\underline{\hat{t}}\geq 0, \underline{s}\geq 0} f(\underline{\hat{t}}B, \underline{\hat{t}}\bar{A} - \underline{s})\alpha_G |\det(B)| dt_1 \ldots dt_h ds_{h+1} \ldots ds_n \quad (22)$$

This leads to

Theorem 4: $\sum_H P_G = 1 - P^u$, where P_G is as defined in (22), H is as defined in Theorem 1 and P^u = Pr{unbounded solution occurs}.

Proof: Similar to that of Theorem 1.

Much as before, in order to find the probability that G is optimal (but not necessarily basic), one only needs to know the corresponding A_k's and rows of A (and generally only a portion of their entries).

Next consider the investigation of the distribution of $Z(\underline{c}) = \max\{z(x) |$ the values of the c_i's are given$\}$, when G is the basis for some $G \in H$. Much as before, the

probability that $Z(\underline{c}) \leq \phi$ where ϕ is some finite real number, can be written as

$$\Pr\{Z(\underline{c}) \leq \phi \text{ and } G \text{ is the basis}\} = P_G(\phi)$$

$$= \alpha_G \iint_{R_\phi} \ldots \int f(\underline{c}) dc_1 dc_2 \ldots dc_n, \quad (23)$$

where $R_\phi = \{\underline{c} \mid (15) \text{ and } (16) \text{ hold and } Z(\underline{c}) \leq \phi\}$.

Let b_j: $j=1,2,\ldots m$, be the (fixed) bounds on the constraints. Since $t_i=0$ for $i \geq k+1$ one has

$$Z(\underline{c}) = \underline{c}_B B^{-1} \underline{b} = \underline{t}\underline{b} = \sum_{i=1}^{m} t_i b_i = \sum_{i=1}^{h} t_i b_i = \hat{\underline{t}}\hat{\underline{b}} ,$$

where $\hat{\underline{b}} = (b_{1i}, \ldots b_h)$. Thus $\Pr\{Z(c) \leq \phi\} = \Pr\{\hat{\underline{t}}\hat{\underline{b}} \leq \phi\}$. Changing the variables of integration of (23) from the c_i's to the corresponding t_i's and s_i's yields

$$P_G(\phi) = \alpha_G \iint_{R'_\phi} \ldots \int f(\hat{\underline{t}}\hat{B}, \hat{\underline{t}}\overline{A} - \overline{\underline{s}}) |\det(\hat{B})| dt_1 \ldots dt_h ds_{h+1} \ldots ds_n, \quad (24)$$

where $R'_\phi = \{(\hat{\underline{t}}, \overline{\underline{s}}) \mid \hat{\underline{t}} \geq 0, \overline{\underline{s}} \geq 0 \text{ and } \hat{\underline{t}}\hat{\underline{b}} \leq \phi\}$.

In order to specify the limits of integration of (24), it is helpful to first reorder the t_i's of $\hat{\underline{t}}$ (and hence the appropriate c_i's). Notice that if $b_k \leq 0$ and $\sum_{i=1}^{h} t_i b_i \leq \phi$ for $t_k = t_k^o$, then $\sum_{i=1}^{h} t_i b_i \leq \phi$ for $t_k \geq t_k^o$. Similar to before, the t_i's in $\hat{\underline{t}}$ are reordered so that those t_i's whose corresponding coefficients in the $\underline{b}$ vector are less than or equal to zero are at the end of the $\hat{\underline{t}}$-vector,

and the c_i's, b_j's and the rows and columns of $\bar{A}$ and $\hat{B}$ are to be reordered correspondingly. Let q be the number of remaining t_i's.

Thus, by the same reasoning as before, (24) can be written in the following definite form:

$$P_G(\phi) = \alpha_G \int_{\underline{s}\geq 0} \cdots \int_{-\infty}^{\infty} \cdots \int_0^{\infty} \int_0^{g_q(t')} \cdots \int_0^{g_k(t')} \cdots \int_0^{g_1(t')} f(\underline{t}\hat{B},\ \underline{t}\bar{A}-\underline{s})\,|\det(\hat{B})|\,dt_1 \cdots dt_q\, dt_{q+1} \cdots dt_h\, ds_{h+1} \cdots ds_n, \quad (25)$$

where $g_k(\underline{t}') = 1/b_k \left[\phi - \sum_{i=k+1}^{h} b_i t_i\right]$ and $\sum_{i=k+1}^{h} b_i t_i = 0$

for k=h.

Since $\lim_{\phi\to\infty} P_G(\phi) = P_G$, $F(\phi) = \Sigma_H P_G(\phi)/(1-P^u)$ is the distribution of the maximum of the objective function by Theorem 4, where this time $P_G(\phi)$ is as defined in (25).

The results can be summarized in the next theorem:

Theorem 5: If the only stochastic vector is $\underline{c}$, the probability that a certain $G \varepsilon H$ is a basis is given by (22). Also, the probability density function of $Z(\underline{c})$ given that G is the basis is $P_G(\phi)/P_G$, where $P_G(\phi)$ is as given in (25). Finally, $F(\phi)$ as defined immediately preceding this theorem is the distribution of the maximum of the objective function.

Similar to the case of the b_j's being stochastic, equations (22) and (25) give a theoretical method for computing P_G and $P_G(\phi)$ provided that the function $f(\underline{c})$ is defined over all of $\mathbb{R}^n$. The discussion concerning $f(\underline{b})$ starting on page 20 applies as well to $f(\underline{c})$.

CHAPTER IV

CASE OF BOTH THE b's AND c's BEING STOCHASTIC

The third and final case is a generalization of the first two cases: consider (1) subject to (2) and (3) letting both $\underline{c}$ and $\underline{b}$ be stochastic with a joint probability density function denoted $f(\underline{b},\underline{c})$. As before, let $\{x_k^o | k=1, 2,\ldots m\}$ be the set of basic variables of the optimal solution after a particular set of values $(b_1^o, \ldots b_m^o)$ and $(c_1^o, \ldots c_n^o)$ replace the b_j's and c_i's in (1) and (2) respectively. Then for the x_k^o's to remain both feasible and optimal, all three equations (4), (15), and (16) must hold. The probability of this is

$$P = \int_S \ldots \iint_R \ldots \int f(\underline{b},\underline{c}) db_1 \ldots db_m dc_1 \ldots dc_n , \quad (26)$$

where S is as in (6) and R as in (17). Making the substitution (5) for $\underline{b}$ and (18) and (19) for $\underline{c}$, one can easily verify that (26) becomes

$$P = \int \cdots \int_{r\geq 0, \hat{t}\geq 0, \bar{s}\geq 0} \iint \cdots \int f(B\underline{r}, \hat{t}\hat{B}, \hat{t}\bar{A}-\bar{s}) |\det(B)|$$

$$|\det(\hat{B})| dr_1 \cdots dr_{\cdot} dt_1 \cdots dt_h ds_{h+1} \cdots ds_n. \quad (27)$$

In contrast to the first two cases, if G is any m-element subset of $\{x_1, x_2, \cdots x_m, \cdots x_{m+n}\}$, then (27) is the probability that G is a basis, since it is immediately possible to find values of the b_j's and c_i's to make G a basis. Hence Pr{G is a basis} is as given in formula (27).

To find the distribution of $Z(\underline{b},\underline{c}) = \max\{z(x) | \text{the values of the } b_j\text{'s and } c_i\text{'s are given}\}$ with G as a basis, note that $Z(\underline{b},\underline{c}) = \underline{c}_B B^{-1}\underline{b} = \bar{t}\underline{b} = \bar{t}B\underline{r}$, where $\bar{t}$ is defined as $(t_1, t_2, \cdots t_h, 0, \cdots 0) = (\hat{t}, 0, \cdots 0)$. Hence

$$\Pr\{Z(\underline{b},\underline{c}) \leq \phi\} = \Pr\{\bar{t}B\underline{r} \leq \phi\} = P_G(\phi) \quad (28)$$

$$= \iint \cdots \iint_{T_\phi} \cdots \int f(\underline{b},\underline{c}) db_1 \cdots db_m dc \cdots dc_n,$$

where $T_\phi = \{(\underline{b},\underline{c}) |$ (4) holds for $\underline{b}$, (15) and (16) hold for $\underline{c}$, and $\underline{c}_B B^{-1}\underline{b} \leq \phi\}$, which, after the appropriate substitutions are made, is equivalent to

$$P_G(\phi) = \int \cdots \iint_{T'_\phi} \cdots \int f(B\underline{r}, \hat{t}\hat{B}, \hat{t}\bar{A}-\bar{s}) |\det(B)| \quad (29)$$

$$|\det(\hat{B})| dr_1 \cdots dr_m dt \cdots dt_h ds_{h+1} \cdots ds_n,$$

where $T'_\phi = \{(\underline{r},\hat{t},\bar{s}) \mid \underline{r}\geq 0, \hat{t}\geq 0, \bar{s}\geq 0, \text{ and } \bar{t}B\underline{r} \leq \phi\}$.

However, there is no simple way to define the limits

of even (29), much less (28), and so one must use a modified version of the Monte Carlo method given in Appendix B.

This yields the following variation of Theorems 2 and 4:

Theorem 6: If both $\underline{b}$ and $\underline{c}$ are stochastic, then the probability that G is a basis is given by P_G as shown in (27). Also, the probability density of $Z(\underline{b},\underline{c})$ when G is a basis is $P_G(\phi)/P_G$, where $P_G(\phi)$ is given by (29). Moreover, the probability density function of the maximum of the objective function, $F(\phi)$, clearly equals to $\Sigma_H P_G(\phi)/(1-P^- - P^u)$, where H and P^- are defined exactly as in Theorem 1 and P^u as in Theorem 4.

Note: The foregoing work with the stochastic b's and c's implicitly assumes that all of them can vary from $-\infty$ to ∞. If there are restrictions on the distributions of b's and c's, one must be sure that these restrictions are reflected in their appropriate substitutions: for instance, if $f(\underline{c})>0$ only for $\underline{c}>0$, then one must have $\hat{t}\hat{B}\geq 0$ and $\underline{\hat{t}}\overline{A}-\overline{\underline{s}} \geq 0$.

CHAPTER V

CONCLUSION

The foregoing results allow one to theoretically find the exact distribution of the maximum of the objective function $F(\phi)$ whenever either $\underline{b}$ or $\underline{c}$, or both, are stochastic vectors. There are three cases to be considered. In each, a basis $G = \{x_k \mid k = 1, 2, \ldots m\}$ is presumed and $Z_G(b,c)$ designates the maximum value of the objective function.

Case 1. Only the $\underline{b}$-vector is stochastic. In this case, one has to find the (square) matrix B which consists of the column vectors of A whose entries are coefficients of an element of G, and α_G, which is an indication of whether or not G can be made basic. One then computes $Z_G(b,c) = \underline{c}_B \underline{r}$, P_G by formula (9), $P_G(\phi)$ by formula (13), and $F(\phi) = \Sigma_H P_G(\phi)/(1-P^-)$.

Case 2. Only the $\underline{c}$-vector is stochastic. In this case, the matrices needed for computation are $\overline{A}$ and $\hat{B}$. $\overline{A}$ is a submatrix of A whose rows are those of A in which

the appropriate slack variables are not in G and whose columns are those A_i's (or subsets of them) having coefficients of x_i's that are not elements of G. The matrix $\hat{B}$ consists of the columns A_i (or subsets of them) having as entries coefficients of an element of G and of rows that are exactly like those of $\bar{A}$. Also α_G is as in case 1. Then $Z_G(\underline{b},\underline{c}) = \underline{\bar{t}}\underline{b}$, where for $\underline{\bar{t}} = (t_1, t_2, \ldots t_n)$, $t_i = 0$ whenever $x_i \notin G$, P_G is computed by formula (21), $P_G(\phi)$ is found by formula (23) $F(\phi) = \Sigma_H P_G(\phi)/(1-P^u)$.

Case 3. Both the $\underline{b}$ and the $\underline{c}$ vectors are stochastic. In this case, define the B, $\hat{B}$, and $\bar{A}$ matrices as they are in cases (1) and (2), set $Z_G(\underline{b},\underline{c}) = \underline{\bar{t}}B\underline{r}$, where $\underline{\bar{t}}$ is defined as in case (2), and compute P_G by formula (23) and $P_G(\phi)$ by formula (29). Unfortunately, formula (29) is not as straightforward in integration as in formula (13) or formula (26) and may have to be approximated by Monte Carlo techniques. However, it is easier to apply this technique to formula (29) than to formula (28), which is how $P_G(\phi)$ is normally expressed, also

$$F(\phi) = \Sigma_H P_G(\phi)/(1-P^{-}-P^u).$$

In the first two cases, the advantages of computing $F(\phi)$ are twofold: (1) it is theoretically exact, and (2) unlike the Monte Carlo methods and those of Bereanu, no

linear programming is needed for the computation except possibly for determining α_G. In the last case, although a version of the Monte Carlo method is needed, no linear programming is required (see Appendix B).

The first example starting on page 42 and especially the last example starting on page 48 give indications of the ease of computation that the change of variables as presented in this dissertation can effect. However, as the second example on page 45 illustrates, the computation need not be very efficient when the distribution has different functional forms for different regions of its domain. For this type of distribution, though, no method yet known is particularly efficient.

Although no extensive numerical calculations of P_G and $P_G(\phi)$ were made the computations of these quantities using numerical analysis may be a useful alternative to approximating $F(\phi)$ by the method of sample points, whose drawbacks have already been noted.

APPENDIX A

Let $G_p = \{x_1, x_2, \ldots x_m\}$ (x_j variable) be two different optimal bases and $D_p = \{\underline{b} | G_p$ is an optimal basis$\}$ for p = r,s.* The purpose of this appendix is to show that $D_r \bigcap D_s$ has zero content (a set in n dimensions has zero content if it cannot contain an n-cube. See [2], page 317). Define for a particular $\underline{b} \varepsilon D_r$, $G_r(\underline{b}) = (\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m, \ldots \bar{x}_{m+n})$ as an (m+n)-component vector of specific values of the x_i's making z(x) optimal under the constraining coefficients $(b_1, b_2, \ldots b_m)^T = \underline{b}$. Also define $\bar{\underline{c}} = (c_1, c_2, \ldots c_n, 0, \ldots 0)$, whence $z(D_r, \underline{b}) = \bar{\underline{c}}\, \bar{G}_r^T(\underline{b})$ is the optimal value of z(x) for $\underline{b} \varepsilon D_r$. Similarly for $\bar{G}_s(\underline{b})$ and $z(D_s, \underline{b})$. Finally, define $\bar{x}_j$ as corresponding to x_i (or let $\bar{x}_j$ correspond to x_i) if $\bar{x}_j$ is the specific value that replaces the variable x_i.

Because G_r and G_s are different primal bases, both contain variables not found in the other. Denote $L = \{x_k |$

---

*If two or more bases can be optimal and feasible simultaneously, change the objective function as given on pages 15 and 16, and choose the optimal bases that result.

$x_k \in G_r \bigcup G_s - (G_r \bigcap G_s)\}$. Given a $\underline{b}$, $z(D_r,\underline{b}) = \underline{\bar{c}}\ \bar{G}_r^T(\underline{b}) = \underline{\bar{c}}\ \bar{G}_s^T(\underline{b}) = z(D_s,\underline{b})$ implies that $G_r^T(\underline{b}) = G_s^T(\underline{b})$, (this follows from the definition of $z(D_r,\underline{b})$ and the fact that no two bases can be optimal and non-coincident simultaneously), and thus $\bar{x}_k=0$ for all corresponding $x_k \varepsilon L$. Hence if $\underline{b} \varepsilon D_r \bigcap D_s$, then both $\bar{G}_r^T(\underline{b})$ and $\bar{G}_s^T(\underline{b})$ have fewer than m nonzero elements, and all of these elements correspond to those found in $G_r \bigcap G_s$.

Since (A,I) is a matrix of constants, (A,I) represents a linear mapping from the set $\gamma = \{(\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m, \ldots \bar{x}_{m+n})^T\}$ to the set $\beta = \{\underline{b} = (b_1, b_2, \ldots b_m)^T\}$ as follows:

$$(A,I)\bar{G}^T(\underline{b}) = \underline{b}, \quad \bar{G}^T(\underline{b}) \in \gamma .$$

It has been shown that if $\underline{b}^* \varepsilon D_r \bigcap D_s$, then $\bar{G}^T(\underline{b}^*)$ must have fewer than m nonzero elements and they correspond to those of $G_r \bigcap G_s$. Let $P = \{\bar{G}^T \varepsilon \gamma \mid \bar{x}_j=0$ if the corresponding $x_i \notin (G_r \bigcap G_s)\}$. Then $\bar{G}^T(\underline{b}^*) \varepsilon P$ and $(A,I)P \supset D_r \bigcap D_s$. Since $\dim(P)<m$, $\dim((A,I)P)<m$, whence $\dim(D_r \bigcap D_s)<m$. Because $D_r \bigcap D_s \subset \beta$, β is m-dimensional, and $D_r \bigcap D_s$ cannot contain an m-cube, $D_r \bigcap D_s$ has zero content. q.e.d.

A similar result can be shown for the case of the c_i's being stochastic simply by considering the dual.

APPENDIX B

The purpose of this appendix is to demonstrate a version of the Monte Carlo method to be used for equation (29). Due to the definition of T' (i.e., $\underline{r} \geq 0$, $\underline{\hat{t}} \geq 0$, and $\underline{\bar{s}} \geq 0$), one does not need to consider negative numbers. First pick a basis, which determines the matrix B, and T'_ϕ. Next, given an $\varepsilon > 0$, pick numbers $M_1, \ldots M_m, M_{m+1}, \ldots M_{m+n}$ so that $\Pr\{u_i \geq M_i\} = \frac{\varepsilon}{m+n}$, where $u_i = r_i$ for $1 \leq i \leq m$, $u_i = t_i$ for $m+1 \leq i \leq m+h$, and $u_i = s_i$ for $m+h+1 \leq i \leq m+n$ (all matrices and vectors are assumed to have been rearranged as discussed in the case of the c_i's being stochastic). Hence, $\Pr\{0 \leq u_i \leq M_i \text{ for } i = 1, 2, \ldots m, \ldots m+n\} \geq \Pr\{u_i \geq 0\} - \sum_{i=1}^{m+n} \Pr\{u_i \geq M_i\} \geq \Pr\{u_i \geq 0\} - \varepsilon$.

Before describing the Monte Carlo method used on (29), some observations are in order. Let a function $f(\underline{x})$ be defined over some (finite) set in hyperspace, V. To approximate the integral of $f(\underline{x})$ over V, one can randomly pick a number of points, $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N$, from V. Then

$$\left(\sum_{i=1}^{N} \frac{f(x_i)}{N}\right) \times \text{(hypervolume of V)}$$

is an estimation of the desired integral.

Let $P = \{(\underline{r},\hat{\underline{t}},\bar{\underline{s}}) = (u_1, \ldots u_m, \ldots u_{m+n}) \mid 0 \le u_i \le M_i\}$ and $Q = \{(\underline{r},\hat{\underline{t}},\bar{\underline{s}}) \in P \mid \bar{\underline{t}}B\underline{r} \le \phi\}$. Define for a set W, V(W) to be the hypervolume of W in hyperspace. Finally, let $g(\underline{r},\hat{\underline{t}},\bar{\underline{s}}) = f(B\underline{r}, \hat{\underline{t}}\bar{B}, \underline{t}\bar{A}-\bar{\underline{s}})$.

Equation (28) is simply $Pr\{\underline{r} \ge 0, \hat{\underline{t}} \ge 0, \bar{\underline{s}} \ge 0$ and $\bar{\underline{t}}B\underline{r} \le \phi\} \approx Pr\{(\underline{r},\hat{\underline{t}},\bar{\underline{s}}) \in Q\}$. Let $\eta_1, \eta_2, \ldots \eta_N$ be a sequence of m+n random numbers:

$$\eta_i = (\zeta_1^i, \ldots \zeta_m^i, \ldots \zeta_{m+n}^i),$$

where $0 \le \zeta_j^i \le 1$. Define $\underline{v}_i = (M_1\zeta_1^i, \ldots M_m\zeta_m^i, \ldots M_{m+n}\zeta_{m+n}^i) = (u_1^i, \ldots u_m^i, \ldots u_{m+n}^i)$, $\underline{r}_i = \{u_j^i \mid 1 \le j \le m\}$, $\hat{\underline{t}}_i = \{u_j^i \mid m+1 \le j \le m+h\}$, and $\bar{\underline{s}}_i = \{u_j^i \mid m+h+1 \le j \le m+n\}$. Designate $\underline{v}_i$ as being in the set U_ϕ, denoted $v_i \in U_\phi$, if $\bar{\underline{t}}_i B\underline{r}_i \le \phi$, and let N_ϕ be the number of such vectors in U_ϕ.

Due to the above results, $Pr\{(\underline{r}, \hat{\underline{t}}, \bar{\underline{s}}) \in Q\} \approx \sum_{\underline{v}_i \in U_\phi} [g(\underline{r}_i, \hat{\underline{t}}_i, \bar{\underline{s}}_i)/N_\phi]V(Q)$. For a large number of vectors $\underline{v}_i$, $V(Q) \approx$ [(number of $\underline{v}_i$ in Q)/(number of $\underline{v}_i$ in P)] $\times$ V(P) = $[N_\phi/N] \prod_{j=1}^{m+n} M_j$. Hence, equation (29) is approximated by

$$\sum_{\underline{v}_i \in U_\phi} [g(\underline{r}_i, \hat{\underline{t}}_i, \bar{\underline{s}}_i)/N] \prod_{j=1}^{m+n} M_j .$$

EXAMPLES

As a first example, consider

$$\max z(x_1,x_2) = 2x_1 + 3x_2$$

subject to

$$x_1 + x_2 + x_3 = k_1$$

$$x_1 + 2x_2 + x_4 = k_2 ,$$

where $\Pr\{k_1 \leq b_1\} = F_1(b_1) = \begin{cases} 1 - e^{-b_1}, & b_1 \geq 0 \\ 0, & \text{otherwise} \end{cases}$

and

$$\Pr\{k_2 \leq b_2\} = F_2(b_2) = \begin{cases} 1 - e^{-2b_2}, & b_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

As shown in Figure 1, the possible bases are (x_1,x_2), (x_1,x_3), and (x_2,x_4). The joint probability distribution is $f(b_1,b_2) = f_1(b_1)f_2(b_2) = 2e^{-(b_1+2b_2)}$.

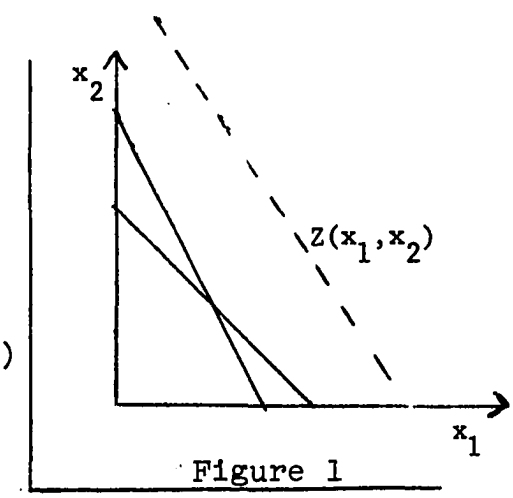


Figure 1

The probability that $G_1 = (x_1,x_2)$ forms a basis is now computed:

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B\underline{r} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = (r_1 + r_2,\ r_1 + 2r_2) = (b_1, b_2).$$

Hence,

$$f(B\underline{r}) = 2e^{-[(r_1+r_2) + 2(r_1+2r_2)]} = 2e^{-(3r_1+5r_2)}$$

$$|J_r| = |\det(B)| = 1.$$

Thus, applying the formula,

$$P_{G_1} = \int_0^\infty \int_0^\infty 2e^{-(3r_1+5r_2)} (1)\, dr_1 dr_2 = \frac{2}{15}$$

If the probability that $G_2 = (x_1, x_3)$ forms a basis is desired, one has

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B\underline{r} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = (r_1 + r_2,\ r_1)$$

$$f(B\underline{r}) = 2e^{-(3r_1+r_2)}, \quad |J_r| = 1,$$

and so $P_{G_2} = \int_0^\infty \int_0^\infty 2e^{-(3r_1+r_2)} (1)\, dr_1 dr_2 = \frac{2}{3}$

Finally for $G_3 = (x_2, x_4)$,

$$B\underline{r} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = (r_1,\ 2r_1 + r_2),$$

and so

$$P_{G_3} = \int_0^\infty \int_0^\infty 2e^{-(5r_1+2r_2)} (1)\, dr_1 dr_2 = \frac{1}{5}$$

The $Pr\{\max z(x_1,x_2) \leq \phi\}$ is computed next:

For G_1, the space of integration is bounded by $r_1 = 0$, $r_2 = 0$, and $\underline{C}_B\underline{r} = 2r_1+3r_2 \leq \phi$. Hence, for $\phi \geq 0$,

$$P_{G_1}(\phi) = \int_{r_2=0}^{1/3\phi} \int_{r_1=0}^{1/2\phi-3/2r_2} 2e^{-(3r_1+5r_2)}\, dr_1 dr_2$$

$$= \int_{r_2=0}^{1/3\phi} \frac{2}{3}[e^{-5r_2} - e^{(-3/2\phi-1/2r_2)}]dr_2$$

$$= \frac{2}{15} - \frac{2}{15} e^{-5/3\phi} - \frac{4}{3} e^{-3/2\phi} + \frac{4}{3} e^{-5/3\phi},$$

and $P_{G_1}(\phi) = 0$ for $\phi<0$.

For G_2, the space of integration is bounded by $r_1=0$, $r_2=0$, and $\underline{C}_B r = 2r_1 \leq \phi$. Thus, for $\phi \geq 0$,

$$P_{G_2}(\phi) = \int_0^{\infty} \int_0^{\phi/2} 2e^{-(3r_1+r_2)}\, dr_1 dr_2$$

$$= \int_0^{\infty} 2/3\, [e^{-2r_2} - e^{(-3/2\phi-r_2)}]dr_2$$

$$= 2/3 - 2/3\, e^{-3/2\phi}$$

Similarly for G_3:

$$P_{G_3}(\phi) = \int_0^{\infty} \int_0^{\phi/3} 2d^{-(5r_1+2r_2)}\, dr_1 dr_2$$

$$= 1/5 - 1/5\, e^{-5/3\phi}$$

$$F(\phi) = P_{G_1}(\phi) + P_{G_2}(\phi) + P_{G_3}(\phi) = 1 + e^{-5/3_\phi} - 2e^{-3/2_\phi}$$

for $\phi \geq 0$.

As a second example, consider

$$\max \quad z(x) = K_1 x_1 + K_2 x_2$$

subject to

$$x_1 + 2x_2 + x_3 = 3$$

$$3x_1 + x_2 + x_4 = 5$$

$$x_1, x_2 \geq 0$$

where

$$F_1(c_1) = \Pr\{K_1 \leq c_1\} = \begin{cases} c_1^2, & 0 \leq c_1 \leq 1 \\ 0, & c_1 < 1 \\ 1, & c_1 > 1 \end{cases}$$

and

$$F_2(c_2) = \Pr\{K_2 \leq c_2\} = \begin{cases} c_2^2, & 0 \leq c_2 \leq 1 \\ 0, & c_2 < 1 \\ 1, & c_2 > 1 \end{cases}$$

Hence,

$$f(c_1, c_2) = \begin{cases} 4c_1c_2, & 0 \leq c_1 \leq 1 \text{ and } 0 \leq c_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The probability that $G_1 = (x_1, x_2)$ is in solution is calculated first. It is computed thus:

$$\bar{A} = \{\phi\}$$

$$\hat{B} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$(c_1, c_2) = \hat{t}\hat{B} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = (t_1 + 3t_2, 2t_1 + t_2)$$

$$\det(B) = 1 - 6 = -5$$

$$f(c_1, c_2) = f(\underline{\hat{t}}\hat{B}, \underline{\hat{t}}\overline{A} - \underline{\overline{s}}) = 4(t_1+3t_2)(2t_1+t_2)$$
$$= 4(2t_1^2 + 7t_1t_2 + 3t_2^2).$$

$$P_{G_1} = \int_0^\infty \int_0^\infty 4(2t_1^2+7t_1t_2+3t_2^2)\alpha_R \cdot 5dt_1dt_2,$$

where

$$\alpha_R = \begin{cases} 1, \text{ if } 0 \le t_1+3t_2 \le 1 \text{ and} \\ \qquad 0 \le 2t_1+t_2 \le 1 \\ 0, \text{ otherwise} \end{cases}$$

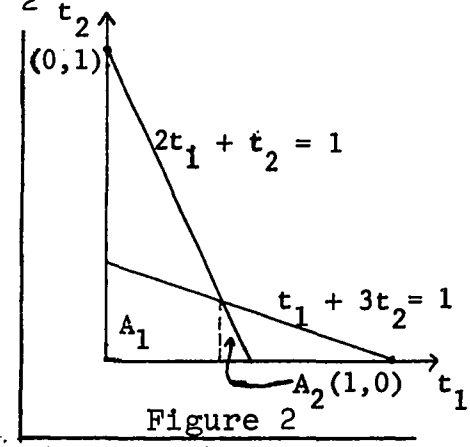


Figure 2

(Recall that $f(c_1, c_2)$ is nonzero only if $0 \le c_1 \le 1$ and $0 \le c_2 \le 1$ and that $(c_1,c_2) = (t_1+3t_2, 2t_1+t_2)$).

$$P_{G_1} = 20 \iint_{A_1} (2t_1^2 + 7t_1t_2 + 3t_2^2)dt_2dt_1$$
$$+ 20 \iint_{A_2} (2t_1^2 + 7t_1t_2 + 3t_2^2)dt_2dt_1,$$

where A_1 and A_2 are as shown in Figure 2. Carrying out the integration yields

$$P_{G_1} \approx .82.$$

If the probability that $G_2 = (x_1, x_3)$ is desired, one has, $\alpha_{G_2} = 1$ by reference to Figure 3, $\overline{A} = [1]$ $\hat{B} = [3]$

$\underline{\hat{t}}\hat{B} = 3t_1 \quad \underline{\hat{t}}\bar{A} = \underline{s} = t_1 - s_1$

$(c_1, c_2) = (3t, t_1 - s_1)$

$\det(B) = (-1)^{2-1}(3) = -3$

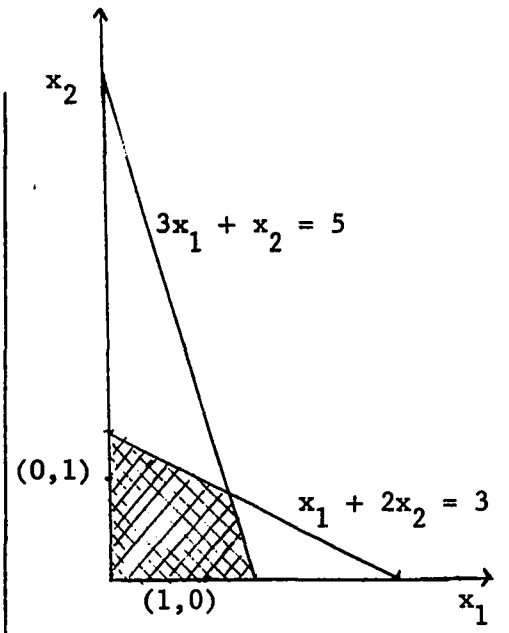
$f(c_1, c_2) = f(\underline{\hat{t}}\hat{B}, \underline{t}\bar{A} - \bar{\underline{s}})$

$= 4(3t_1)(t_1 - s_1) = 12(t_1^2 - t_1 s_1)$

$$P_{G_2} = \int_0^\infty \int_0^\infty 12(t_1^2 - t_1 a_1) \cdot 3\alpha_A \, ds_1 dt_1,$$

where

$$\alpha_A = \begin{cases} 1, \text{ if } t_1 - s_1 \leq 0 \text{ and } t_1 \leq \frac{1}{3} \\ 0, \text{ otherwise} \end{cases}$$



Feasible region is crosshatched.

Figure 3

Refer to Figure 4.

$$P_{G_2} = \int_0^{1/3} \int_0^{t_1} 36(t_1^2 - t_1 s_1) ds_1 dt_1$$

$$= \frac{1}{18} \approx .06$$

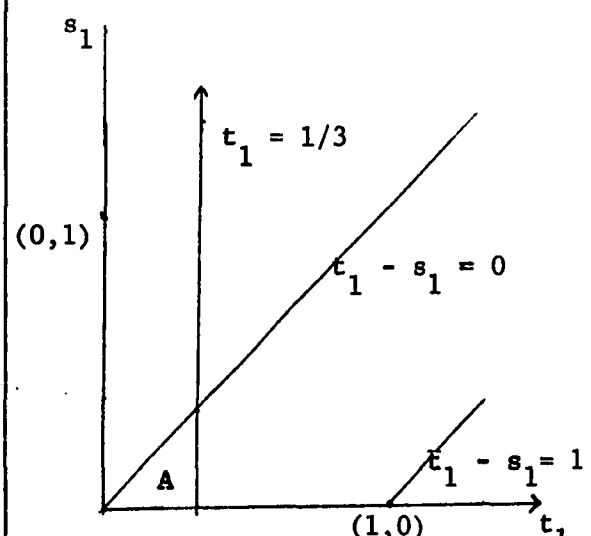


Figure 4

Finally one can find the distribution of $Z(\underline{c})$, given $G_2 = (x_1, x_3)$ is the basis. In this case, the probability that $Z(\underline{c}) = \hat{\underline{t}} \cdot \hat{\underline{b}} = 3t_1 + 5t_2 = 3t_1 \leq \phi$ is desired (recall that $t_2 = 0$ in this case.

Therefore, $P_{G_2}(\phi) = \int_0^\infty \int_0^\infty 12(t_1^2 - t_1 s_1)\alpha_A 3 ds_1 dt_1$, where α_A is as defined before.

$$P_G(\phi) = 36 \int_0^{\phi/3} \int_0^{t_1} \alpha_A(t_1^2 - t_1 s_1) ds_1 dt_1$$

$$= \frac{\phi}{18} , \quad 0 \leq \phi \leq 1$$

$$\frac{1}{18} , \quad \phi > 1$$

$$0 , \quad \phi < 0.$$

This example problem shows that the change of variables as described in this dissertation can ease considerably the computation of some probabilities. Suppose that a maximization problem had the following constraints:

$$\begin{aligned}
x_1 + 2x_2 + 4x_3 + 3x_4 + x_5 &= b_1 \\
3x_1 + 7x_2 + 9x_3 + 6x_4 + x_6 &= b_2 \\
2x_1 + 3x_2 + 12x_3 + 9x_4 + x_7 &= b_3 \\
x_1 + 11x_3 + 10x_4 + x_8 &= b_4 ,
\end{aligned}$$

where

$$f(\underline{b}) = \begin{cases} 4e^{-(b_1+2b_2+2b_3+b_4)} & \text{for } b_1, b_2, b_3, b_4 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Further suppose that $G = \{x_1, x_2, x_3, x_4\}$ is an optimal basis and it is desired to find P_G.

In this situation, $B = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 7 & 9 & 6 \\ 2 & 3 & 12 & 9 \\ 1 & 0 & 11 & 10 \end{bmatrix}$,

and making the change of variables, one has

$$B\underline{r} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 7 & 9 & 6 \\ 2 & 3 & 12 & 9 \\ 1 & 0 & 11 & 10 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} r_1 + r_2 + 4r_3 + 3r_4 \\ 3r_1 + 7r_2 + 9r_3 + 6r_4 \\ 2r_1 + 3r_2 + 12r_3 + 9r_4 \\ r_1 + 11r_3 + 10r_4 \end{bmatrix}$$

and note that $B\underline{r} \geq 0$ for $\underline{r} \geq 0$. Thus $f(\underline{b}) = 4e^{-(b_1+2b_2+2b_3+b_4)} = 4e^{-(12r_1+21r_2+57r_3+43r_4)}$, $|J_r| = 1$, and

$$P_G = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(B\underline{r})\, dr_1 dr_2 dr_3 dr_4 = \frac{4}{(12)(21)(57)(43)} .$$

If the integration is done directly without a change of variables, then

$$P_G = \int_{B^{-1}\underline{b} \geq 0} \int \int \int f(\underline{b})\, db_1 db_2 db_3 db_4 .$$

$$B^{-1} = \begin{bmatrix} 75 & -21 & -1 & -9 \\ -24 & 7 & 0 & 3 \\ -5 & 1 & 1 & 0 \\ -2 & 1 & -1 & 1 \end{bmatrix} \text{ and so } B^{-1}\underline{b} = \begin{bmatrix} 75 & -21 & -1 & -9 \\ -24 & 7 & 0 & 3 \\ -5 & 1 & 1 & 0 \\ -2 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$= \begin{bmatrix} 75b_1 - 21b_2 - b_3 - 9b_4 \\ -24b_1 - 7b_2 - 3b_4 \\ 5b_1 + b_2 + b_3 \\ -2b_1 + b_2 - b_3 + b_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Unlike the situation after the change of variables, the determination of the limits for b_1, b_2, b_3, and b_4 present great difficulties.

BIBLIOGRAPHY

1. Babbar, M. M., "Distributions of Solutions of a Set of Linear Equations with an Applications to Linear Programming, Journal of the American Statistical Association, Vol. 50 (1955), pages 854-869.

2. Bartle, R. G., The Elements of Real Analysis (ed.) John Wiley and Sons, Inc., (1964).

3. Bereanu, B., "On Stochastic Linear Programming. I. Distribution Problems: A Single Random Variable," Revue Roumaine de Mathematiques Pures et Appliquees, Vol. 8 (1963), pages 683-697.

4. Bereanu, B., "On Stochastic Linear Programming. II. Distribution Problems, Nonstochastic Technological Matrix," Revue Roumaine de Mathematiques Pures et Appliquees, Vol. 11 (1966), pages 713-725.

5. Bereanu, B., "On Stochastic Linear Programming. The Laplace Transform of the Distribution of the Optimum and Applications," Journal of Mathematical Analysis and Applications, Vol. 15 (1966), pages 280-294.

6. Bracken, J., and Soland, R. M., "Statistical Decision Analysis of Stochastic Linear Programming Problems," Naval Research Logistics Quarterly, Vol. 13 (1966), pages 205-225.

7. Charnes, A., and Cooper, W. W., "Deterministic Equivalents for Optimizing and Satisfying Under Chance Constraints," Operations Research, Vol. 11 (1963), pages 18-39.

8. Charnes, A., and Kirby, M., "Some Special P-Models in Chance-Constrained Programming," Management Science, Vol. 14 (1967-68), pages 183-195.

9. El Agizy, M., "Two-Stage Programming Under Uncertainty with Discrete Distribution Function," Operations Research, Vol. 15 (1967), pages 55-70.

10. Elmaghraby, S. E., "An Approach to Linear Programming Under Uncertainty," Operations Research, Vol. 7 (1959), pages 208-216.

11. Elmaghraby, S. E., "Allocation Under Uncertainty When the Demand has Continuous d.f.," Management Science, Vol. 6 (1959-60), pages 270-294.

12. Geoffrion, A. M., "Stochastic Programming with Aspiration or Fractile Criteria," Management Science Vol. 13 (1966-67), pages 672-679.

13. Kataoka, S., "A Stochastic Programming Model," Econometrica, Vol. 3 (1963), pages 181-196.

14. Madansky, A., "Methods of Solution of Linear Programs Under Uncertainty," Operations Research, Vol. 10 (1962), pages 463-471.

15. Miller, B. L., and Wagner, H. M., "Chance Constrained Programming with Joint Constraints," Operations Research, Vol. 13 (1965), pages 930-945.

16. Naslund, B., "A Model of Capital Budgeting Under Risk," Journal of Business, Vol. 39 (1966), pages 257-271.

17. Prekopa, A., "On the Probability Distribution of the Optimum of a Random Linear Program," S.I.A.M. Journal on Control, Vol. 4 (1966), pages 211-222.

18. Radner, R., "The Applications of Linear Programming to Team Decision Problems," Management Science Vol. 5 (1958-59), pages 143-150.

19. Sengupta, J. K., "Safety-First Rules under Chance-Constrained Linear Programming," Operations Research, Vol. 17 (1969), pages 112-132.

20. Sengupta, J. K., Tintner, G., and Millham, C., "On Some Theorems of Stochastic Linear Programming with Applications," Management Science, Vol. 10 (1963-64), pages 143-159.

21. Sengupta, J. K., Tintner, G., and Morrison, B., "Stochastic Linear Programming with Applications to Economic Models," Economica, Vol. 30 (1963), pages 262-276.

22. Symonds, G. H., "Deterministic Solutions for a Class of Chance-Constrained Programming Problems," Operations Research, Vol. 15 (1967), pages 495-512.

23. Szwarc, W., "The Transportation Problem with Stochastic Demand," Management Science, Vol. 11 (1964-65), pages 33-50.

24. Taha, H. A., Operations Research, An Introduction (ed.) The MacMillan Company, New York (1971).

25. Walkup, D. W., and Wets, R. J. B., "Stochastic Programs with Recourse," S.I.A.M. Journal on Applied Mathematics, Vol. 15 (1967), pages 1299-1314.

26. Wets, R. J. B., "Programming Under Uncertainty: The Equivalent Convex Program," S.I.A.M. Journal on Applied Mathematics, Vol. 14 (1966), pages 89-105.

27. Wets, R. J. B., "Programming Under Uncertainty: The Solution Set," S.I.A.M. Journal on Applied Mathematics, Vol. 14 (1966), pages 1143-1151.

28. Williams, A. C., "A Stochastic Transportation Problem," Operations Research, Vol. 11 (1963), pages 759-770.

29. Williams, A. C., "On Stochastic Linear Programming," S.I.A.M. Journal on Applied Mathematics, Vol. 13 (1965), pages 927-940.

30. Williams, A. C., "Approximation Formulas for Stochastic Linear Programming," S.I.A.M. Journal on Applied Mathematics, Vol. 14 (1966), pages 668-677.

31. Wilson, R., "On Programming Under Uncertainty," Operations Research, Vol. 14 (1966), pages 652-657.

32. Zinn, Clyde Dale, "An Algorithm for the Solution of the Distribution Problem of Probabilistic Linear Programming," unpublished Ph.D. dissertation, University of Oklahoma, Norman, Oklahoma, 1971.