A Method of Calculating the Direct Shortwave Radiation Income of Slopes

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ABSTRACT

A formula has been developed whereby variations in the shortwave radiation income on different slopes in any latitude can be easily determined from a knowledge of surface geometry and the sun's declination. This is achieved by expressing slope characteristics and the sun's position as unit vectors in the coordinates of a common system, and multiplying the cosine of the angle between the two vectors by a factor involving the solar constant, atmospheric transmissivity and the optical air mass. Integration of the formula permits the topographic variations of direct shortwave radiation income over specified areas to be calculated for daily or longer periods from radiation observations at a single site. Thus, a device has been developed which has applications in many aspects of pure and applied microclimatology, and which differs from other similar formulae in that it is at the same time both integrable and applicable under different atmospheric transmissivities.

1. Introduction

The important role played by radiation in the heat budget of the earth's surface has long been recognized. This radiation consists of shortwave radiation—direct, diffuse, and reflected—and longwave radiation, both from the earth's surface, and received at the earth's surface by emission from the atmosphere. Among these components the shortwave global radiation (direct and diffuse) is fundamentally important since, in one way or another, the other components depend on it. Moreover, within a given area, the intensity of the direct, as distinct from diffuse, shortwave radiation on a surface varies considerably according to the geometry of the terrain. The other components of radiation income, by contrast, vary only slightly from one slope to another and in a way directly related to the angle of slope (Kondrat'yev, 1965, Chapter 7). Thus, the calculation of how direct solar radiation varies topographically over an area is primary to understanding the details of the surface heat budget. The present article describes a way of making this calculation from the observations of a single representative site.

2. Fundamental formulae

The instantaneous intensity of direct shortwave radiation on any slope is a composite of the energy being delivered by the sun's rays and its modification by the angle and azimuth of the slope in question. The monochromatic intensity I_λ of the solar beam at the surface is given by

$$I_\lambda = I_{\lambda 0} p_\lambda{}^m, \quad (1)$$

where $I_{\lambda 0}$ is the monochromatic intensity of the solar beam at the top of the atmosphere, p_λ the monochromatic zenith-path transmissivity, and m the optical air mass.

The integration of (1) with respect to wavelength provides the total incident intensity at the ground surface. This integration permits the derivation of a simple law of transmission (Haltiner and Martin, 1957) which, if used to denote the direct solar energy falling on 1 cm² for 1 min, can be conveniently written as[1]

$$I_m = I_0 p^m, \quad (2)$$

where I_m is the direct solar radiation (ly min^{-1}) reaching a surface normal to the sun's rays, I_0 is the solar constant, and p the mean-zenith-path transmissivity of the atmosphere.

The modification to I_m introduced by a given slope is a function of the relation between the angle and azimuth of the slope on the one hand, and the azimuth and height of the sun on the other. If these respective characteristics of slope and sun are expressed by unit coordinate vectors, the flux per minute I_s of direct shortwave radiation on a given slope is the product of I_m and the cosine of the angle between the two vectors, such that

$$I_s = I_0 p^m \cos(\mathbf{X} \wedge \mathbf{S}), \quad (3)$$

where **X** is a unit coordinate vector normal to the slope and pointing away from the ground, **S** a unit coordinate vector expressing the height and position of the sun, and $\wedge$ is a symbol denoting the angle between **X** and **S**.

For (3) to have universal applicability, it is necessary to express **X** and **S** in the coordinates of a common

[1] Haltiner and Martin describe the formula as "an approximate, but adequate, form of the law of transmission of solar radiation." Reference to tables available in de Brichambaut (1963) suggests that this form of the law is accurate to the third decimal place for calculations over a period of 1 min.

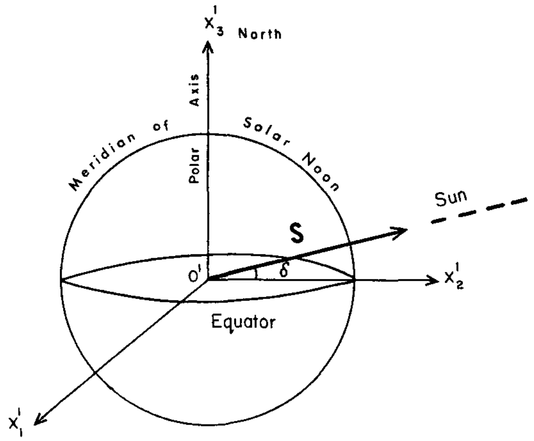

FIG. 1. Coordinate axes for the position of the sun.

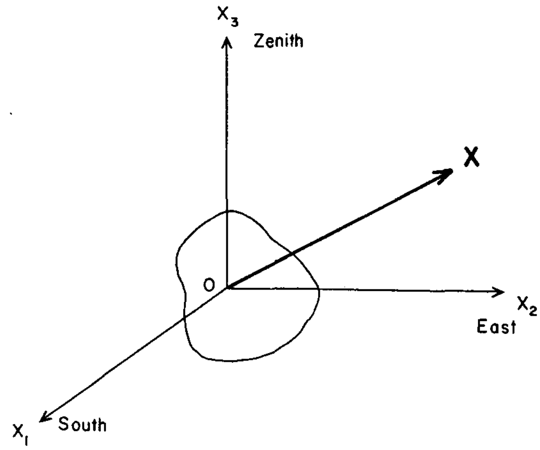

FIG. 2. Coordinate axes for expressing the character of a slope in a local system.

system and to integrate the equation for periods of time >1 min. This procedure is simplified by treating the sun as moving north and south with the seasons along the plane of the meridian of solar noon for the slope, and by treating the slope as rotating about the earth's polar axis.

3. Expression of $\cos(X \wedge S)$

By considering the sun as moving north and south along the meridian of solar noon, the unit coordinate vector **S** can be specified solely in terms of the sun's declination. This is clear from Fig. 1 in which X'_3 represents a line on the earth's polar axis and pointing north, X'_2 is at right angles to X'_3 and cuts the meridian of solar noon, and X'_1 is vertical to the plane of the meridian of solar noon. Thus,

$$\mathbf{S}=(0, \cos\delta, \sin\delta), \tag{4}$$

where δ is the sun's declination, positive when the sun is north of the equator and negative when the sun is south of the equator.

The unit coordinate vector **X** may be expressed in global terms and its changing relationships with respect to the position of the sun demonstrated, by first specifying **X** in terms of a local system as shown in Fig. 2, and then transferring the origin of the axes of the local system to the center of the earth and conceiving the system as rotating once a day around the polar axis. The axes chosen for the local system enable **X** to be expressed in terms of the slope's angle and azimuth in the form

$$\mathbf{X}=[(-\cos A \sin Z_x), (\sin A \sin Z_x), \cos Z_x], \tag{5}$$

where A is the azimuth of the slope measured from north through east, and Z_x is the zenith angle of the vector **X**.

The transfer of this local system into a global system can then be effected by following the principles of vector algebra with the result that the expression $\cos(\mathbf{X}\wedge\mathbf{S})$ becomes

$$\begin{aligned}\cos(\mathbf{X}\wedge\mathbf{S})=[&(\sin\phi \cos H)(-\cos A \sin Z_x)\\ &-\sin H(\sin A \sin Z_x)+(\cos\phi \cos H)\cos Z_x]\cos\delta\\ &+[\cos\phi(\cos A \sin Z_x)+\sin\phi \cos Z_x]\sin\delta,\end{aligned} \tag{6}$$

where ϕ is the latitude of the slope, and H the hour angle measured from solar noon, positively toward west.

4. Integration

The integration of (3) for daily totals may be achieved by choosing sufficiently small intervals of time H in the equation

$$I_d=I_0\int_{H_1}^{H_2} p^m \cos(\mathbf{X}\wedge\mathbf{S})dH, \tag{7}$$

in which I_d is the daily total of direct radiation in langleys and H_1 and H_2 are the times when the sun shines on the slope for the first and last times each day.[2]

To perform this integration it is necessary to express the optical air mass m as a function of time H. This may be done by using the secant approximation for m, from which it follows that

$$m=\sec Z_s=1/\cos Z_s, \tag{8}$$

where Z_s is the zenith angle of the sun.[3] For a horizontal

[2] Since (7) is not mathematically integrable its solution must be achieved by summation for small intervals of H; a value of $H=20$ min is the largest interval of H which can be used in order to give daily totals which are the same to the first decimal place as those achieved using $H=1$ min. Accordingly, an interval of $H=20$ min has been used to calculate I_d for the purposes described in this article.

[3] It is recognized that in calculating m it is customary to allow for atmospheric pressure so that $m=\sec Z_s(P/1000)$ where P is pressure in millibars. The pressure factor has been neglected in the present discussion for two reasons: the modification of m under normal pressure ranges is less than ±2%, which is insignificant for the overall results, and if the formula is used in conjunction with measured radiation at a single, representative site, the pressure correction for m is, in any case, built in.

surface

$$\cos Z_s = \cos(\mathbf{X} \wedge \mathbf{S}). \quad (9)$$

Since for such a surface the components of **X** in (5) are 0, 0, 1, the value of m by substitution in (6) becomes

$$m = 1/(\cos\delta \cos\phi \cos H + \sin\delta \sin\phi). \quad (10)$$

The effective formula for calculating daily totals of direct radiation is, therefore, that shown in Eq. (7), using the values for m and $\cos(\mathbf{X} \wedge \mathbf{S})$ given in (10) and (6) respectively, together with a recognized value for I_0, such as $I_0 = 2.00$ ly min^{-1} (Johnson, 1954).

Calculating m in (7) by means of (10) underestimates solar energy when $Z_s > 70°$, since the secant approximation for m becomes inaccurate at low sun altitudes (Haltiner and Martin, 1957; de Brichambaut, 1963). In general, the lower the sun the larger is the error, although the extent of the error varies from slope to slope. Assuming that a maximum underestimation of 10% in daily totals is acceptable, it can be shown that for horizontal surfaces this degree of error is never exceeded by using (10) in (7) for areas within 35° of the equator. At 40N the whole year is free of error except for two weeks on either side of the winter solstice, but at 80N the error-free period is only from 20 May to 20 July. During periods of potential error, therefore, the value of m should be either calculated more exactly or else read from tables (List, 1963, Table 137).

5. Atmospheric transmissivity

The only element of (7) which is not readily obtainable from tables, earth/sun relationships, or terrain analysis, is atmospheric transmissivity. This varies considerably from place to place according to the weather and air mass conditions. For completely cloudless days it is possible to obtain good approximations of atmospheric transmissivity by standard methods (Fritz, 1951) and in the absence of measured radiation data, such estimates must be used. However, the formula developed in this article is intended to be used in conjunction with measured radiation data from a single site, representative of whatever area is being studied. Such data can be used to obtain the atmospheric transmissivity either by sampling through the day or by obtaining an "equivalent atmospheric transmissivity" term by reference to the daily radiation total, and the atmospheric transmissivity equivalent to this total given in tables prepared from (7) (see Table 1). Both of these methods depend on being able to separate direct from diffuse shortwave radiation in the radiation data being used.

6. Computer programming

High speed computer facilities are essential if the formula suggested here is to be put to practical use. Such facilities enable tables to be prepared showing the direct shortwave radiation income on different slopes for different dates anywhere on earth. An example of such a table, prepared for use in latitude 45N on a clear equinoctial day, is given in Table 1. The computations involved were effected by an IBM 360 computer using Fortran 4 programming. Although the programming can be in any desired degree of detail, in practice it has been found sufficient to prepare tables for 10° intervals for the slope angle and azimuth characteristics, and to compute the results for atmospheric transmissivity p from 0.50–0.85 at intervals of 0.05. Calculating for $p < 0.50$ is not worthwhile since at such low atmospheric transmissivities diffuse radiation contributes too high a

TABLE 1. Daily totals of direct solar radiation (ly day^{-1}) at the equinoxes for 45N for an atmospheric transmissivity of 0.75.

Angle of slope	Azimuth of slope 0	10	20	30	40	50	60	70	80	90
0.0	369.6	369.6	369.6	369.6	369.6	369.6	369.6	369.6	369.6	369.6
10.0	299.8	300.7	303.6	308.4	314.8	322.8	332.1	342.3	353.1	364.2
20.0	220.8	222.8	228.7	238.6	252.3	269.0	287.9	308.3	329.6	351.2
30.0	135.2	138.3	149.1	166.9	189.7	215.7	244.0	273.7	304.0	334.0
40.0	45.4	54.4	77.1	105.1	136.3	170.0	205.2	241.5	277.9	314.0
50.0	0.0	9.3	33.6	63.8	97.5	134.0	172.4	212.0	251.9	291.4
60.0	0.0	2.4	16.4	40.7	71.6	106.8	145.1	184.8	225.6	266.2
70.0	0.0	1.0	9.3	27.6	53.7	85.5	121.4	159.5	198.9	238.2
80.0	0.0	0.5	6.0	19.5	40.9	68.4	100.2	135.3	171.3	208.1
90.0	0.0	0.3	3.9	13.9	30.8	53.8	81.4	111.7	144.0	176.1

Angle of slope	Azimuth of slope 90	100	110	120	130	140	150	160	170	180
0.0	369.6	369.6	369.6	369.6	369.6	369.6	369.6	369.6	369.6	369.6
10.0	364.2	375.4	386.1	396.2	405.3	413.2	419.6	424.3	472.2	428.2
20.0	351.2	372.4	393.0	412.1	429.5	444.5	457.0	466.2	471.9	473.8
30.0	334.0	363.6	391.8	418.2	442.1	463.2	480.7	493.9	502.2	505.0
40.0	314.0	349.3	383.0	414.6	443.4	468.8	490.3	506.8	517.3	520.8
50.0	291.4	330.0	366.9	401.5	433.2	461.4	485.5	504.3	516.6	520.8
60.0	266.2	305.7	343.5	379.2	412.0	441.2	466.5	486.8	500.2	505.0
70.0	238.2	276.8	313.8	348.2	380.4	409.1	434.0	454.5	468.7	473.8
80.0	208.1	243.9	278.2	310.2	339.7	366.2	389.4	409.0	423.0	428.2
90.0	176.1	208.2	238.2	266.3	291.5	314.1	334.2	351.4	364.5	369.6

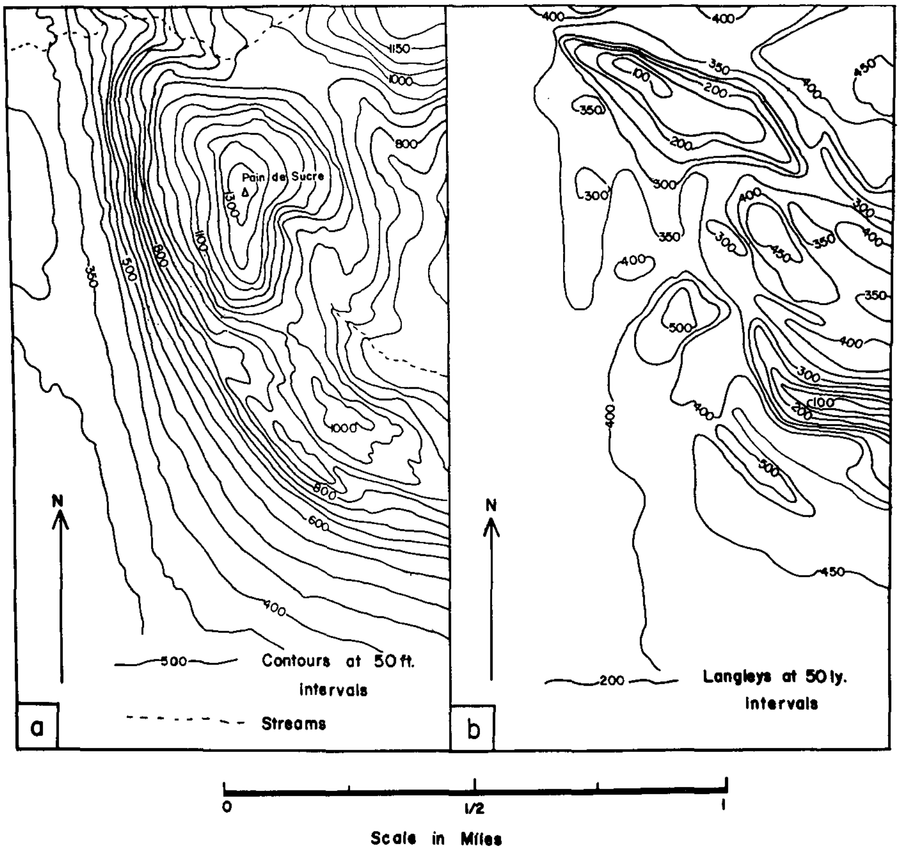

FIG. 3. The variation of direct solar radiation on part of Mont St. Hilaire, Quebec, on 19 September 1967: a., topographic base map; b., radiation map.

percentage of total shortwave radiation for the formula to be used effectively.

7. Application of the formula

Since the formula has been derived chiefly as a means of perceiving the topographic variations of shortwave radiation income, its primary use is for making maps of such variations. An example of such a map is given in Fig. 3. It shows how shortwave radiation income varied over a small part of Mont St. Hilaire, 25 mi east of Montreal, on 19 September 1967. The day was clear and the radiation record at the McGill University Mont St. Hilaire climatology station revealed an atmospheric transmissivity of 0.75.

To draw the radiation map, the topographic map (Fig. 3a) was covered with a grid of squares with sides representing 150 yards on the ground. At each intersection of the grid the radiation for the day was noted, having regard to the slope angle and azimuth at the point, and also to any necessary correction for skyline obstruction from neighboring relief features.[4] Isolines were then drawn and the resulting map is given in Fig. 3b. It shows considerable variations in shortwave radiation over the area. The range is from over 500 to under 100 ly. In addition, some well-marked zoning is apparent. The land west of the mountain, for example, has under 400 ly whereas the southwest slopes exceed this figure. The north-facing sides of the valleys on either side of Pain de Sucre are, by contrast, poorly irradiated, receiving less than half the shortwave radiation experienced both on slopes across the two valleys and generally elsewhere in the area mapped. Variations such as these have important effects on the heat budget from place to place, and noticeably influence the fluxes of sensible and latent heat. By using the formula proposed, moreover, the evaluation of these fundamentally important elements of microclimate becomes possible without the use of a dense network of observation points.[5]

Although the formula can be used most fundamentally to map distributions, it has other applications. Fig. 4, for example, shows the accumulated values through the day for different atmospheric transmis-

[4] A simple technique for making this correction has been developed but its explanation is too long for inclusion here.

[5] A larger program is at present being developed which will enable diffuse sky radiation to be included in the mapping. In addition, there will be provision for the reflection due to surface albedo (Ohmura, 1968).

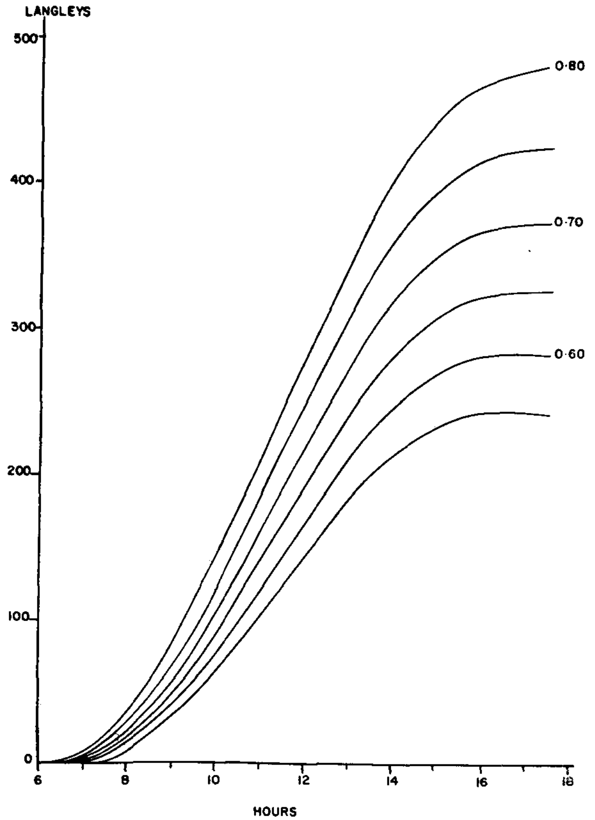

FIG. 4. Daily accumulation of direct solar radiation at the equinoxes for different atmospheric transmissivities (0.55–0.80) on a 10° slope facing 60° south of east at 45N.

sivities at the equinox for a given slope. The computer programming can be easily adapted for providing accumulations of this type over short or long periods to any degree of detail required. Such diagrams suggest many possible additional applications of the formula, not only in respect of biological and agricultural research on such matters as plant growth or crop water balance, but also in respect of urban planning, house siting, and even house design.

8. Conclusion

It is suggested that the formula outlined in this article for calculating direct shortwave radiation, provides a valuable tool for estimating variations of this element in space and time in a number of ways. It can be used to prepare appropriate tables and diagrams for generalized use on a monthly or seasonal basis in different latitudes. It is not, of course, the first formula which has been devised for such purposes. However, the present proposal seems particularly noteworthy in two respects: it permits integration over any period of time, whereas most other formulae are for instantaneous values only; and in developing the integration, atmospheric transmissivity has been allowed for, whereas other integrations (e.g., Frank and Lee, 1966) tend to reduce their usefulness by neglecting this aspect of reality.

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