A METHOD OF DETERMINING EXPLICITLY THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION

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1. Introduction. When an investigator is interested in all of the latent roots of the characteristic equation of a matrix and not in its latent vectors, it is sometimes desirable to expand out the determinental equation in order to determine explicitly the polynomial coefficients $(p_1, p_2, \cdots, p_n)$ in the expression

$$D(\lambda) = |\lambda I - a| = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n . \tag{1}$$

This can be done in a variety of ways, all of which are necessarily somewhat tedious for high order matrices. Except for sign the coefficients are respectively the sum of a's principal minors of a given order. These can be computed efficiently by "pivotal" methods [1]. Alternatively through the utilization of the Cayley-Hamilton theorem, whereby a matrix satisfies its own characteristic equation, the p's appear as the solution of n linear equations [2, 3]. In a third method Horst has employed Newton's formula concerning the powers of roots to derive the p's as the solution of a triangular set of equations, the coefficients of the latter only being attained after considerable matrix multiplication [4]. A fourth method suggested to me by Professor E. Bright Wilson, Jr. of Harvard University, consists of evaluating $D(\lambda)$ for n values of λ, presumably by efficient "Doolittle" methods; to these n points, Lagrange's interpolation formula is applied to determine the n coefficients explicitly.

2. The New Method. The present paper describes a new computational method based upon well-known dynamical considerations. A single nth order differential equation can be converted into "normal" form, involving n first order differential equations. This is easily done by defining appropriate new variables. If the original nth order differential equation is written as

$$X^{(n)}(t) + p_1X^{(n-1)}(t) + \cdots + p_{n-1}X'(t) + p_n = 0, \tag{2}$$

then the new normal system can be written as

$$X_i'(t) = \sum_1^n b_{ij}\, X_j(t), \qquad (i = 1, \cdots n) \tag{3}$$

where

$$[b_{ij}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{bmatrix} \tag{4}$$

is the so-called companion matrix to the polynomial in question.

The reverse process of going from a normal system in many variables to a single high order equation is not so simple. Yet it can be done, and in so doing we attain the required polynomial coefficients [5]. If

$$x'(t) = ax(t) \tag{5}$$

represents the normal system in matrix form, then symbolically

$$D\left(\frac{d}{dt}\right) X_1(t) = X_1^{(n)}(t) + p_1 X_1^{(n-1)}(t) + \cdots + p_{n-1} X_1'(t) + p_n . \tag{6}$$

Because we wish to find out the expanded form of $D(\lambda)$, this relationship is of no use to us. Since similar matrices have the same characteristic equation, ours is the problem of finding a non-singular matrix C, such that

$$C^{-1}aC = b, \tag{7}$$

where b is of the form given in equation (4).

This problem can be approached from an elementary algebraic viewpoint. The relationships in (5) represent n linear equations between $2n$ variables, $[X_1(t), X_2(t), \cdots, X_n(t), X_1'(t), X_2'(t), \cdots, X_n'(t)]$. These are not sufficient to eliminate the $2(n-1)$ variables not involving the subscript 1. However, inasmuch as (5) holds for all values of t we may differentiate it repeatedly until we finally have the system of equations

$$\begin{aligned}
-X_1^{(n)} + \cdots + a_{11}X_1^{(n-1)} + \cdots + a_{1n}X_n^{(n-1)} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
-X_n^{(n)} + a_{n1}X_1^{(n-1)} + \cdots + a_{nn}X_n^{(n-1)} &= 0 \\
-X_1^{(n-1)} + \quad + a_{11}X_1^{(n-2)} + \cdots + a_{1n}X_n^{(n-2)} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
-X_n^{(n-1)} + a_{n1}X_1^{(n-2)} + \cdots + a_{nn}X_n^{(n-2)} &= 0 \\
-X_1' + \cdots + a_{11}X_1 + \cdots + a_{1n}X_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
-X_n' + a_{n1}X_1 + \cdots + a_{nn}X_n &= 0
\end{aligned} \tag{8}$$

These are n^2 linear equations in $n^2 + n$ variables. We wish to eliminate all variables which have a subscript other than one; namely, $(X_2, \cdots, X_n, X_2', \cdots, X_n', \cdots, X_2^{(n)}, \cdots X_n^{(n)})$. These are $(n+1)(n-1) = n^2 - 1$ in number. We may utilize all but one of the n^2 equations to perform this elimination. The remaining equation after substitution will be the desired high order equation, and its coefficients are the polynomial coefficients.

Ordinarily one would solve all but one of the equations for the values of the variables to be eliminated. These would then be substituted into the remaining equation. Actually from the computational standpoint it is unnecessary to solve completely for any unknowns. The so-called "forward" solution of the usual Gauss-Doolittle technique automatically performs the elimination or

substitution, without necessary recourse to a "back" solution for the values of the eliminated variables. These values are in any case of no interest.

There is no unique order in which the equations must be reduced. Indeed, when one order fails because a leading principal minor vanishes, we may switch to another. A suggested convenient order is given below. Let

$$\left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right] = \left[\begin{array}{c|c} a_{11} & R \\ \hline S & M \end{array}\right]; \qquad I = (\delta_{ij}), \qquad (i,j = 1, \cdots, n-1)$$

Then, consider the partitioned matrix

$$W = \left[\begin{array}{cccccc|cccccc} -I & M & 0 & \cdots & 0 & 0 & 0 & -S & 0 & \cdots & 0 \\ 0 & -I & M & \cdots & 0 & 0 & 0 & 0 & -S & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -I & M & 0 & 0 & 0 & \cdots & -S \\ 0 & 0 & 0 & \cdots & 0 & R & 0 & 0 & 0 & \cdots & -a_{11} \\ 0 & 0 & 0 & \cdots & R & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & R & \cdots & 0 & 0 & 0 & 1 & -a_{11} & \cdots & 0 \\ \hline 0 & R & 0 & \cdots & 0 & 0 & 1 & -a_{11} & 0 & \cdots & 0 \end{array}\right] \tag{9}$$

It is simply the matrix of the equations in (8) with the variables $(X_1, X_1', \cdots, X_1^{(n)})$ shifted over to the right-hand side, and with the equations in which the variable one leads off being placed at the bottom.

If the usual "forward" Doolittle technique is followed, then the final elements computed, corresponding to the elements in the lower right-hand box, are the coefficients $(1, p_1, p_2, \cdots, p_n)$. It is the present writer's experience that the Crout form [6], like Dwyer's [7] the last word in Doolittle abbreviation, is to be recommended, particularly since we are dealing with an asymmetrical matrix. A clerk masters its ritual in a few minutes, and the speeds achieved once the operations become mechanical are impressive.

For the trivial case of determining the coefficients corresponding to a two by two matrix the W matrix is of the form

$$\left[\begin{array}{ccc|ccc} -1 & a_{22} & 0 & 0 & -a_{21} & 0 \\ 0 & -1 & a_{22} & 0 & 0 & -a_{21} \\ 0 & 0 & a_{12} & 0 & 1 & -a_{11} \\ \hline 0 & a_{12} & 0 & 1 & -a_{11} & 0 \end{array}\right] \tag{10}$$

The Auxiliary Crout matrix becomes

$$\left[\begin{array}{ccc|ccc} -1 & a_{22} & 0 & 0 & -a_{21} & 0 \\ 0 & -1 & a_{22} & 0 & 0 & -a_{21} \\ 0 & 0 & a_{12} & 0 & 1 & -a_{11} \\ \hline 0 & -a_{12} & a_{22} & 1 & (-a_{11} - a_{22}) & (-a_{12}a_{21} + a_{11}a_{22}) \end{array}\right] \tag{11}$$

The answer in the lower right-hand box will immediately be recognized as the correct one. I have found it convenient to vary the precise Crout routine by dividing vertical columns by the "leading" diagonal element, rather than horizontal columns. This is a matter of indifference and saves some computations. As in the higher order cases, the presence of the identity matrix along the diagonal reduces most of the computations to mere copying. Actually the intelligent computer will soon notice that most of the copying may be eliminated since the numbers in question are to be added in later in other sums of products. After eliminating unknowns corresponding to the equations above the line on which (9) is written, there results the system

$$(12)\qquad \left[\begin{array}{l|cccccccc} R & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -a_{11} \\ RM & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -a_{11} & -RS \\ RM^2 & 0 & 0 & 0 & 0 & \cdots & 1 & -a_{11} & -RS & -RMS \\ \cdot & & & & & & & & & \\ \cdot & & & & & & & & & \\ \cdot & & & & & & & & & \\ RM^{n-1} & 1 & -a_{11} & -RS & -RMS & \cdots & & \cdots & \cdots & -RM^{n-2}S \end{array}\right]$$

Thus, it would be simpler to start from this stage, avoiding unnecessary copying.

This remark shows that the present method is related to the Cayley-Hamilton methods described in [2] and [3], since the above set is derivable from the set

$$(13)\qquad \left[\begin{array}{ll|ccccc} e_1' & A^0 & 1 & 0 & 0 & \cdots & 0 \\ e_1' & A^1 & 0 & 1 & 0 & \cdots & 0 \\ e_1' & A^2 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ e_1' & A^n & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

The last named set appears in the Cayley-Hamilton method when the first row of the powers of the original matrix are used in setting up n equations to determine our n unknowns. Although related, the two methods are distinct since in the Cayley-Hamilton method one would arrive at a different set of equations after straightforward elimination of one variable, and since it would be shorter to dispense with the identity matrix used in the Aitken method in favor of the solution of a single set of equations by the usual Doolittle "back-solution."

The reader will easily see how the method may be modified to handle the more general case of determining the coefficients of

$$(14)\qquad D(\lambda) = |c\lambda + a| = 0,$$

where c and a are any matrices. The method also can be used to reduce a polynomial equation involving a determinant of the nth order, each of whose coefficients are of a given degree in λ, to a lower order determinant whose coefficients are of higher degree in λ.

The present method derives the p's as the algebraic solution of high order linear equations. It would therefore seem inferior to those methods which need only solve a system of n equations. However, two remarks are in order. The matrix of the high order system can be written down immediately without computation. Furthermore, most of the elements in the matrix are zeros, so that a mere counting of the equations is not a true indication of the labor involved.

3. Some comparsions between present method and other methods. Within the brief compass of the present work it is not possible to give an exhaustive appraisal of the comparative computational efficiencies of the methods mentioned. In general, a computing method is to be judged in terms of the number of multiplications that it involves, although other considerations such as the number of additions, the magnitude and sign of the numbers handled, the repetitiveness of the operations involved, the adaptability to punch card machinery, etc. are modifying factors. In this discussion the *power* of a method will be taken to be an inverse function of the number of multiplications that it involves.

It may be said first of all that inasmuch as the minimum number of multiplications involved in computing an nth order determinant is of the order of n^3, even with the most efficient "pivotal" methods, direct computation of the coefficients by principal minors involves, for sufficiently large n, computation of the order of n^4. The same is true of the Wilson method described above. The Horst method, and any other that requires the explicit n powers of an nth order matrix, also asymptotically requires multiplications of the order of n^4. This does not mean that the above three methods are equally powerful for small n, nor even asymptotically, since the coefficients of the n^4 term in the formula for the requisite number of multiplications may not be equal. In fact, Riersol [1] has shown that his method is better than Horst's for small n, but asymptotically less powerful.

It can also be shown that the Cayley-Hamilton methods which simply involve products of the powers of a matrix with row or column vectors are asymptotically more powerful than any of the above methods, the work only increasing as the cube of n. This is true whether the longer Aitken form of reduction is employed or whether the usual Doolittle back-solution is followed. The present method is also an efficient one in the sense that its requisite number of multiplications increases with the cube of n. For small values of n and asymptotically it can be shown to be more powerful than the Cayley-Hamilton method which uses the Aitken method of reduction, although in the limit as n becomes large the ratio of the powers of the two methods approaches unity.

It is of the greatest interest to compare the power of the new method with the shorter Doolittle *C-H* method. It can easily be shown that the coefficients of n^3 in the expressions giving the respective requisite number of multiplications differ in such a way as to make the *C-H* method more powerful after some value of n,

the ratio of the respective powers approaching the limit 8/9. However, for low order matrices the new method is the more powerful. The reader may easily verify this for the case of a second order matrix. Below a sixth order matrix the present method seems to involve the smaller number of multiplications. For a sixth order matrix the two methods seem to involve the same number of multiplications (multiplications by unity not being counted). For matrices of the seventh order or higher the *C-H* method seems to be optimal.

As compared to an explicit evaluation of the coefficients by a straightforward computation of principal minors according to the fundamental definition of a determinant as the sum of signed products of elements, all of the methods discussed are efficient, since the work in the former increases faster than any power of n. However, for each of the methods discussed, in singular cases the method of reduction may fail so that modified procedures will be necessary. In actual practice such singularities will "almost never" be encountered. But in the neighborhood of such singular points the computations become extremely sensitive to any rounding off of digits. Consequently, it is from the nature of the case impossible ever to develop exact rules for the maximum error involved in any given calculation.

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