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A Method to Expand Family of Continuous Distributions based on Truncated Distributions

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Abstract. A new method to generate various family of distributions is introduced. This method introduces a new two-parameter extension of the exponential distribution to illustrate its application. Some statistical and reliability properties of the new distribution, including explicit expressions for the moments, quantiles, mode, moment generating function, mean residual lifetime, stochastic orders, order statistics and some entropies are discussed. Maximum likelihood method is used to estimate the unknown parameters and the Fisher information matrix is given. The obtained results are validated using a real data set and it is shown that the new family provides a better fit than some other known distributions.

Keywords. Exponential distribution; hazard rate function; truncated exponential-exponential distribution; maximum-likelihood estimation; survival function.

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1 Introduction

Many researchers are interested to expand family of distributions in order to obtain better fit for data analyzing. In the last few years, numerous distributions have been proposed based on an extension of known distributions. So, several ways for generating new distributions from classic ones were developed. Among these generators, we point out the class of Beta-G distributions that was introduced by Eugene et al. (2002). Alexander et al. (2012) defined a new generalized class of McDonald distributions. Cordeiro and de Castro (2011) defined the Kumaraswamy-G class of distributions and Zografos and Balakrishnan (2009) and Ristić and Balakrishnan (2012) proposed a family of univariate distributions generated by gamma random variables.

These generators were applied to create new distributions such as beta modified Weibull (Silva et al., 2010), beta Weibull geometric (Cordeiro et al., 2013), McDonald Gumbel (de Brito et al., 2016), Kumaraswamy modified Weibull (Cordeiro et al., 2014), Gamma-Generated Logistic (Castellares et al., 2015), gamma Birnbaum-Saunders (Cordeiro et al., 2016) distributions.

In this paper, we propose the new generator that work with the cumulative distribution function (CDF) of truncated random variable U on $(0, 1)$. We expect that it will attract wider applications in biology, medicine and reliability, and other areas of research.

Based on this generator, we introduce a new distribution, so called the truncated exponential-exponential (TEE) distribution which can be used quite effectively in analyzing several lifetime data, particularly in place of Weibull distribution. The Weibull distribution which contains the exponential and Rayleigh distributions, as special cases, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates. For modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes.

We consider a classic analysis for the TEE distribution. The inferential part is carried out using the asymptotic distribution of the maximum likelihood estimators (MLEs).

The rest of the paper is organized as follows. In Section 2, we introduce truncated method to generate distributions. In Section 3, we introduce the TEE distribution. A range of mathematical properties of the new distribution is considered in Sections 4. These include quantile function, simulation, mode, moments, mean residual lifetime and order statistics. Stochastic orders are discussed in Section 5. The Rényi and Shannon entropies are calculated in Section 6. The maximum likelihood estimate and their properties are discussed in Section 7. In Section 8, we provide a real data set in order to indicate the capacity of the proposed model. Finally, concluding remarks are provided in Section 9.

2 Truncated Method

Let U be a random variable with support (a, b), where $a \leqslant 0$ and $b \geqslant 1$, and CDF F. Then the CDF of truncated random variable U on $(0, 1)$ is given by

$$F_{U_t}(u) = \frac{F(u) - F(0)}{F(1) - F(0)}. \tag{1}$$

Using (1), we introduce the new truncated F-G family of distributions. For each absolutely continuous G distribution (here and henceforth "G" denotes the baseline distribution), we associate the TF-G distribution. The CDF of the TF-G class of distributions is defined by

$$G_X(x) = \frac{F(G(x)) - F(0)}{F(1) - F(0)}, \tag{2}$$

where G is the CDF of random variable V which is used to generate a new distribution.

The probability density function (PDF), $f(x)$, survival function, $S(x)$, and the hazard rate function (HRF), $h(x)$, using (2), are

$$f_X(x) = \frac{g(x)f(G(x))}{F(1) - F(0)},$$

$$S_X(x) = \frac{F(1) - F(G(x))}{F(1) - F(0)}$$

and

$$h_X(x) = \frac{g(x)f(G(x))}{F(1) - F(G(x))},$$

where f and g are the PDFs of random variables U and V, respectively.

3 Truncated Exponential-Exponential Distribution

Now, we introduce new family of distributions called Truncated exponential-exponential (TEE) distribution by taking $F(x)$ and $G(x)$ in (2) as CDFs of exponential distributions with means $1/\alpha$ and $1/\lambda$, respectively.

The non-negative random variable X has the TEE distribution denoted by $TEE(\alpha, \lambda)$, with the shape and scale parameters as $\alpha > 0$ and $\lambda > 0$, respectively, if the CDF of X is

$$F_X(x) = \frac{e^{\alpha} - e^{\alpha e^{-\lambda x}}}{e^{\alpha} - 1}, \qquad x > 0. \tag{3}$$

Note that even if $\alpha < 0$, then (3) is still a bona fide CDF. Hence, we can consider $\mathbb{R} - \{0\}$ as the space of the parameter α. Therefore, in order to receive more flexibility from the distribution in question we consider $\alpha \neq 0$ and $\lambda > 0$ as the shape and scale parameters in the rest of the paper. The PDF of TEE distribution is given by

$$f_X(x) = \frac{\alpha\lambda e^{-\lambda x} e^{\alpha e^{-\lambda x}}}{e^{\alpha} - 1}, \qquad x > 0.$$

It is easy to show when α converges to 0, then TEE distribution converges to exponential distribution with mean $1/\lambda$. The survival and hazard rate functions for the TEE distribution are given in the following forms

$$S_X(x) = \frac{e^{\alpha e^{-\lambda x}} - 1}{e^{\alpha} - 1}, \qquad x > 0,$$

$$h_X(x) = \frac{\alpha\lambda e^{-\lambda x} e^{\alpha e^{-\lambda x}}}{e^{\alpha e^{-\lambda x}} - 1}, \qquad x > 0.$$

Figure 1 shows some of the different shapes of $TEE(\alpha, \lambda)$ for selected values of α and λ. Figure 2 shows some of the different shapes of HRF for selected values of α and λ. If $\alpha > 0$, then the hazard function is a non-increasing function which converges to λ as α tends to infinity. If $\alpha < 0$, then it is a non-decreasing function which converges to λ as α tends to infinity.

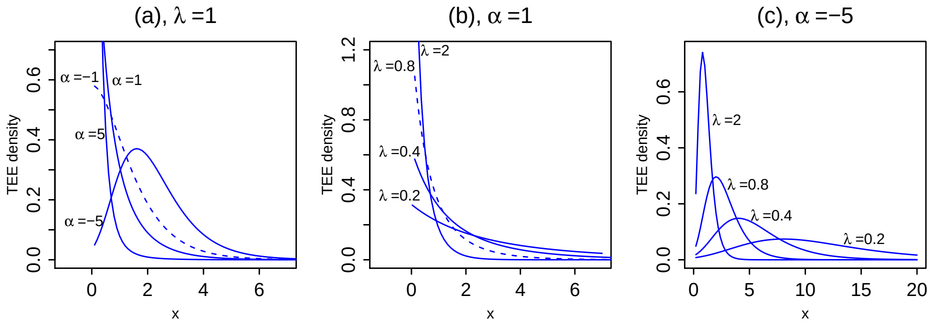

Figure 1. (a): The PDF of TEE distribution with various shape parameter and fixed scale parameter $\lambda = 1$. (b) and (c): The PDF of TEE distribution with various scale parameter and fixed shape parameter $\alpha = 1$ and $\alpha = -5$, respectively.

4 Statistical Properties and Order Statistics

4.1 Quantile Function and Simulation

The quantile function of TEE distribution is given by

$$x_p = \frac{-1}{\lambda} \log \left[\frac{1}{\alpha} \log \left\{ e^{\alpha}(1-p) + p \right\} \right]. \tag{4}$$

One of the advantages of the TEE distribution is that its CDF has a closed form which helps us to generate random variables by using the following simple formula

$$X = \frac{-1}{\lambda} \log \left[\frac{1}{\alpha} \log \left\{ e^{\alpha}(1-U) + U \right\} \right],$$

where U is a uniformly distributed random variable on (0, 1).

4.2 Mode and Median

The mode of TEE exists for any $\alpha < -1$ in point of

$$M = \frac{-1}{\lambda} \log \left(\frac{-1}{\alpha} \right).$$

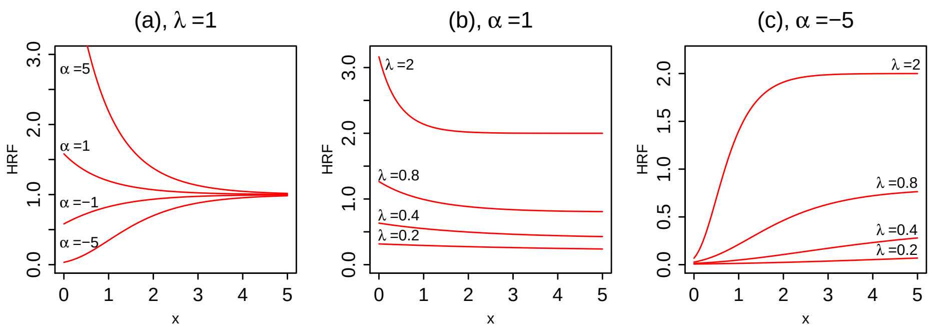

Figure 2. (a): The HRF of TEE distribution with various shape parameter and fixed scale parameter $\lambda = 1$. (b) and (c): The HRF of TEE distribution with various scale parameter and fixed shape parameter $\alpha = 1$ and $\alpha = -5$, respectively.

The median can be derive from (4) by considering $p = 0.5$ as

$$m = \frac{-1}{\lambda} \log \left(\frac{1}{\alpha} \log \frac{e^{\alpha} + 1}{2} \right).$$

It is not possible to compute the mean of the TEE distribution explicitly, but from Figure 3 (a) it is observed that

$$\text{mode} < \text{median} < \text{mean}, \quad \text{if} \quad \alpha < -1,$$

and

$$\text{median} \leqslant \text{mean}, \quad \text{if} \quad \alpha \geqslant -1.$$

Figure 3 (b) shows the variance of TEE distribution for different values of α, when $\lambda = 1$. It is observed that the mean and the variance are decreasing functions of α.

4.3 Moment-Generating Function and Moments

Using the series representations

$$e^{u} = \sum_{k=0}^{\infty} \frac{u^{k}}{k!}, \tag{5}$$

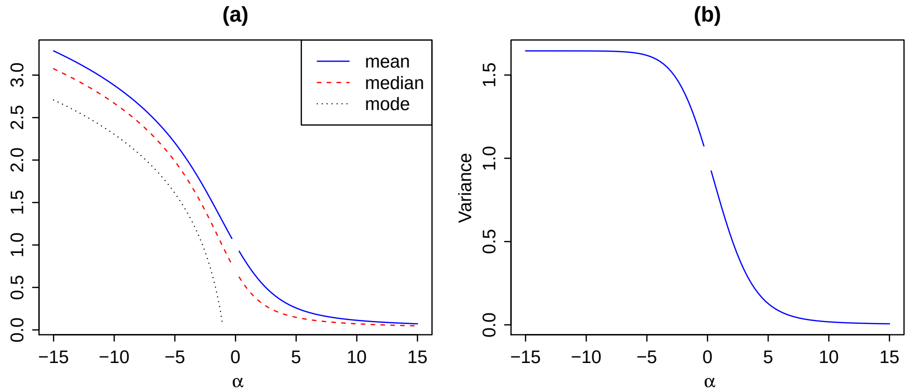

Figure 3. (a): The mean, median and mode of TEE distribution for different values of α, when $\lambda = 1$. (b): The variance of TEE distribution for different values of α, when $\lambda = 1$.

and

$$\int u^n (\log u)^m du = u^{n+1} \sum_{k=0}^{m} (-1)^k \frac{m!(\log u)^{m-k}}{(m-k)!(n+1)^{k+1}}, \tag{6}$$

(Gradshteyn and Ryzhik, 2014, p. 238, formula. 2.722) we derive two infinite expansions for the moment-generating function (MGF) and the n-th moment as

$$M_X(t) = \frac{\lambda}{e^{\alpha} - 1} \sum_{k=0}^{\infty} \frac{\alpha^{k+1}}{(k\lambda - t + \lambda)k!}, \qquad t < \lambda,$$

and

$$E(X^n) = \frac{n!}{\lambda^n (e^{\alpha} - 1)} \sum_{k=1}^{\infty} \frac{\alpha^k}{k! k^n}.$$

4.4 Mean Residual Lifetime

The expected additional lifetime given that a component has survived until time t is called mean residual life (MRL). It describes the aging process. Therefore, it plays an important role in reliability and survival analysis. The MRL function, $\mu(t)$, for random variable X is defined as

$$\mu(t) = \frac{1}{S(t)} \int_t^{\infty} S(x) dx, \qquad t \geqslant 0. \tag{7}$$

Using (7) the MRL function of $X \sim TEE(\alpha, \lambda)$ is given by

$$\mu(t) = \frac{1}{\lambda(e^{\alpha e^{-\lambda t}} - 1)} \sum_{k=1}^{\infty} \frac{\alpha^k}{k!k} e^{-k\lambda t}, \qquad t \geqslant 0, \alpha \neq 0, \lambda > 0.$$

Ghai and Mi (1999) have shown that if HRF is increasing (decreasing), then MRL function is decreasing (increasing). Therefore, the function $\mu(t), t \geqslant 0$ is increasing in t for $\alpha > 0$ and it is decreasing in t for $\alpha < 0$.

4.5 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. We now give the PDF of the k-th order statistic $Y = X_{k:n}$ in a random sample of size n from the TEE distribution as follows

$$\begin{aligned} f_Y(y) =& \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\{1 - F(y)\}^{n-k} f(y) \\ =& \frac{n!\alpha\lambda}{(e^{\alpha} - 1)^n (k-1)!(n-k)!} e^{(\alpha e^{-\lambda y} - \lambda y)} (e^{\alpha} - e^{\alpha e^{-\lambda y}})^{k-1} (e^{\alpha e^{-\lambda y}} - 1)^{n-k}. \end{aligned}$$

Using (5) and (6) and binomial expansion, the q-th moment of Y can be expressed as

$$E(Y^q) = \frac{n!q!}{\lambda^q (e^{\alpha} - 1)^n} \sum_{l=0}^{k-1} \sum_{m=0}^{n-k} \sum_{r=0}^{\infty} \frac{(-1)^{n+l-m-k} e^{\alpha(k-l-1)} \alpha^{r+1} (l+m+1)^r}{l!m!r!(k-l-1)!(n-k-m)!(r+1)^{q+1}}.$$

5 Stochastic Orders

Let us give a quick review of stochastic orders and some notions which are relevant in the context of this paper. Let X and Y be two random variables with distribution functions F and G and density functions f and g, respectively.

Usual stochastic order (denoted by $X \leqslant_{st} Y$): X is said to be stochastically smaller than Y if for all x, $F(x) \geqslant G(x)$.

Hazard rate ordering (denoted by $X \leqslant_{hr} Y$): X is smaller than Y in hazard rate ordering if $h_X(x) \geqslant h_Y(x)$, where $h_X(x)$ and $h_Y(x)$ are the hazard rate functions of random variables X and Y, respectively.

Likelihood ratio ordering (denoted by $X \leqslant_{lr} Y$): X is smaller than Y in likelihood ratio ordering if $\dfrac{g(x)}{f(x)}$ is an increasing function of x.

The following implications hold among these stochastic orders: $X \leqslant_{lr} Y \Rightarrow X \leqslant_{hr} Y \Rightarrow X \leqslant_{st} Y$. For further results see Shaked and Shanthikumar (2007).

Suppose random variables X and Y are distributed according to $TEE(\alpha_1, \lambda_1)$ and $TEE(\alpha_2, \lambda_2)$, respectively. If $\lambda_1 = \lambda_2$, then it is easy to show that $\dfrac{g(x)}{f(x)}$ is a decreasing function of x if and only if $\alpha_1 < \alpha_2$. So, $Y \leqslant_{lr} X$ implies $Y \leqslant_{hr} X$ and hence we see that $Y \leqslant_{st} X$ for $\alpha_1 < \alpha_2$. If $\alpha_1 = \alpha_2 = \alpha$ then

$$\frac{g(x)}{f(x)} = \frac{\lambda_2}{\lambda_1} h(x), \tag{8}$$

where

$$h(x) = e^{(\lambda_1 - \lambda_2)x} e^{\alpha(e^{-\lambda_2 x} - e^{-\lambda_1 x})}.$$

Hence, we have

$$\frac{\partial \log h(x)}{\partial x} = q(\lambda_1) - q(\lambda_2),$$

where

$$q(\lambda) = \lambda(1 + \alpha e^{-\lambda x}). \tag{9}$$

It can be shown that if $\alpha > -1$ then (9) is an increasing function of λ (see Appendix A for more details) and it follows that (8) is a decreasing function of x if $\lambda_1 < \lambda_2$. So, $Y \leqslant_{lr} X$ implies $Y \leqslant_{hr} X$ and hence we see that $Y \leqslant_{st} X$ for any $\alpha > -1$ and $\lambda_1 < \lambda_2$.

6 Entropies

Entropy has been used in various situations in Science and Engineering. The entropy of a random variable X with PDF $f(x)$ is a measure of variation of the uncertainty. There exist many entropy definitions and they are not equally good for all applications. While the most famous (and most liberal) Shannon (1951) Entropy, which quantifies the encoding length, is extremely useful in information theory. Shannon showed important applications of this entropy in communication theory and many applications have been used in different areas such as Engineering, Physics, Biology and Economics. Using

(5) and (6), we obtain the Shannon entropy as

$$E\{-\log f(X)\} = \log\left(\frac{e^{\alpha}-1}{\lambda\alpha}\right) + \frac{1}{e^{\alpha}-1}\sum_{k=1}^{\infty}\left\{\frac{\alpha^k}{k!k} - \frac{\alpha^{k+1}}{(k-1)!(k+1)}\right\}.$$

A generalized definition of entropy that stems from modifying the additivity postulate and results in a class of information measures that contain Shannons definitions as special cases is Rényi (1961) entropy. If X has the PDF $f(x)$ then Rényi entropy is defined by

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int f(x)^{\rho}dx\right\}, \tag{10}$$

where $\rho > 0$ and $\rho \neq 1$. Using (5), the integral in $I_R(\rho)$ for the TEE distribution can be reduced to

$$\int_0^{\infty} f(x)^{\rho}dx = \left(\frac{\alpha}{e^{\alpha}-1}\right)^{\rho}\lambda^{\rho-1}\sum_{k=0}^{\infty}\frac{(\rho\alpha)^k}{k!(\rho+k)}.$$

So, one obtains the Rényi entropy as

$$I_R(\rho) = \left(\frac{\rho}{1-\rho}\right)\log\left(\frac{\alpha}{e^{\alpha}-1}\right) - \log\lambda + \frac{1}{1-\rho}\log\left\{\sum_{k=0}^{\infty}\frac{(\rho\alpha)^k}{k!(\rho+k)}\right\}.$$

7 Maximum Likelihood Estimation

We now determine the MLEs of the parameters of the TEE distribution from complete samples only. Let $x_1, x_2, \ldots, x_n$ be a sample from $TEE(\alpha, \lambda)$ distribution. Then, the log-likelihood function is

$$\log L = n\log\alpha - n\log(e^{\alpha}-1) + n\log\lambda - \lambda\sum_{i=1}^{n}x_i + \alpha\sum_{i=1}^{n}e^{-\lambda x_i}.$$

The first order derivatives of $\log L$ are

$$\frac{\partial \log L}{\partial\alpha} = \frac{n}{\alpha} - \frac{ne^{\alpha}}{e^{\alpha}-1} + \sum_{i=1}^{n}e^{-\lambda x_i}, \tag{11}$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} x_i e^{-\lambda x_i}. \tag{12}$$

The MLEs of the parameters can be obtained in the following manner. First, we solve Equation (12) for α and obtain

$$\alpha = \frac{n - \lambda \sum_{i=0}^{n} x_i}{\lambda \sum_{i=0}^{n} x_i e^{-\lambda x_i}} = A(\lambda). \tag{13}$$

Now, by substituting Equation (13) in Equation (11), we obtain the equation

$$0 = \frac{n}{A(\lambda)} - \frac{n e^{A(\lambda)}}{e^{A(\lambda)} - 1} + \sum_{i=1}^{n} e^{-\lambda x_i}.$$

By solving this non-linear equation for λ, we obtain the MLE of λ denoted by $\hat{\lambda}$. Finally, replacing $\hat{\lambda}$ in Equation (13), we obtain the MLE of the parameter α denoted by $\hat{\alpha}$. In this paper we use the optim function from the statistical software R (Team, 2013) to maximize the logarithm of the likelihood function.

Under conditions that are fulfilled for the parameters in the interior of the parameter space, the asymptotic distribution of $(\hat{\alpha}, \hat{\lambda})$ as $n \to \infty$ is bivariate normal with mean (α, λ) and variance co-variance matrix $\mathbf{I}^{-1}(\hat{\alpha}, \hat{\lambda})$,

$$(\hat{\alpha} - \alpha, \hat{\lambda} - \lambda) \longrightarrow N_2(\mathbf{0}, \mathbf{I}^{-1}(\hat{\alpha}, \hat{\lambda})),$$

where $\mathbf{I}(\alpha, \lambda)$ is the Fisher information matrix. Using (11) and (12) the elements of $\mathbf{I} = (I_{ij})$ are given by

$$I_{11} = \frac{n}{\alpha^2} - \frac{n e^{\alpha}}{(e^{\alpha} - 1)^2},$$

$$I_{22} = \frac{n}{\lambda^2} - \frac{2n}{\lambda^2 (e^{\alpha} - 1)} \sum_{k=0}^{\infty} \frac{\alpha^{k+2}}{k!(k+2)^3},$$

and

$$I_{12} = I_{21} = \frac{n}{\lambda (e^{\alpha} - 1)} \sum_{k=0}^{\infty} \frac{\alpha^{k+1}}{k!(k+2)^2}.$$

The $100(1-\gamma)\%$ two sided asymptotic confidence intervals for α and λ are given, respectively, by $\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \hat{se}(\hat{\alpha})$ and $\hat{\lambda} \pm Z_{\frac{\gamma}{2}} \hat{se}(\hat{\lambda})$, where $\hat{se}(\cdot)$ is the square root of the diagonal element of $\mathbf{I}^{-1}(\hat{\alpha}, \hat{\lambda})$ corresponding to each parameter, and $Z_{\frac{\gamma}{2}}$ is the upper $(\frac{\gamma}{2})$ th percentile of a standard normal distribution.

8 Application: Coal-Mining Dataset

The following 109 data points represent the intervals in days between 109 successive coal-mining disasters in Great Britan, for the period 1875-1951, published by Maguire et al. (1952). The sorted data are given as follows: 1 4 4 7 11 13 15 15 17 18 19 19 20 20 22 23 28 29 31 32 36 37 47 48 49 50 54 54 55 59 59 61 61 66 72 72 75 78 78 81 93 96 99 108 113 114 120 120 120 123 124 129 131 137 145 151 156 171 176 182 188 189 195 203 208 215 217 217 217 224 228 233 255 271 275 275 275 286 291 312 312 312 315 326 326 329 330 336 338 345 348 354 361 364 369 378 390 457 467 498 517 566 644 745 871 1312 1357 1613 1630.

In order to identify the shape of the hazard function, we shall consider a graphical method based on the Total Time on Test (TTT) plot. In its empirical version the TTT plot is given by

$$T(r/n) = \frac{(\sum_{i=1}^{r} Y_{i:n} + (n-r)Y_{r:n})}{\sum_{i=1}^{r} Y_{i:n}},$$

where $r = 1, 2, \ldots, n$ and $Y_{i:n}$ represents the ith order statistic of the sample. If the empirical TTT transform is convex, concave, first convex then concave, and first concave then convex, the shape of the corresponding hazard rate function is, respectively, decreasing, increasing, bathtub, and unimodal (for more details, see Aarset, 1987). Figure 4 shows the empirical TTT plots for the coal-mining disasters data, which is convex indicating a decreasing failure rate function, which can be properly accommodated by TEE distribution. We compare TEE distribution with four other two-parameter models

- Gamma distribution with PDF $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $\alpha > 0, \lambda > 0, x > 0$.
- Weibull distribution with PDF $f(x) = \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha}$, $\alpha > 0, \lambda > 0, x > 0$.
- Generalized exponential distribution (GE) introduced by Gupta and Kundu (1999) with PDF $f(x) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}$, $\alpha > 0, \lambda > 0, x > 0$.

- Weighted exponential distribution (WE) proposed by Gupta and Kundu (2009) with PDF $f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1 - e^{-\alpha\lambda x})$, $\alpha > 0$, $\lambda > 0$, $x > 0$.

To see which one of these models is more appropriate to fit data. The MLEs of parameters, Kolmogorov-Smirnov statistics and p-values are obtained.

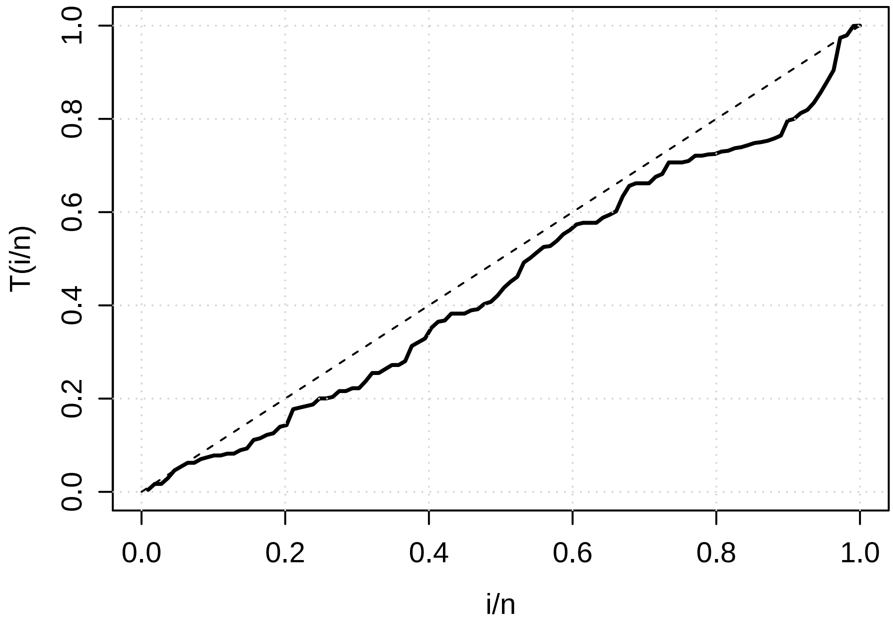

Figure 4. Empirical TTT plot for the coal-mining data.

The results are given in Table 1. By comparing the log-likelihood values and p-values based on the Kolmogorov Smirnov test, we see that TEE distribution gives a satisfactory fit to data. The relative histogram and the fitted TEE distribution are plotted in Figure 5 (a). In order to assess if the model is appropriate, the plots of the fitted survival functions and empirical survival functions for fitted models are displayed in Figure 5 (b)-(f).

Table 1. The maximum likelihood estimates, the corresponding standard errors are given in parentheses and Kolmogorov-Smirnov statistics and p-values for coal-mining data.

The model	MLEs of the parameters	Log-likelihood	K-S statistic	p-value
gamma	$\hat{\alpha} = 0.8555, \hat{\lambda} = 0.0037$	-702.4007	0.0823	0.4517
	$(0.100670, 0.000575)$			
Weibull	$\hat{\alpha} = 0.8848, \hat{\lambda} = 0.0046$	-701.7724	0.0784	0.5135
	$(0.0633, 0.0005)$			
GE	$\hat{\alpha} = 0.8605, \hat{\lambda} = 0.0039$	-702.5524	0.0830	0.4402
	$(0.1051036589, 0.0005061644)$			
WE	$\hat{\alpha} = 704.9214, \hat{\lambda} = 0.00429201$	-703.2087	0.0836	0.4313
	$(232.9913, 0.00003390085)$			
TEE	$\hat{\alpha} = 2.0594, \hat{\lambda} = 0.0024$	-700.6492	0.0725	0.6154
	$(1.60496444, 0.00118128)$			

9 Conclusion

In this paper, we propose a new method to generalize family of distributions, based on truncated continuous random variable on support $(0, 1)$. As an application a new two-parameter family of distributions, namely TEE distribution, is introduced which may sometimes be a competitor to the Weibull, gamma and other two-parameter life time models. Various properties of the new distribution are obtained. These properties include moments, quantiles, mode, moment generating function, mean residual lifetime, stochastic orders, order statistics and some entropies. We discuss maximum likelihood estimation of the model parameters and derive the observed information matrix. An application of the TEE family is demonstrated in a real dataset. The development and investigate the behavior of the proposed method to expand other family of distributions will be taken up in a future work.

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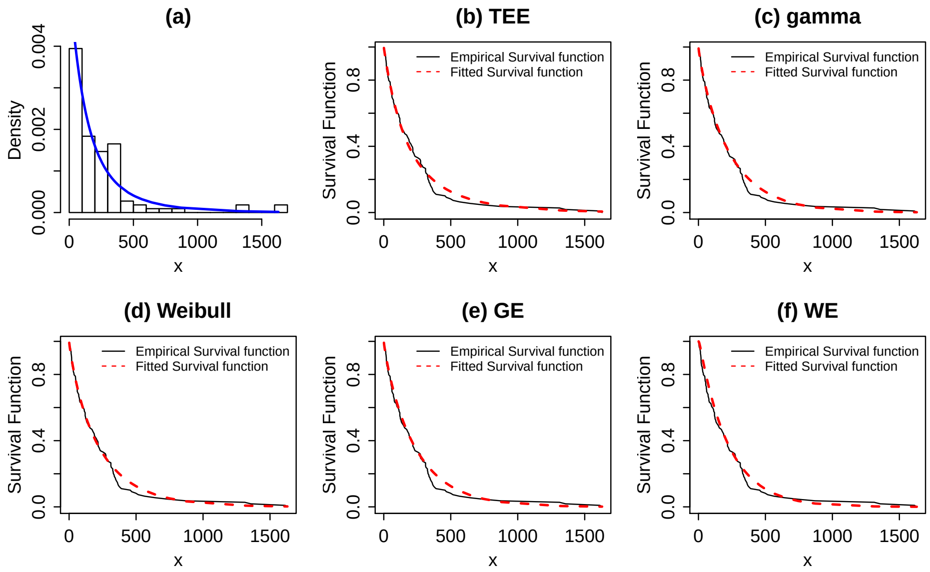

Figure 5. (a): The histogram and the fitted TEE distribution. (b)-(f): The fitted survival functions and empirical survival functions for fitted models.

References

Aarset, M.V. (1987). How to Identify a Bathtub Hazard Rate. *IEEE Transactions on Reliability, IEEE*, **36**, 106-108.

Alexander, C., Cordeiro, G.M., Ortega, E.M.M. and Sarabia, J.M. (2012). Generalized Beta-generated Distributions. *Computational Statistics and Data Analysis*, **56**, 1880-1897.

de Brito, E., Silva, G.O., Cordeiro, G.M. and Demetrio, C.G.B. (2016). The McDonald Gumbel Model. *Communications in Statistics - Theory and Methods*, **45**, 3367-3382.

Castellares, F., Santos, M.A.C., Montenegro, L.C. and Cordeiro, G.M. (2015). A Gamma-Generated Logistic Distribution: Properties and Inference. *American Journal of Mathematical and Management Sciences*, **34**, 14-39.

Cordeiro, G.M. and de Castro, M. (2011). A New Family of Generalized Distributions. *Journal of Statistical Computation and Simulation*, **81**, 883-898.

Cordeiro, G.M., Lima, M.C.S., Cysneiros, A.H.M.A., Pascoa, M.A.R., Pescim, R.R. and Ortega, E.M.M. (2016). An Extended Birnbaum–Saunders Distribution: Theory, Estimation, and Applications. *Communications in Statistics - Theory and Methods*, **45(8)**, 2268-2297.

Cordeiro, G.M., Ortega, E.M.M. and Silva, G.O. (2014). The Kumaraswamy Modified Weibull Distribution: Theory and Applications. *Journal of Statistical Computation and Simulation*, **84**, 1387-1411.

Cordeiro, G.M., Silva, G.O. and Ortega, E.M.M. (2013). The Beta-Weibull Geometric Distribution. *Statistics*, **47**, 817-834.

Eugene, N., et al. (2002). Beta-normal Distribution and Its Applications. *Communications in Statistics-Theory and Methods*, **31**, 497-512.

Ghai, G.L. and Mi, J. (1999). Mean Residual Life and Its Association with Failure Rate. *IEEE Transactions on Reliability*, **48**, 262-266.

Gradshteyn, I.S. and Ryzhik, I.M. (2014). *Table of Integrals, Series, and Products*. Academic press.

Gupta, R.D. and Kundu, D. (1999). Generalized Exponential Distributions. *Australian and New Zealand Journal of Statistics*, **41**, 173-188.

Gupta, R.D. and Kundu, D. (2009). A New Class of Weighted Exponential Distributions. *Statistics*, **43**, 621-634.

Maguire, B., et al. (1952). The Time Intervals between Industrial Accidents. *Biometrika*, **39**, 168-180.

Rényi, A. (1961). On Measures of Entropy and Information. *In Proceedings of the 4th berkeley symposium on mathematical statistics and probability*, **1**, 547-561. Berkeley: University of California Press.

Ristić, M.M. and Balakrishnan, N. (2012). The Gamma-exponentiated Exponential Distribution. *Journal of Statistical Computation and Simulation*, **82**, 1191-1206.

Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer, New York.

Shannon, C.E. (1951). Prediction and Entropy of Printed English. *The Bell System Technical Journal*, **30**, 50-64.

Silva, G.O., Ortega, E.M.M. and Cordeiro, G.M. (2010). The Beta Modified Weibull Distribution. *Lifetime Data Analysis*, **16**, 409-30.

R Core Team (2013). R: A language and environment for statistical computing. *R Foundation for Statistical Computing*, Vienna, Austria. URL http://www.R-project.org/.

Zografos, K. and Balakrishnan, N. (2009). On Families of Beta- and Generalized Gamma-generated Distributions and Associated Inference. *Statistical Methodology*, **6**, 344-362.

Appendix A

$$\frac{\partial q(\lambda)}{\partial \lambda} = 1 + \alpha e^{-\lambda x} - \alpha \lambda x e^{-\lambda x},$$

by taking $u = \lambda x$

$$\frac{\partial q(\lambda)}{\partial \lambda} = 1 + \alpha e^{-u} - \alpha u e^{-u}.$$

It follows that $\frac{\partial q(\lambda)}{\partial \lambda} > 0$ if

$$\alpha > \frac{-e^{u}}{1-u}; \qquad 0 < u < 1,$$

or

$$\alpha < \frac{e^{u}}{1-u}; \qquad u > 1.$$

By considering following equations

$$\lim_{x \to 0^+} (\frac{e^{u}}{1-u}) = 1,$$

$$\lim_{x \to 1^-} (\frac{e^{u}}{1-u}) = +\infty,$$

$$\lim_{x \to +\infty} (\frac{e^{u}}{1-u}) = -\infty,$$

and

$$\lim_{x \to 1^+} (\frac{e^{u}}{1-u}) = -\infty,$$

it follows that $\frac{\partial q(\lambda)}{\partial \lambda} > 0$ for $\alpha > -1$.

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